Transcendental Hubbard Trees

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Post-critically finite polynomials

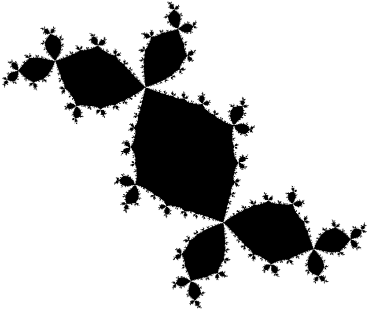


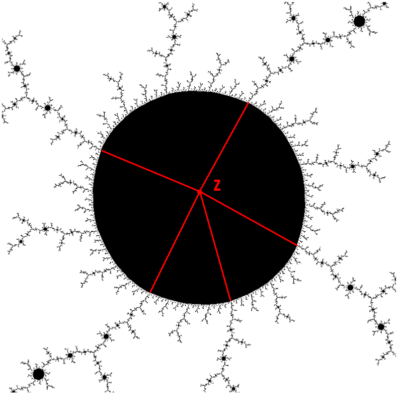
Figure: The Douady rabbit is the filled in Julia set of the polynomial $z \mapsto z^2 + c$, $c \approx -0.12 + 0.74i$.

Let $p\colon \mathbb{C} \to \mathbb{C}$ be post-critically finite.

- The Julia set $J(p)$ and the filled-in Julia set $K(p)$ are connected and locally connected.
- The filled-in Julia set $K(p)$ is full. Its complement $I(p) = \mathbb{C} \setminus K(p)$ is the escaping set.
- The filled-in Julia set is uniquely arcwise connected up to homotopy.

Let $U \subset K(p) \setminus J(p)$ be a bounded Fatou component of p.

- U is a Jordan domain
- The intersection $\Omega(C(p)) \cap U = \{z\}$ is a singleton. We call z the **center** of U.
- Let $\varphi \colon U \to \mathbb{D}$ be a Riemann map, $\varphi(z) = 0$. For $\theta \in \mathbb{R}/\mathbb{Z}$, the arc $\gamma := \varphi^{-1}([0, e^{2\pi i\theta}))$ is called an **internal ray** of U.
- Internal rays are dynamically invariant



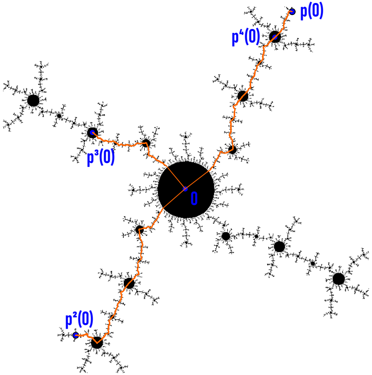


Figure: Filled-in Julia set of a degree 4 unicritical polynomial $z \mapsto z^4 + c$ in black and its Hubbard Tree in orange.

The **Hubbard Tree** of a post-critically finite polynomial $p\colon \mathbb{C} \to \mathbb{C}$ is the unique smallest embedded tree $H \subset \mathbb{C}$ satisfying:

- $C(p) \subset H$, i.e., H contains all critical points of p.
- $p(H) \subset H$.
- Let U be a bounded Fatou component. The intersection of H with U is either empty, a singleton, or it consists of internal rays of U.

Naive generalization

We want to extend the definition to the transcendental case.

Definition (Naive definition of Transcendental Hubbard Trees)

The Hubbard Tree of a post-singularly finite entire function $f : \mathbb{C} \to \mathbb{C}$ is the unique smallest embedded tree $H \subset \mathbb{C}$ satisfying:

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There exist transcendental entire functions without critical points, e.g., $C(\lambda \exp) = \emptyset$.

Singularities of the inverse function

For $f \in \mathcal{S}$, let V be a small disk around $a \in S(f)$. Let U be a connected component of $f^{-1}(V)$ such that $f|_U$ is *not injective*.

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Algebraic singularity

$$\begin{array}{ccc} U & \xrightarrow{\psi} & \mathbb{D} \\ \big\downarrow f & & \big\downarrow z \mapsto z^d \\ V & \xrightarrow{\varphi} & \mathbb{D} \end{array}$$

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We form an extension $\mathbb{C}_f \supset \mathbb{C}$ of the complex plane by adding all logarithmic singularities.

