Infinite-Dimensional Topology and the Hilbert-Smith Conjecture

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Hilbert-Smith Conjecture

- **Hilbert's Fifth Problem** (1900)**.** Let G be a compact group acting on the manifold M. Suppose that

$$G \times M \to M$$

is effective. Then G is a Lie group. That is, the space of G is a differentiable manifold with multiplication and inversion differentiable functions. The action is also differentiable.

Focus on the Group

- When is a locally compact group a Lie group?

Solutions

- John von Neumann (1929). If G is a compact group such that G is locally Euclidean, then G is a Lie group.
- Equivalently, if G is a compact group that has no small subgroups, then G is a Lie group.
- Equivalently, if G is finite dimensional and locally connected, then G is a Lie group.

Solutions

- Lev Pontryagin (1934). If G is a locally compact Abelian group with no small subgroups, then G is a Lie group.

- Andrew Gleason, Dean Montgomery, Leo Zippin (1950's). If G is a locally compact group with no small subgroups, then G is a Lie group.

Solutions

- Hidehiko Yamabe (1953). A locally compact connected group G is the inverse limit of Lie groups. If it has no small subgroups, then it is a Lie group.

Hilbert Space

- What can be said about Hilbert space manifold groups?

- **Theorem** (Bessaga and Pelczynski, 1972). Let *X* be a complete separable metric space. Then the space of measureable functions from [0,1] to *X* is Hilbert space.

$$\mathfrak{M}([0,1],X) \approx \ell_2$$

$$\mathfrak{M}([0,1],G)$$

Hilbert Space

- Examples.

$$\mathfrak{M}([0,1], Z_2)$$

$$\mathfrak{M}([0,1], Q)$$

$$\pi_1\left({\mathfrak{M}([0,1], Q)}\middle/{Q}\right) \cong Q$$

Hilbert-Smith

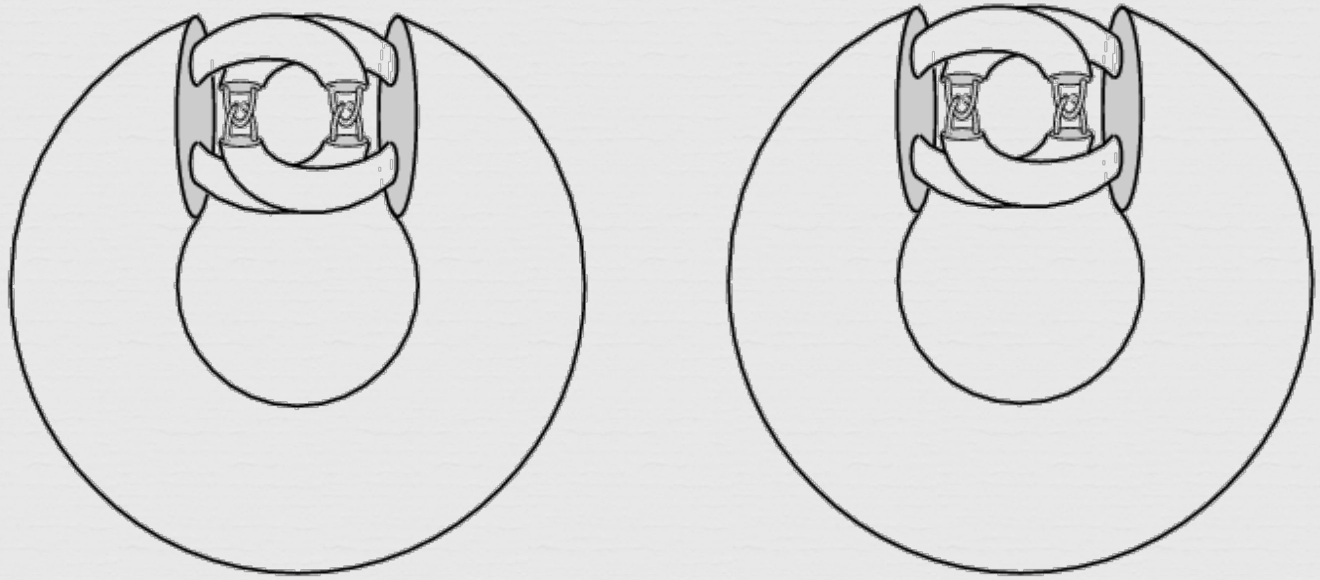
- **The other side of Hilbert's Fifth Problem.** If

$$G \times M \to M$$

is an effective action by a compact group *G* on a differentiable manifold *M*, then *G* is Lie and the action is differentiable.

