Bounded Hyperbolic Components of Bicritical Rational Maps

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Definitions

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- Each component of the set of hyperbolic maps is called a *hyperbolic component*.
- *type D* hyperbolic component: each map has maximal number of *disjoint* attracting cycles.
  strict type D hyperbolic component: type D + each attracting cycle has period at least 2.

Bounded hyperbolic components

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Theorem (Epstein, '00)

Let $\mathcal{H}$ be a strict type D hyperbolic component in rat_2. Then $\mathcal{H}$ is bounded.

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It seems not easy to reproduce this argument for rational maps of higher degree.

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- Choose suitable coordinates so that $\mathcal{F} = \mathbb{C}^2 - \{2 \text{ lines}\} \subset \mathbb{C}^2 \subset \mathbb{P}^2$.
- Let $\mathcal{M}_d$ be the moduli space of bicritical rational maps of degree d. Then a hyperbolic component $\mathcal{H} \subset \mathcal{M}_d$ lifts to a hyperbolic component $\widetilde{\mathcal{H}} \subset \mathcal{F}$.

Main Result

Theorem (N.-Pilgrim)

Let $\mathcal{H} \subset \mathcal{M}_d$ be a strict type D hyperbolic component. Then $\mathcal{H}$ is bounded in $\mathcal{M}_d$.

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In summary, if $\mathcal{H}$ is unbounded, we can find a holomorphic family $\{f_t\}_{t\in\mathbb{D}^*} \subset \mathcal{F}$ such that for some $t_k \to 0$, $f_{t_k} \in \widetilde{\mathcal{H}}$ and $[f_{t_k}] \to \infty$ in rat_d.

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From now on, we assume $\mathcal{H}$ is unbound and consider the family $\{f_t\}$.

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- The holomorphic family $\{f_t\}$ induces a rational map

$$\mathbf{f}(z) \in \mathbb{C}((t))(z) \subset \mathbb{C}\{\{t\}\}(z) \subset \mathbb{L}(z),$$

where $\mathbb{C}((t))$ is the field of Laurent series, $\mathbb{C}\{\{t\}\}$ is the field of Puiseux series, and $\mathbb{L}$ is the completion of $\mathbb{C}\{\{t\}\}$ w.r.t the natural non-Archimedean absolute value.

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- The map $\mathbf{f}$ extends to an endomorphism on Berkovich space $\mathbf{P}^1$ over $\mathbb{L}$.
  (The Berkovich space $\mathbf{P}^1$ is a compact, Hausdorff, uniquely path-connected topological space with tree structure.)

Sketch of proof (cont.)

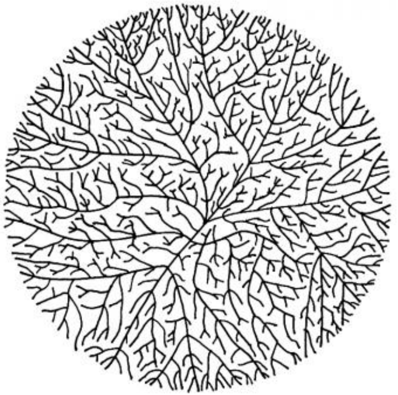

Figure 1: The Berkovich space $\mathbf{P}^1$ (see book "Berkovich Spaces and Applications")

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- It follows that $\mathbf{f}$ has a repelling q-cycle for some $q \geq 2$ where the reduction G of $\mathbf{f}^q$ is a degree d bicritical rational map with a multiple fixed point $\hat{z}$.
- The limit of the cycle $\langle z_t \rangle$ (resp. of $\langle w_t \rangle$) is either $\{\hat{z}\}$, contains a cycle disjoint from $\hat{z}$, or contains a preperiodic critical point that iterates under G to $\hat{z}$.

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- an arithmetic result of Rivera-Letelier: number of fixed points in a Berkovich Fatou component.
- Epstein's refined version of the Fatou-Shishikura Inequality: relations on the numbers of critical points and non-repelling cycles.

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We derive an over-determined set of constraints on the critical dynamics of G.

Thank you.