The definition of transcendental Hubbard Trees

Definition (Hubbard Trees for psf entire functions)

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Work in progress:

- If $AV(f) = \emptyset$, i.e., if $\mathbb{C}_f = \mathbb{C}$, then f has a Hubbard Tree.
- Even if $AV(f) \neq \emptyset$, the map f has a Hubbard Tree as long as post-singular points are not separated by logarithmic singularities.

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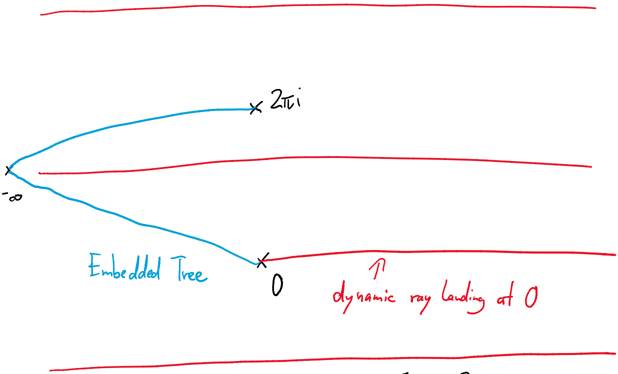
But: There are psf entire functions that do not have a Hubbard Tree in the above sense, e.g., exponential maps. See

Pfrang, David; Rothgang, Michael; Schleicher, Dierk. Homotopy Hubbard Trees for post-singularly finite exponential maps. arXiv:1812.11831 [math.DS]

No invariant tree



$E_{2\pi i}: z \mapsto 2\pi i e^z, \quad P(E_{2\pi i}) = \{0, 2\pi i\},$

$C(\hat{E}_{2\pi i}) = \{-\infty\}$

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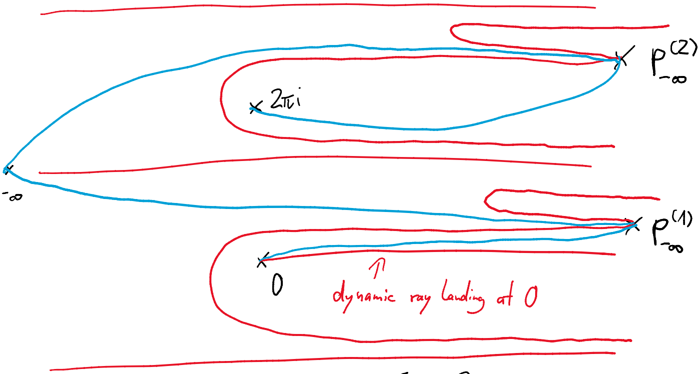

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Definition (Homotopy Hubbard Trees)

Let f be a post-singularly finite entire function. A *(reduced) Homotopy Hubbard Tree* for f is a *finite* embedded tree $H \subset \mathbb{C}$ such that

- All endpoints of H are post-singular points.
- H is forward invariant up to homotopy rel $P(f)$.
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Why is this concept useful?

Theorem (P., 2019)

Every post-singulary finite entire function has a Homotopy Hubbard Tree and this tree is unique up to homotopy relative to the post-singular set.

- Homotopy Hubbard Trees are a tool to prove the existence of actual Hubbard Trees (in the cases where they exist).

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- H is a *finite* embedded tree. The full Hubbard Tree is, in general, infinite. The full tree can easily be recovered.
- Natural for *Thurston Theory*. The reduced tree gives rise to a *finite* combinatorial object that distinguishes functions with the same "geometry".

Classification via Hubbard Trees

Theorem (P., Rothgang, Schleicher - 2018)

Every post-singularly finite exponential map has a Homotopy Hubbard Tree. This tree is unique up to homotopy relative to the post-singular set.
For every abstract exponential Hubbard Tree, there is a unique post-singularly finite exponential map realizing it.

