Constructing a new quasiregular map in dimension 3.

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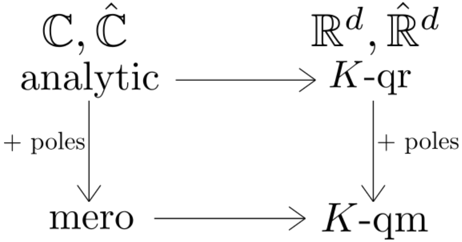
Qr and qm mappings

Informally, a continuous map $f : \mathbb{R}^d \to \mathbb{R}^d$, with $d \geq 2$, is quasiregular (qr) if it maps small spheres to small ellipsoids of bounded eccentricity.

For $K \geq 1$, f is K-qr if the amount of local stretching is uniformly bounded by K.



Informally, $g : \mathbb{R}^d \to \hat{\mathbb{R}}^d$ is quasimeromorphic (qm) if it is qr away from poles.

Some properties of qr maps

- non-constant qr maps are open, discrete, sense-preserving and differentiable a.e.
- When $d = 2$, analytic functions = 1-qr and mero functions = 1-qm.
- injective qr = quasiconformal (qc).
- Compositions: $qr \circ qr = qr$, $qm \circ qr = qm$, and $M\ddot{o}bius \circ qr = qm$.
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- Cannot represent qr or qm maps as power series.
- Some topological problems in dimension 3 and higher.
- For f, g qr maps, $f + g$ is *not* qr if they are 'too similar'....

but $f + g$ *is* qr on a domain D if f 'dominates' g on D.

Motivation for new example

- Not many examples of trans qr and trans qm maps exist.
- Many trans entire maps on $\mathbb{C}$ with a value taken finitely often, such as ze^z...
- Currently no explicit examples of a trans qr map on $\mathbb{R}^d, d \geq 3$ with a value taken finitely often!

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Theorem (W.)

There exists a trans qr map $f : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $f(x) = 0$ if and only if $x = 0$.

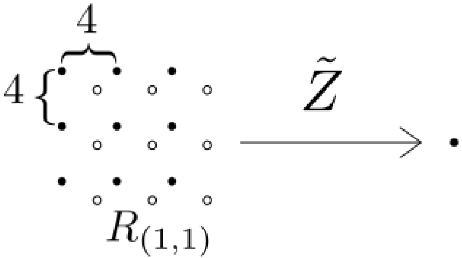
Note: No known examples of a trans qm map in dimension 3 with $\mathcal{O}^-(\infty)$ finite either (easiest mero example on $\mathbb{C}$ is e^z/z).
If $M : \hat{\mathbb{R}}^3 \to: \hat{\mathbb{R}}^3$ is a sense-preserving Möbius map s.t. $M(0) = \infty$ and $M(\infty) = 0$, then $M \circ f$ will be qm trans with $\mathcal{O}^-(\infty)$ finite.

Qr maps $\tilde{Z}$ and g

Zorich-type maps form the higher dimensional analogues of e^z. We will consider a particular version $\tilde{Z} : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$.
Denote the point reached by rotating x by π about the line $(1, 1, x_3)$ by $R_{(1,1)}(x)$.

Properties of $\tilde{Z} : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$

(i) $\tilde{Z}$ is 4-periodic in x_1 and x_2 directions,
(ii) $\tilde{Z}(x) = \tilde{Z}(R_{(1,1)}(x))$ for all $x \in \mathbb{R}^3$.



Qr maps $\tilde{Z}$ and g

Nicks and Sixsmith constructed a qr trans map $g : \mathbb{R}^3 \to \mathbb{R}^3$ whilst studying periodic domains of qr maps.

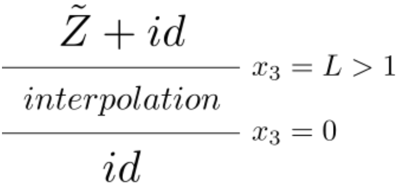
Properties of $g : \mathbb{R}^3 \to \mathbb{R}^3$

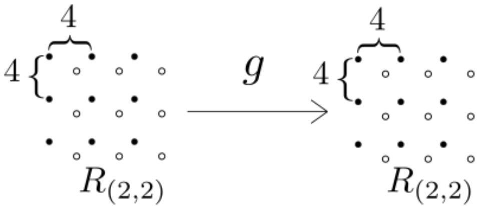
(i) $g(x) = x$ on $\{x_3 \leq 0\}$,

(ii) there is some constant $L > 1$ s.t. $g(x) = \tilde{Z}(x) + x$ on $\{x_3 \geq L\}$,

(iii) $g(x + (4n, 4m, 0)) = g(x) + (4n, 4m, 0)$ for all $n, m \in \mathbb{Z}$,

(iv) $g(R_{(2,2)}(x)) = R_{(2,2)}(g(x))$ for all $x \in \mathbb{R}^3$.