R. H. Bing

- There is a Z_2 – action on S^3 that cannot be differentiable. (1952)



http://mathworld.wolfram.com/AlexandersHornedSphere.html

Hilbert-Smith Conjecture

- **Conjecture.** If $G \times M \rightarrow M$ is an effective action of a compact group G on a manifold M, then G is a Lie group.

- **Equivalent.** There is no effective action of a p-adic group, Δ_p, on a manifold for any p.

Adding Machine

$$\alpha = (p_0, p_1, \ldots)$$

$$\Delta_\alpha = \lim_{n\to\infty}\left\{Z_{p_0 p_1 \cdots p_n} \leftarrow Z_{p_0 p_1 \cdots p_{n+1}}\right\}$$

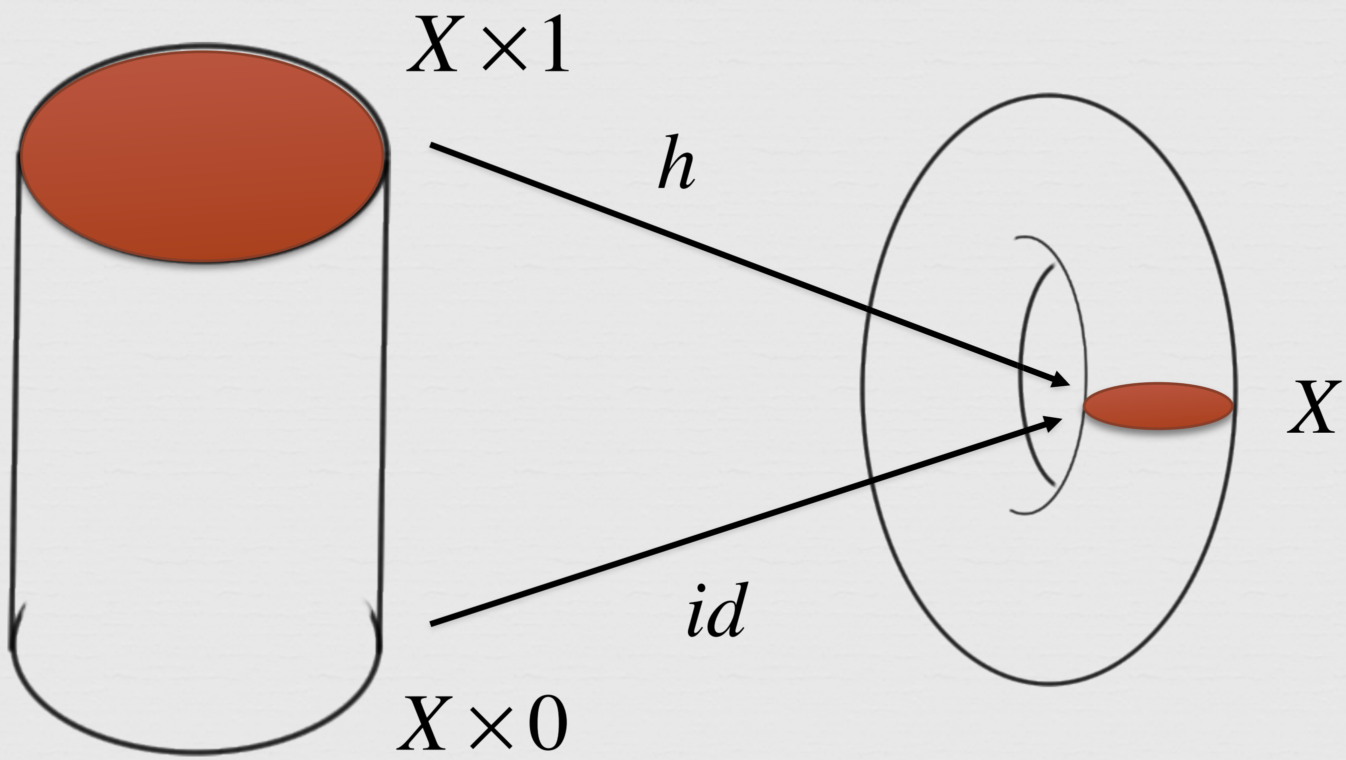
$$Z_{p_0} \longleftarrow Z_{p_0 p_1} \longleftarrow Z_{p_0 p_1 p_2} \longleftarrow \circ \circ \circ \; \Delta_\alpha$$

