Constrained optimization simple exercises

Solve the following exercices analitycally or geometrically by using the KKT conditions.

(1) $\max \; x_1 - 2x_1^2 + 2x_2 - x_2^2 + x_1x_2,$

$$\text{s.t.} \begin{cases} x_1^2 - x_2 \leq 0, \\ 2x_1 - x_2 \geq 0. \end{cases}$$

(2) $\min \; x_1,$

$$\text{s.t.} \begin{cases} (x_1 - 1)^2 + (x_2 + 2)^2 \leq 16, \\ x_1^2 + x_2^2 \geq 13. \end{cases}$$

(3) $\min \; x_1 + 3x_2,$

$$\text{s.t.} \begin{cases} x_1 + \frac{1}{10}x_2 \geq 1, \\ \frac{1}{2}x_1 + \frac{1}{9}x_2 \geq 1, \\ \frac{1}{3}x_1 + \frac{1}{8}x_2 \geq 1, \\ \frac{1}{4}x_1 + \frac{1}{7}x_2 \geq 1, \\ \frac{1}{5}x_1 + \frac{1}{6}x_2 \geq 1, \\ \frac{1}{6}x_1 + \frac{1}{5}x_2 \geq 1, \\ \frac{1}{7}x_1 + \frac{1}{4}x_2 \geq 1, \\ \frac{1}{8}x_1 + \frac{1}{3}x_2 \geq 1, \\ \frac{1}{9}x_1 + \frac{1}{1}x_2 \geq 1, \\ \frac{1}{10}x_1 + x_2 \geq 1, \\ x_1 \leq 20, \\ x_2 \geq 0. \end{cases}$$

(4) $\min \; x_1 + 3x_2,$

$$\text{s.t.} \begin{cases} x_1 + 2x_2 \geq 2, \\ x_1 - 3x_2 \leq 2, \\ -x_1 + 3x_2 \leq 12, \\ x_1, x_2 \geq 0. \end{cases}$$

(5) $\min\ x_1^2 + x_2^2,$

$$\text{s.t.} \begin{cases} \frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 2, \\ x_1 \leq 2, \\ x_2 \geq 1. \end{cases}$$

(6) $\min\ x_1^2,$

$$\text{s.t.} \begin{cases} \frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 2, \\ x_1 \leq 2, \\ x_2 \geq 1. \end{cases}$$

(7) $\min\ x\sin(x) + y\sin(y),$

$$\text{s.t.} \begin{cases} x \geq \frac{1}{3}, \\ y \geq \frac{3}{4}, \\ x - \sin(y) \geq 0, \\ x^2 + y^2 \leq 5. \end{cases}$$

(8) $\min\ (x+1)^2 + \frac{1}{2}y^2,$

$$\text{s.t.} \begin{cases} x \leq 3, \\ y \geq 0, \\ \frac{1}{8}x^3 - y \geq 0. \end{cases}$$

(9) Solve graphically:

$$\min\ \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}(x_2 - 5)^2,$$

$$\text{s.t.} \begin{cases} -2x_1 + x_2 \leq 2, \\ -x_1 + x_2 \leq 3, \\ x_1 \leq 3, \\ x_1, x_2 \geq 0. \end{cases}$$

Simple numerical problems. Use simple programs to solve them.

(1) $f(x_1, x_2) := 2(x_1 + 1)^2 + 8(x_2 + 3)^2 + 5x_1 + x_2$

(a) Solve the problem to minimize f over $\mathbb{R}^2$ using the steepest descent and Newton's (unit step) methods. Start at the points $(10, 10)$ and $(-5, -5)$. Towards which point do the methods converge? How many iterations are required?

(b) Is the point obtained an optimal point (globally or locally)? Why?

(c) Why does Newton's method converge in one iteration?

(2) **Rosenbrock's function**

$$f(x_1, x_2) := 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

(a) Solve the problem to minimize f over $\mathbb{R}^2$ using the steepest descent and Newton's (unit step, Levenberg–Marquardt, and modified) methods. Start at the point $(-1.5, -1)$. Towards which point do the methods converge? How many iterations are required for the different methods?

(b) (Where) Is f convex? Is the point obtained a global minimum?

(3) $f(x_1, x_2) := \cos(x_1^2 - 3x_2) + \sin(x_1^2 + x_2^2)$

(a) Solve the problem to minimize f over $\mathbb{R}^2$ using the steepest descent and Newton's (unit step, Levenberg–Marquardt, and modified) method. Start at the point $(1.2, 0.5)$. Towards which points do the methods converge? How many iterations are required for the different methods? Does something unexpected happen? What is the explanation?

(b) Is f convex? Are the points obtained global minima?

(4) $\min\ (x_1 - 2)^2 + (x_2 - 1)^2,$

$$\text{s.t.} \begin{cases} -x_1 - x_2 + 2 \geq 0, \\ -x_1^2 + x_2 \geq 0. \end{cases}$$

(5) $\min\ (x_1 - x_2)^2 + (x_2 - x_3)^3 + (x_3 - x_4)^4 + (x_4 - x_5)^4,$

$$\text{s.t.} \begin{cases} x_1 + x_2^2 + x_3^3 - 3 = 0, \\ x_2 - x_3^2 + x_4 - 1 = 0, \\ x_1 x_5 - 1 = 0. \end{cases}$$