The classification cycle:

$$\begin{array}{ccc} \text{Hubbard tree} & \xleftarrow[\text{Forgetful}]{} & \text{Holomorphic post-singularly finite map} \\ \Big\downarrow{\scriptstyle \text{Forgetful}} & & {\scriptstyle \text{Thurston rigidity}}\Big\uparrow \\ \text{Abstract Hubbard tree} & \longrightarrow & \text{Topological post-singularly finite map} \end{array}$$

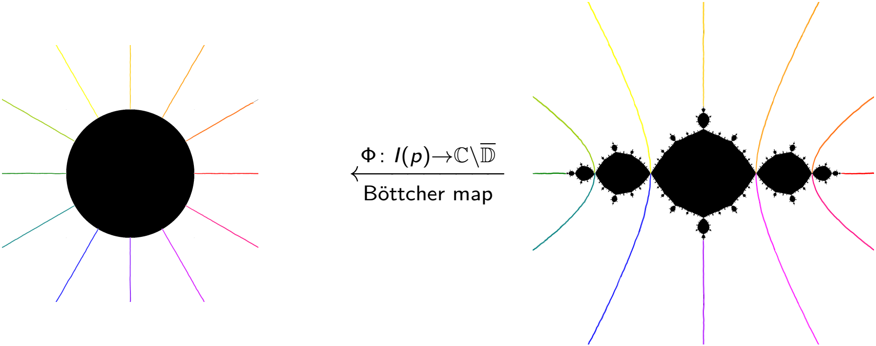
Construction of Homotopy Hubbard Trees

> How can we prove the existence of (Homotopy) Hubbard Trees for transcendental maps?

In the polynomial case, the topology of the Julia set was used to prove existence and uniqueness of Hubbard Trees.

For a post-singularly finite transcendental entire function f, the structure of $J(f)$ is, in general, not useful. In many cases, we have $J(f) = \mathbb{C}$.

Escaping sets and dynamic rays of polynomials



The **Böttcher map** Φ is the unique conformal isomorphism from $I(p)$ onto $\mathbb{C} \setminus \overline{\mathbb{D}}$ satisfying $\lim_{z \to \infty} \Phi(z)/z = 1$.

The **dynamic ray** g_θ **of angle** $\theta \in \mathbb{R}/\mathbb{Z}$ is the preimage $g_\theta = \Phi^{-1}((e^{2\pi i \theta}, \infty))$ of the straight radial line of angle θ under the Böttcher map.

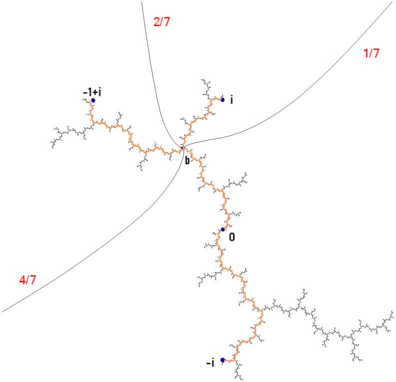
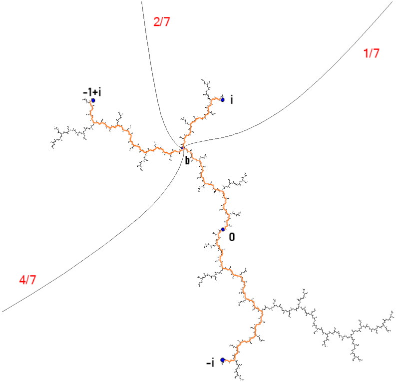


Figure: Julia set and Hubbard Tree of the polnomial $z \mapsto z^2 + i$. The rays of angle $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{4}{7}$ land together.

For a post-critically finite polynomial, every dynamic ray lands at a point in $J(f)$ and every $z \in J(f)$ is the landing point of a dynamic ray.

Landing of dynamic rays



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We call a point $b \in J(f)$ a *branch point* if $J(f) \setminus \{b\}$ has at least three connected components.

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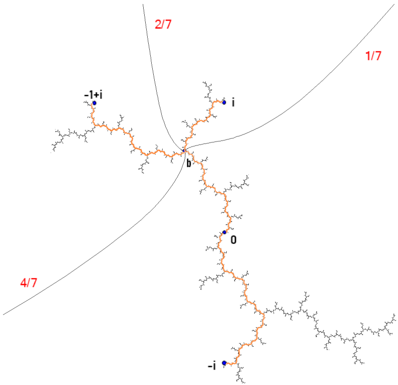

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Every branch point is eventually periodic. All *periodic* branch points of f are contained in its Hubbard Tree.