Adding Machine

$$\Delta_p \xrightarrow{h} \Delta_p$$

$$h(x) = 1 + x$$

Mapping Torus



Mapping Torus

$$\Delta_p \times X \to X$$

$$\Sigma_h = T_h = X \times [0,1] \Big/ (x,0) \approx (h(x),1)$$

$$\Sigma_p \times T_h \to T_h$$

$$R \times T_h \to T_h$$

Action by Solenoid

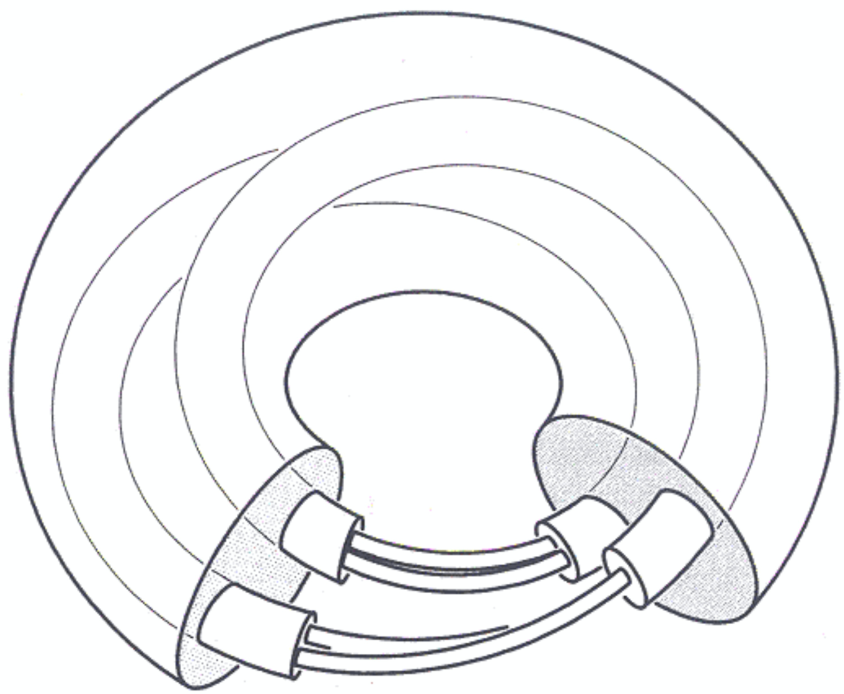
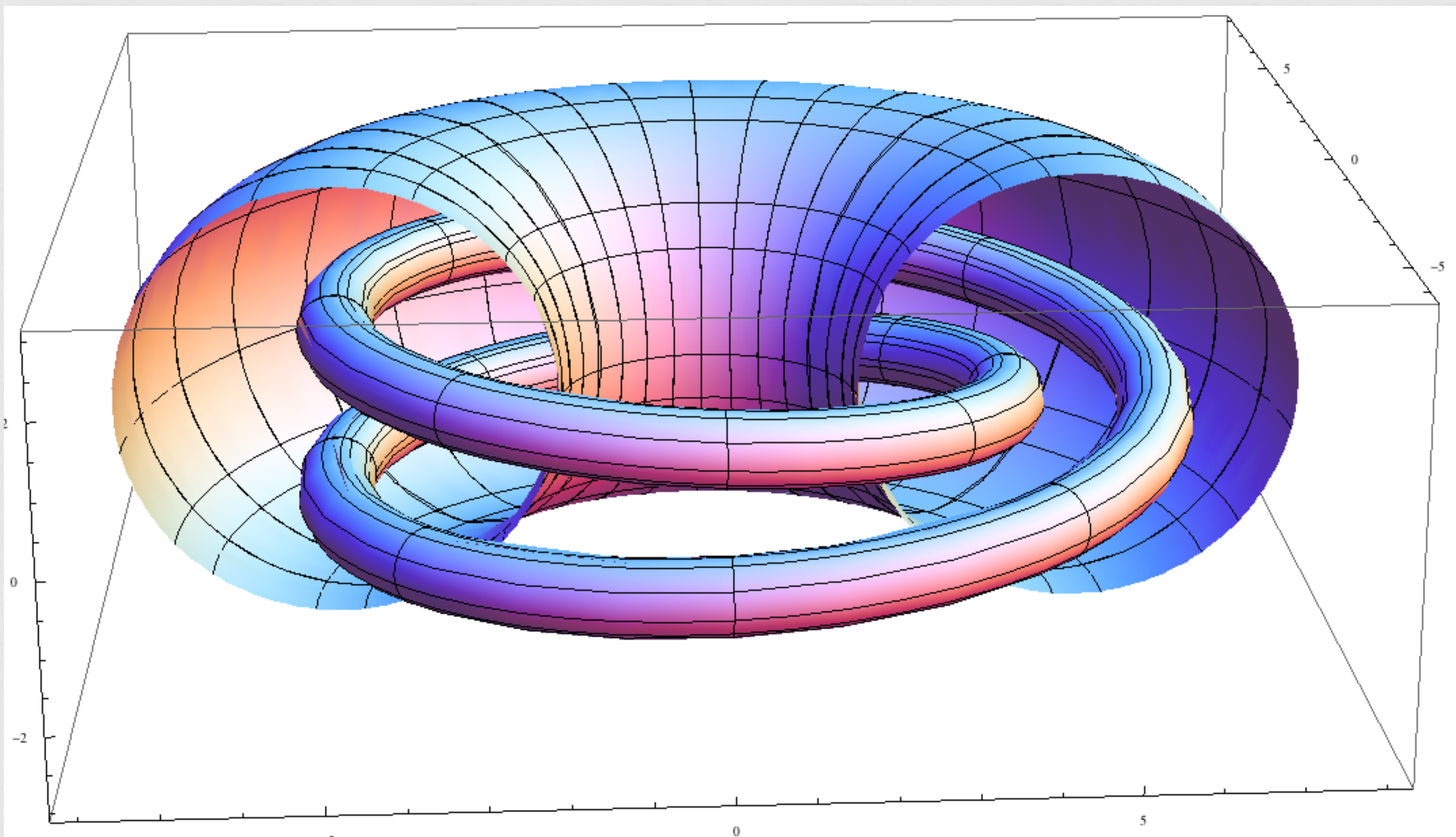
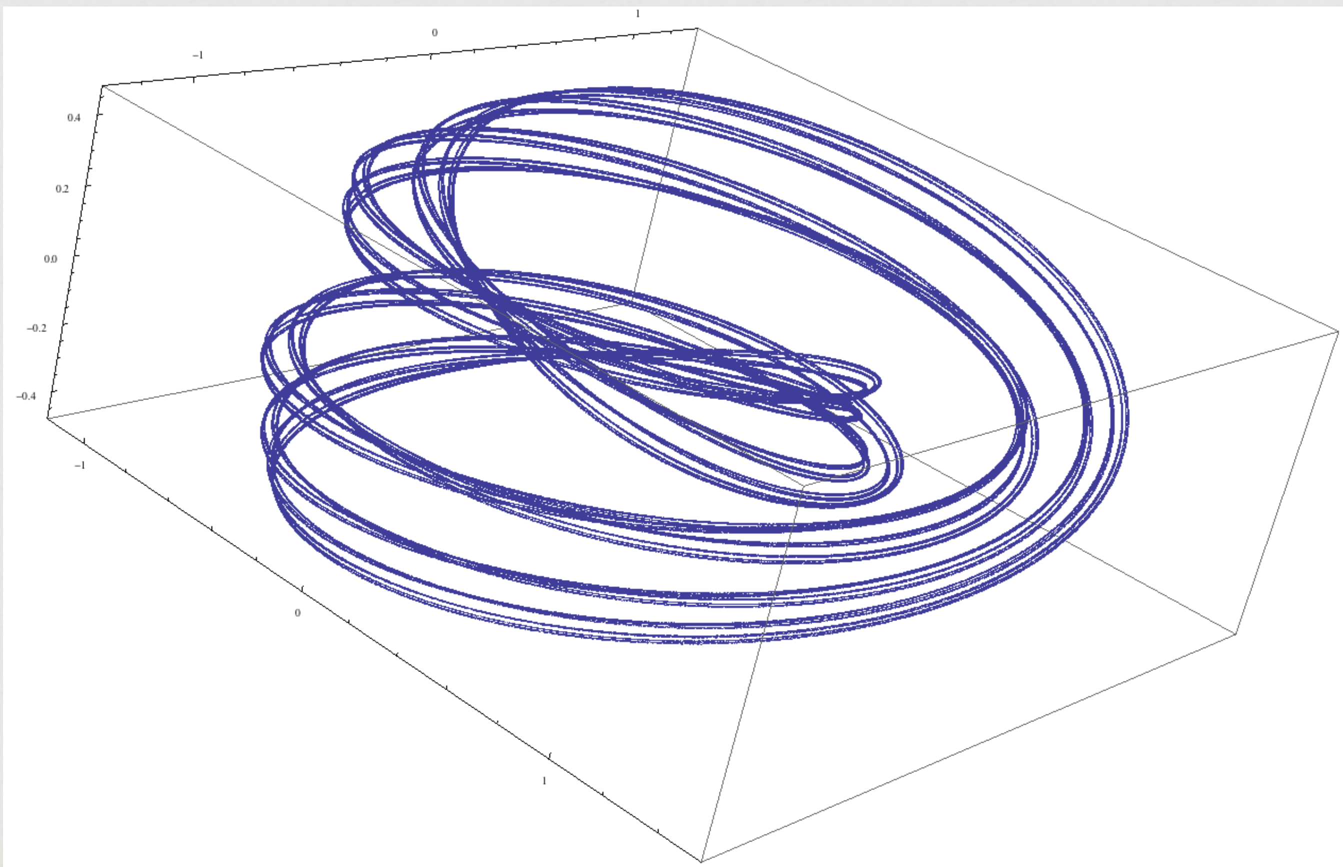


