Periods of periodic orbits for vertex maps on graphs

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Outline

Introduction

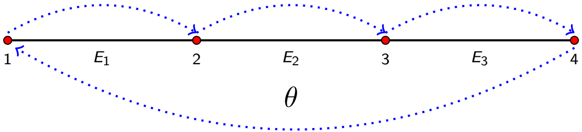
One of the basic starting points for one-dimension combinatorial dynamics is Sharkovsky's Theorem.

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a periodic point of least period v then f also has a periodic point of least period m for any $m \lhd v$, where

$$1 \lhd 2 \lhd 4 \lhd \ldots \ldots 28 \lhd 20 \lhd 12 \lhd \ldots 14 \lhd 10 \lhd 6 \ldots 7 \lhd 5 \lhd 3.$$

an example



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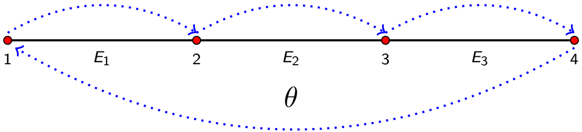

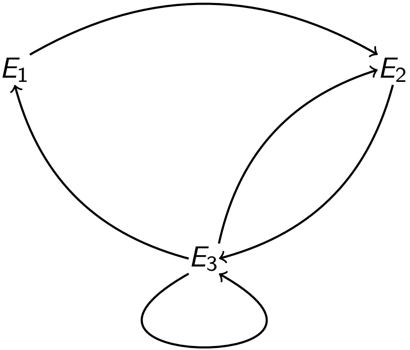

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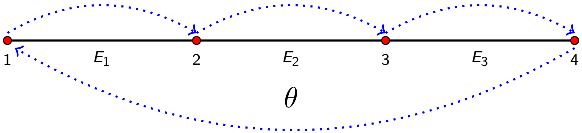

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

a basic result

Theorem

Let M be the Markov matrix associated to a directed graph that has vertices labeled $E_1, \ldots, E_n$, then the ijth entry of M^k gives the number of walks of length k from E_j to E_i.

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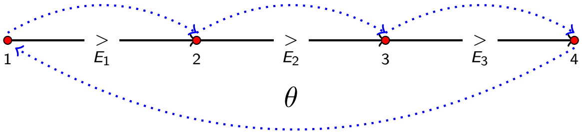
Let M be the Markov matrix associated to a directed graph that has vertices labeled $E_1, \ldots, E_n$, then the ijth entry of M^k gives the number of walks of length k from E_j to E_i.

Corollary

The trace of M^k gives the total number of closed walks of length k.

an example – with orientation



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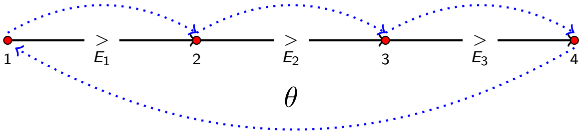

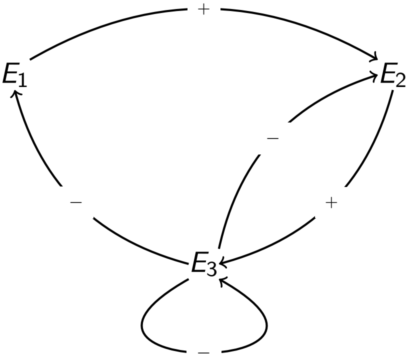

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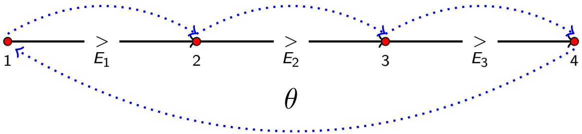
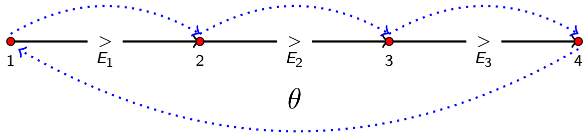


$$M_1(\theta) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

an example – with orientation



$$M_0(\theta) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, M_1(\theta) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

basic properties

Theorem

The ijth entry of $(M_1(\theta))^k$ gives the number of positively oriented walks of length k from E_j to E_i minus the number negatively oriented walks from E_j to E_i.