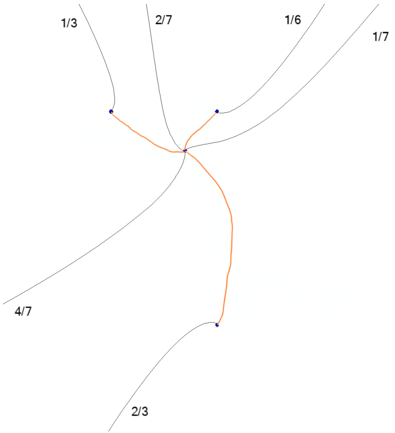
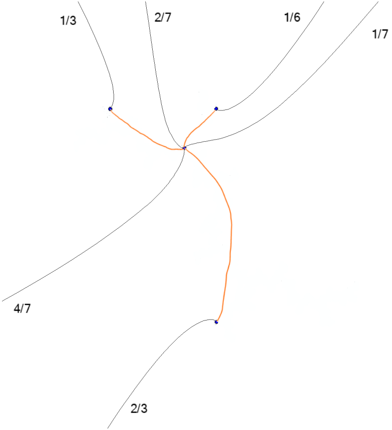


Figure: A Homotopy Hubbard Tree for the polynomial $z \mapsto z^2 + i$

The set of dynamic rays landing at post-singular points and branch points is forward invariant.



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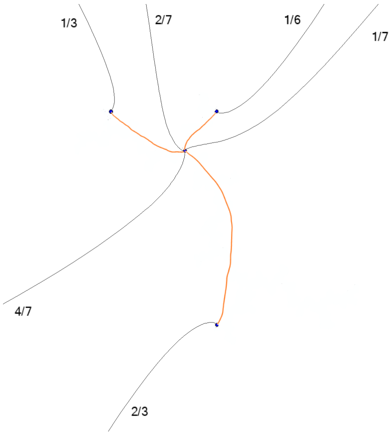

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The preimage tree also does not intersect them. Therefore, T is forward invariant up to homotopy relative to the post-singular set.

Dreadlocks

Theorem (Decomposition of the escaping set, Benini, A.; Rempe-Gillen, L. - 2017)

Let f be a post-singularly bounded. There is a natural decomposition $I(f) = \dot{\bigcup}_{\underline{s}\in\mathcal{S}} G_{\underline{s}}$ into dreadlocks $G_{\underline{s}}$ parametrized by external addresses. For every external address $\underline{s} \in \mathcal{S}$, the dreadlock $G_{\underline{s}}$ is either empty or unbounded and connected.

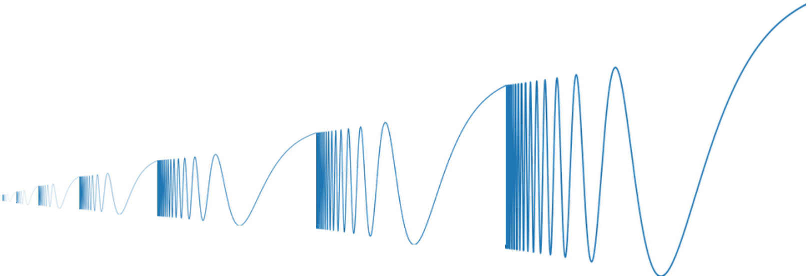


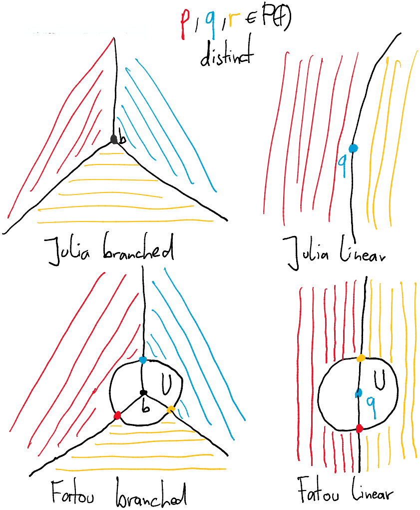
Figure: This continuum, like many others, arises as a periodic Julia continuum of a post-singularly finite entire function.