FIGURE: The dyadic solenoid.

Action by Solenoid



Action by Solenoid



Classic Results

- **Quotient space.** (C. T. Yang, 1960) If $\Delta_p \times M^n \to M^n$ is a free action, then the dimension of the quotient space is $n + 2$ or infinity.

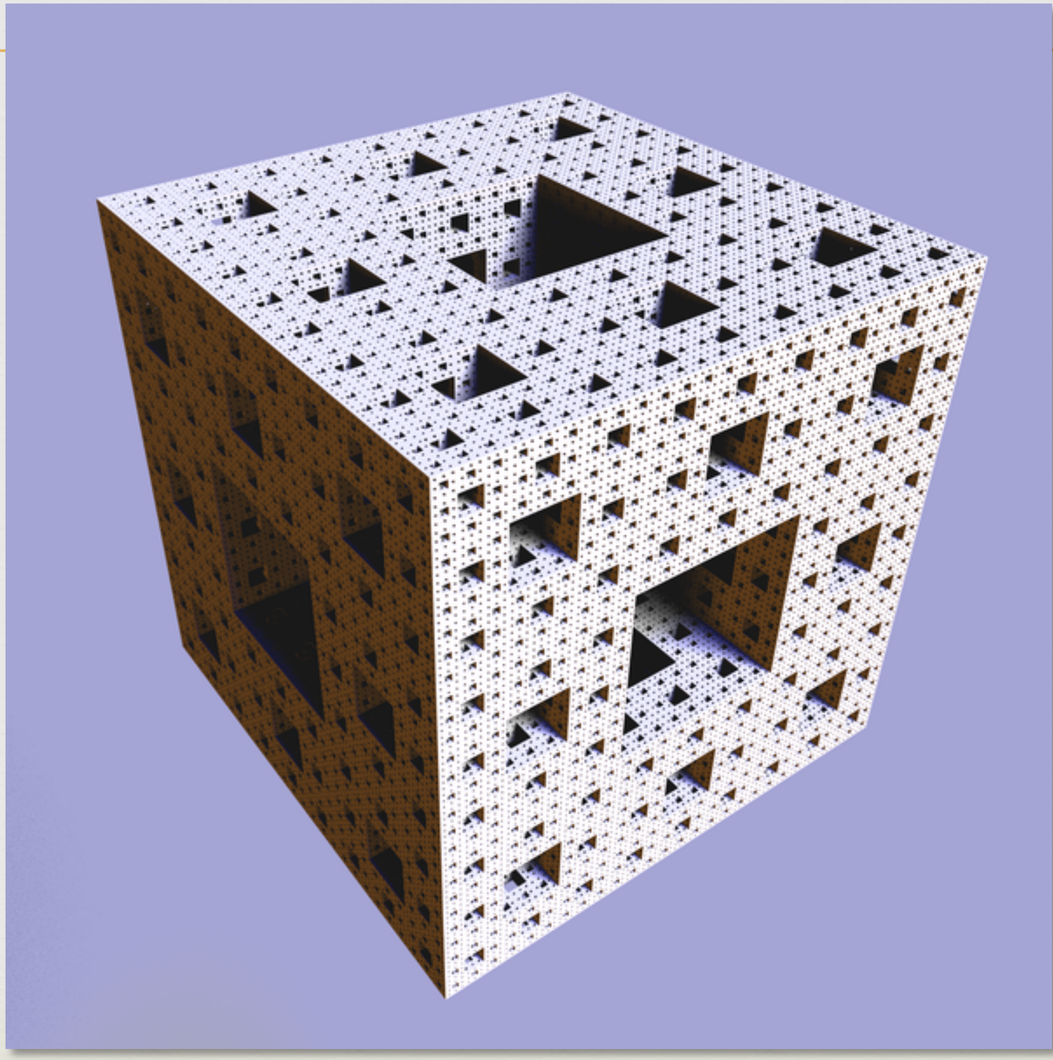
- **Examples.** (D. Wilson, 1970's) For every $n \geq 3$ and every $m \geq n$ there is an open mapping f from I^n onto I^m whose point inverses are Cantor sets.