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The ijth entry of $(M_1(\theta))^k$ gives the number of positively oriented walks of length k from E_j to E_i minus the number negatively oriented walks from E_j to E_i.

Corollary

The trace of $(M_1(\theta))^k$ gives the number of positively oriented closed walks of length k minus the number of negatively oriented closed walks of length k.

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Theorem

1. $(M_0(\theta))^k = M_0(\theta^k)$
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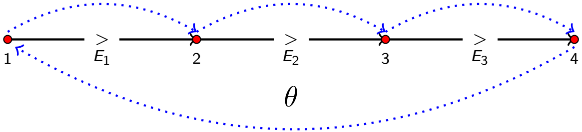

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basic properties

Theorem

$Trace\ (M_0(\theta)) - Trace\ (M_1(\theta)) = 1.$

first lemma

Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that f has a periodic point of period 17. Then f has a periodic point of period 2^k for any non-negative integer k.

first lemma

Proof.

Since 17 is not a divisor of 2^k we know that θ^{2^k} does not fix any of the integers in $\{1, 2, \ldots, 17\}$. So Trace $(M_0(\theta^{2^k})) = 0$.

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□

second lemma

Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that f has a periodic point of period 17. Then f has a periodic point of period m for any non-negative integer $m > 17$.

second lemma

Proof.

Trace $(M_1(\theta)) = -1$. So there vertex E_j in the Markov graph with a closed walk of length one with negative orientation.

second lemma

Proof.

Trace $(M_1(\theta)) = -1$. So there vertex E_j in the Markov graph with a closed walk of length one with negative orientation. $M_1(\theta)^{17}$ is the identity matrix. So there is a closed walk from E_j to itself with length 17 and with positive orientation.

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Trace $(M_1(\theta)) = -1$. So there vertex E_j in the Markov graph with a closed walk of length one with negative orientation. $M_1(\theta)^{17}$ is the identity matrix. So there is a closed walk from E_j to itself with length 17 and with positive orientation. The closed walk of length 17 is not a repetition of the walk of length 1.

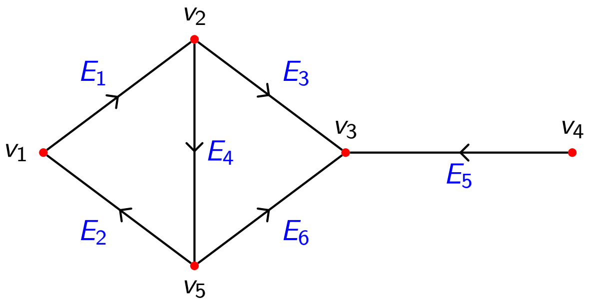
second lemma

Proof.

Trace $(M_1(\theta)) = -1$. So there vertex E_j in the Markov graph with a closed walk of length one with negative orientation. $M_1(\theta)^{17}$ is the identity matrix. So there is a closed walk from E_j to itself with length 17 and with positive orientation. The closed walk of length 17 is not a repetition of the walk of length 1. We can construct a non-repetitive closed walk of length m by going once around the walk of length 17 and then $m - 17$ times around the walk of length 1.

