Non-autonomous exponential maps: Hausdorff dimension of hair

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A general non-autonomous system

Take a one-parameter analytic family $f_\lambda(z) = \lambda e^z$.

Denote by $\Omega = (\mathbb{C} \setminus \{0\})^{\mathbb{N}}$ with the shift action $\sigma\colon \Lambda \to \Lambda$.

For any sequence $\lambda = (\lambda_n)_{n=0}^{\infty} \in \Omega$ define f_λ^n as

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One could also think of this as a skew-product:

$$F(\lambda, z) = (\sigma\lambda, \lambda e^z)$$

Julia and Fatou sets

The Julia and Fatou sets are defined similarly to the typical situation (of equal λ_n). Namely,

Definition

The Fatou set $F(f_\lambda)$ consists of all $z \in \mathbb{C}$ such that for some neighbourhood U of z the sequence $\{f_\lambda^n|_U\}$ forms a normal family.

The Julia set $J(f_\lambda) = \mathbb{C} \setminus F(f_\lambda)$.

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- lack of periodic points
- lack of (pre-) periodic Fatou components
- limited use of critical (asymptotic) values
- Julia set may be empty

Stricter assumptions

In this talk we are going to assume that for all n we have $\lambda_n \in \mathbb{R}$ and $\lambda_n \in [a, b]$, for some $b < +\infty$ and $a > \frac{1}{e}$.

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Theorem (Urbański, Zdunik)

Under assumptions as above $J(f_\lambda) = \mathbb{C}$.

In fact they proved more...

Coding

For any integer c define a horizontal strip

$$P_c = \{z \in \mathbb{C} : (2c-1)\pi < \operatorname{Im}(z) \leq (2c+1)\pi\}.$$

Take any sequence $\bar{c} = (c_n)_{n=0}^{\infty} \in \mathbb{Z}^{\mathbb{N}}$.

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We say that a point $z \in \mathbb{C}$ has a code $\bar{c}$ if for all $n \in \mathbb{N}$

$$f_\lambda^n(z) \in P_{c_n}$$

Denote the set of all the points having a code $\bar{c}$ as $\Lambda_{\bar{c}}$.

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$$f_{\lambda_{n-1}} \circ f_{\lambda_{n-2}} \circ \cdots \circ f_{\lambda_0}(z) = f_{\lambda}^{n}(z) \in P_{c_n}$$

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For a moment assume that $\lambda_n = \mathrm{const} \in \mathbb{R}_+$.
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- If $\lambda > \frac{1}{e}$, then $\Lambda_{\bar{c}}^{\mathrm{bd}}$ is at most one point
- If $\Lambda_{\bar{c}}^{\mathrm{ubd}} \neq \emptyset$, then either $\bar{c}$ is unbounded or it contains infinitely many 0's.

The *most important* code

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Theorem (The strong version of Urbański, Zdunik)

If all $\lambda_n > 0$ and $\mathrm{int}(\Lambda) = \emptyset$, then $J(f_\lambda) = \mathbb{C}$.

Properties of Λ

In the following assume that all $\lambda_n \in [a, b]$.

Theorem (P)

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The set Λ has the Hausdorff dimension equal to 1.
(but the Hausdorff measure is not σ-finite).

Hopes and expectations

The previous results should (?) also hold if we assume:

$$\text{for all } n \in \mathbb{N} \text{ we have } \lambda_n \in [\varepsilon, M]$$
$$\text{and for all } k \in \mathbb{N} \text{ we have } \lim_{n \to +\infty} f^n_{\sigma^k \lambda}(0) = \infty$$