Classic Results

- **Menger Manifolds.** (A. Dranishnikov, 1989) For every n–dimensional Menger manifold μ^n and every p, there is a free action of Δ_p on μ^n.

- (J. Mayer and C. Stark, 1985) There are free actions of Δ_p on μ^n such that the dimension of the quotient is $n + 1$. There are free actions such that the dimension of the quotient is $n + 2$.

Menger Space



Classic Results

- **Cannot Be Smooth Homeomorphisms.** (Bochner and Montgomery, 1946) There cannot be a *p*–adic action on a manifold by smooth maps.

- **Cannot Be Lipschitz Homeomorphisms.** (Repovs and Shchepin, 1997)

Extending Actions to Compactifications

- When can a compact group action extend to a compactification?
- What kind of compact spaces can extend the action of the group.
- Characterize manifold compactifications and actions.

Compactifications

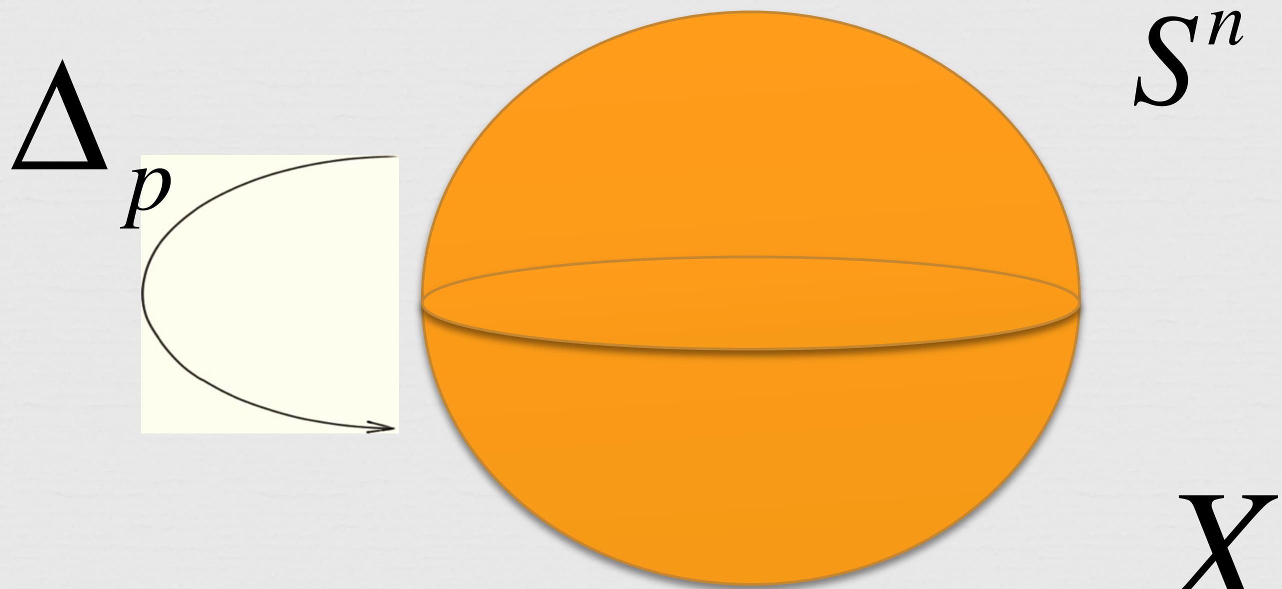
- **Rings of Continuous Functions.**

$$X \quad \subset \quad K \qquad \text{Compactification}$$

$$C^*(X) \leftarrow C(K)$$

- There is a one-to-one correspondence between the compactifications of X and the closed subrings of $C^*(X)$ generating the topology if X.