□

vertex maps on graphs



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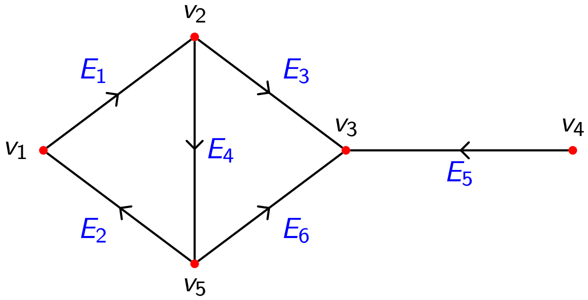

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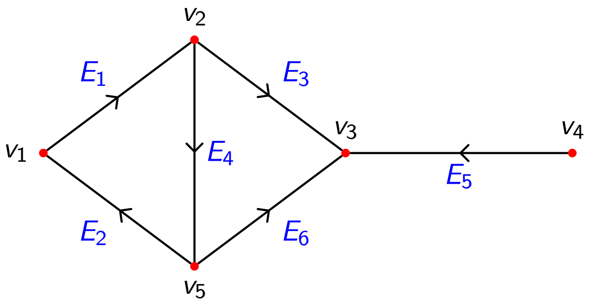

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3. $Trace\ (M_0(\theta)) - Trace\ (M_1(\theta)) = L_f$

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3. $Trace\ (M_0(\theta)) - Trace\ (M_1(\theta)) = L_f$

Corollary

If the underlying map is homotopic to the identity, then $Trace\ (M_0(\theta)) - Trace\ (M_1(\theta)) = v - e$

first lemma – redux

Lemma

Let G be a graph and f a vertex map from G to itself that is homotopic to the identity. Suppose that the vertices form one periodic orbit. Suppose f flips an edge. If v is not a divisor of 2^k, then f has a periodic point with period 2^k.

first lemma – redux

Proof.

Since f flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation.

first lemma – redux

Proof.

Since f flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $Trace(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation.

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Proof.

Since f flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $Trace(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation. By going around each of these loops in the Markov graph twice we can see that there must be at least $e - v + 2$ loops of length 2 that have positive orientation.

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Since $Trace(M_1(f)^2) = e - v$, there must be at least one loop of length 2 with negative orientation. Since it has negative orientation, it cannot be the repetition of a shorter loop.

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Proof.

Since f flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $Trace(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation. By going around each of these loops in the Markov graph twice we can see that there must be at least $e - v + 2$ loops of length 2 that have positive orientation.
Since $Trace(M_1(f)^2) = e - v$, there must be at least one loop of length 2 with negative orientation. Since it has negative orientation, it cannot be the repetition of a shorter loop. So the Markov graph of f has a non-repetitive loop of length 2 with negative orientation.

first lemma – redux

Proof.

Since f flips an edge, there must be at least one loop in the Markov graph that has length 1 and has negative orientation. Since $Trace(M_1(f)) = e - v$, there must be at least $e - v + 1$ loops in Markov graph of length 1 that have positive orientation. By going around each of these loops in the Markov graph twice we can see that there must be at least $e - v + 2$ loops of length 2 that have positive orientation.

Since $Trace(M_1(f)^2) = e - v$, there must be at least one loop of length 2 with negative orientation. Since it has negative orientation, it cannot be the repetition of a shorter loop. So the Markov graph of f has a non-repetitive loop of length 2 with negative orientation.

etc – use induction ☐

second lemma – redux

Lemma

Let G be a graph and f a map from G to itself that is homotopic to the identity. Suppose that the vertices form one periodic orbit. Suppose f flips an edge. If $v = 2^p q$, where $q > 1$ is odd and $p \geq 0$, then f has a periodic point with period $2^p r$ for any $r \geq q$.

second lemma – redux

Lemma

Let G be a graph and f a map from G to itself that is homotopic to the identity. Suppose that the vertices form one periodic orbit. Suppose f flips an edge. If $v = 2^p q$, where $q > 1$ is odd and $p \geq 0$, then f has a periodic point with period $2^p r$ for any $r \geq q$.

Proof.

Similar trace argument. □

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The Sharkovsky ordering can be defined as follows:
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   1. $2^l \triangleleft v$, for all positive integers l.
   2. $2^k r \triangleleft v$, where $r \geq s$.
   3. $2^l r \triangleleft v$, where $l > k$ and such that $2^l r < v$.

final remarks

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