Periodic Dreadlocks

Theorem (Landing Theorem, Benini, A.; Rempe-Gillen, L. - 2017)

Let f be a post-singularly bounded entire function. Every periodic dreadlock of f lands at a repelling or parabolic periodic point. Conversely, every repelling and every parabolic periodic point of f is the landing point of at least one and at most finitely many dreadlocks all of which have the same period.

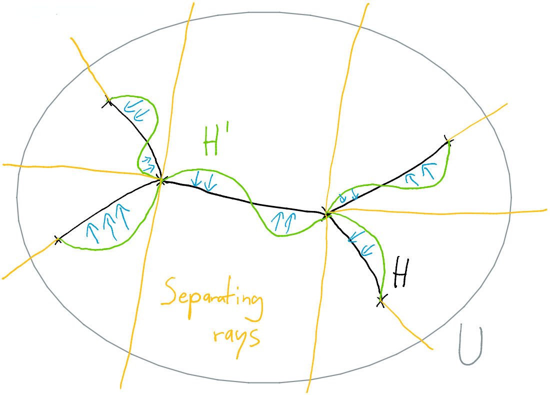
We use **symbolic dynamics** on the space of external addresses to construct (pre-)periodic dreadlocks that land together and separate post-singular points. Their landing points are the **branch points** of the Homotopy Hubbard Tree.



Theorem (Post-singular separation)

Let f be a post-singularly finite entire function, and let $p, q, r \in P(f)$ be distinct post-singular points. Then the three points p, q, and r are separated in one of the four ways drawn to the left.

Exactly invariant trees



Find a domain $U \supset H, H'$ and a conformal metric ρ (orbifold metric, modified hyperbolic metric) such that f is expanding on U w.r.t. ρ.

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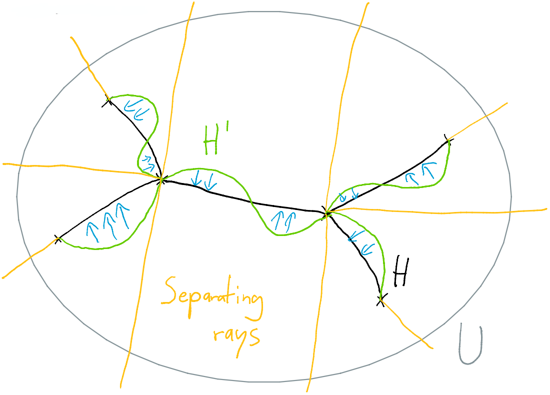
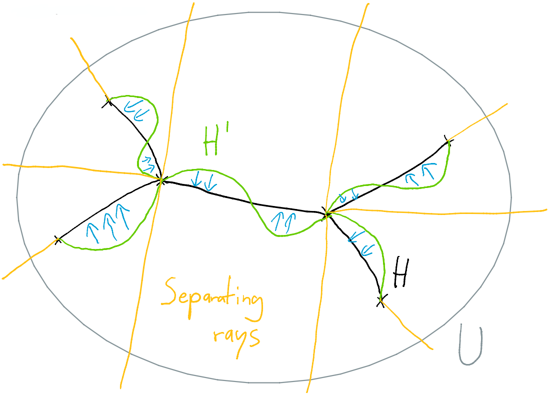


Choose a differentiable homotopy between H and the preimage H' in U.



Iteratively, lift the homotopy, to obtain a forward invariant compact subset as a limit. Separating dreadlocks ensure that the limiting object is a tree.

Thank you for your attention!

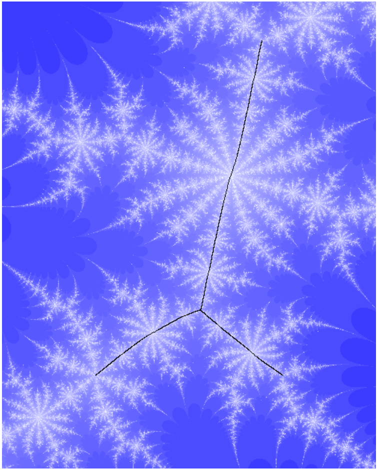


Figure: Hubbard Tree of $f(z) = \cos(c(z+1))$, $c \approx -0.68 + 1.00i$. Picture by Lasse Rempe-Gillen.