Construction

Observation:

$$ze^z = e^{e^{\log z}+\log z} = [\exp \circ (\exp + id) \circ \exp^{-1}](z).$$

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Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the translation $T(x) = x - (1, 1, 0)$, and let $\tilde{V}^{-1} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3$ be an inverse branch of $\tilde{Z}^{-1}$.

Define $f : \mathbb{R}^3 \to \mathbb{R}^3$ by setting $f(0) = 0$, and for $x \in \mathbb{R}^3 \setminus \{0\}$, set $f(x) = [\tilde{Z} \circ T \circ g \circ T^{-1} \circ \tilde{V}^{-1}](x)$.

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Claim

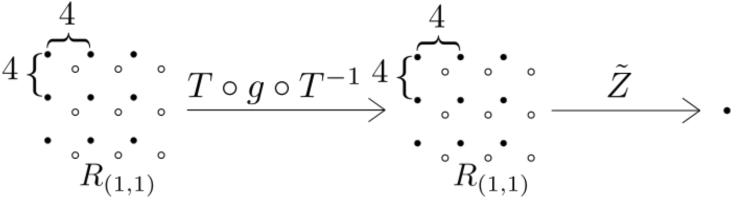
f is trans qr with $f(x) = 0$ if and only if $x = 0$.

Note: semi-conjugacy implies $f^n = \tilde{Z} \circ T \circ g^n \circ T^{-1} \circ \tilde{V}^{-1}$ for all $n \in \mathbb{N}$.

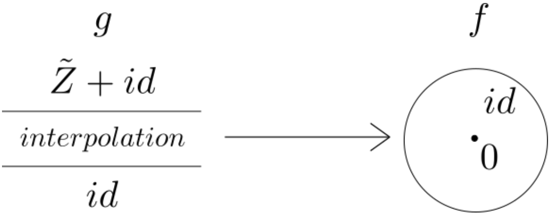
Sketch proof of claim

$f(x) = [\tilde{Z} \circ T \circ g \circ T^{-1} \circ \tilde{V}^{-1}](x)$, and $f(0) = 0$.

Well-defined:



$f(x) = 0$ if and only if $x = 0$:



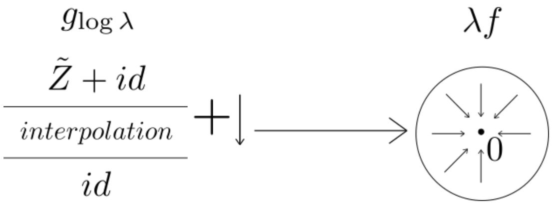
The family $\{\lambda f\}$ - dynamics for small $\lambda > 0$

We can modify g to create more qr examples.

For $\lambda > 0$, define $g_{\log\lambda}(x) = g(x) + (0, 0, \log\lambda)$. Then

$$\lambda f = \tilde{Z} \circ T \circ g_{\log\lambda} \circ T^{-1} \circ \tilde{V}^{-1}.$$

Note: If $\lambda > 0$ is small, then 0 becomes an attracting point for λf.



The family $\{\lambda f\}$ - dynamics for small $\lambda > 0$

Theorem (Nicks, Sixsmith, '18)

For $\lambda > 0$ sufficiently small, $QF(g_{\log \lambda})$ is a single connected domain containing $\{x_3 < 0\}$. Further, for every $x \in QF(g_{\log \lambda})$ there is some $k \in \mathbb{N}$ such that $g^k_{\log \lambda}(x) \in \{x_3 < 0\}$, and all points in $\{x_3 < 0\}$ iterate to infinity.

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By using the semi-conjugacy of f, we get the following dynamics for λf when λ is sufficiently small.

Theorem (W.)

Let $\lambda > 0$ be sufficiently small. Then

(i) $(Z \circ T)(J(g_{\log\lambda})) = J(\lambda f)$,

(ii) $(Z \circ T)(QF(g_{\log\lambda})) = QF(\lambda f) \setminus \{0\}$,

(iii) $QF(\lambda f) = \mathcal{A}_{\lambda f}(0)$ *is connected.*