Compactification



Extending Group Actions

- Suppose that you have a metric compactification K of X and a compact group G acting on X. Can we produce a compactification K' of X above K so that the group action extends continuously to K'?
- Construction:

$$\begin{array}{ccc} X & \subset & K \\ \updownarrow & & \uparrow \\ X & \subset & K' \end{array} \qquad \begin{array}{ccc} G \times X & \to & X \\ \cap & & \cap \\ G \times K' & \to & K' \end{array}$$

Extending Group Actions

$$C(K) = F_0 \subset C^*(X)$$

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset \overline{\bigcup_{i=0}^{\infty} F_i}$$

Example

- There is a separable metric space X and a topological group G that acts continuously on X and a metric compactification K of X such that each element of the group extends to K, but the group action on K is not continuous.

$$X = N \times Z_2 \subset (N \cup p_\infty) \times Z_2 = K$$

$$G = \bigoplus_{i=0}^{\infty} Z_2$$

$$\begin{array}{ccc} G \times X & \rightarrow & X \\ \downarrow & & \downarrow \\ G \times K & \rightarrow & K \end{array}$$

Example

- There is a separable metric space X and a compact group G that acts continuously on X and a metric compactification K of X such that the only compactification K' above K that will allow extension of the action of all of the elements of G on K is $K' = \beta X$.

$$X = N \times Z_2 \subset (N \cup p_\infty) \times Z_2$$

$$G = \prod_{i=0}^{\infty} Z_2$$

Example

Theorem

- **Theorem.** (Maissen) Suppose that G is a compact group acting on the separable metric space X. Suppose that K is a metric compactification of X such that each element of G extends to the compactification. Then the action on K by G is continuous.

Hilbert Space

- **Actions on Hilbert Space.** (Bessaga and Pelczynski, 1972) Let X be a complete separable metric space. Then the space of measureable functions from [0,1] to X is Hilbert space.

$$\mathfrak{M}([0,1], X) \approx \ell_2$$

- There is an action of Δ_p on ℓ_2

$$\mathfrak{M}([0,1], \Delta_p)$$

Hilbert Space

- Alternative actions:

$\mathfrak{M}([0,1],\Sigma_p)$ using the subgroup $\Delta_p \subset \Sigma_p$

$\mathfrak{M}([0,1],\mu^n)$ where μ^n is a Menger Manifold

Hilbert Space

What compactifications can extend the action?

$$\Delta_p \times I^\infty \to I^\infty$$

Free action except for one fixed point.

$$\prod_{i=1}^{\infty} \mu^{n_i}$$

Invariant Hilbert space

Irrationals

- **Actions on the Irrationals.** There is precisely one free action of the *p*–adic group on the irrational numbers.

$$\Delta_p \times Z^\infty \to \Delta_p \times Z^\infty \approx Z^\infty$$

Dense Invariant Subspaces

- Copy of this irrational action in *K*.

$$\begin{array}{ccc} \Delta_p \times K & \rightarrow & K \\ \cup & & \cup \\ \Delta_p \times Z^\infty & \rightarrow & Z^\infty \end{array}$$

Summary

- Hilbert's Fifth Problem has generated a rich fabric of results. The current version continues to do so.
- There are simple obstructions to extending compact group actions on a separable metric space to any metric compactification and a simple theorem when we can extend an action.
- There are many examples of interesting actions of Δ_p on compact metric spaces. All of these are extensions of actions of invariant non-compact subspaces.
- We have characterizations of manifolds and other spaces to help develop a theory.

Extending Actions

- Characterize compact manifolds by the ring of continuous functions. [This would likely use a classical characterization theorems by Frank Quinn.]
- What compact group actions on a separable metric space can extend to a metric compactification?
- For what compact groups G can one extend the action on $\mathfrak{M}([0,1],G)$ to a metric compactification?
- What compactifications extend the action of Δ_p on the irrationals.