Wandering domains for entire functions of finite order in the class $\mathcal{B}$

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– joint work with Mitsuhiro Shishikura –

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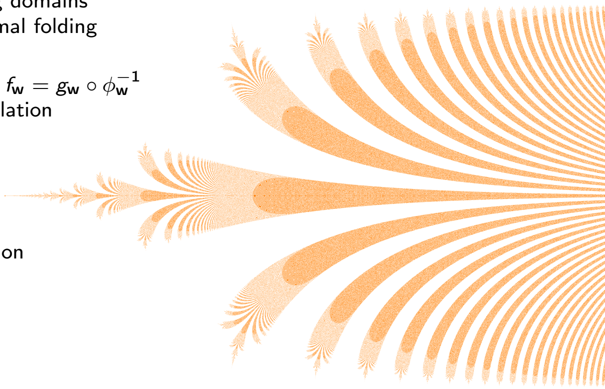
Sketch of the talk

1. Introduction to wandering domains and Bishop's quasiconformal folding
2. Definition of the function $f_{\mathbf{w}} = g_{\mathbf{w}} \circ \phi_{\mathbf{w}}^{-1}$ using quasiregular interpolation
3. Estimates for the quasiconformal map $\phi_{\mathbf{w}}$
4. Diagram of the construction and the domains $\{U_n\}_n$
5. Shrink and shoot



Introduction

Let f be a transcendental entire function. We consider the sets:

- the **Fatou set** of f:

$$F(f) := \{z \in \mathbb{C} \ : \ \{f^n\}_n \text{ is a normal family in an open set } U \ni z\}$$

- the **Julia set** of f:

$$J(f) := \mathbb{C} \setminus F(f)$$

- the **escaping set** of f:

$$I(f) := \{z \in \mathbb{C} \ : \ f^n(z) \to \infty, \text{ as } n \to \infty\}$$

- the **set of bounded orbits** of f:

$$K(f) := \{z \in \mathbb{C} \ : \ \exists R = R(z) > 0, \ |f^n(z)| < R \text{ for all } n \in \mathbb{N}\}$$

- the **set of unbounded non-escaping orbits** of f (a.k.a. the **bungee set** of f):

$$BU(f) := \mathbb{C} \setminus (I(f) \cup K(f)).$$

Thus, we have two partitions

$$\mathbb{C} = F(f) \cup J(f) = I(f) \cup BU(f) \cup K(f).$$

SO18 D. J. Sixsmith and J. W. Osborne, *On the set where the iterates of an entire function are neither escaping nor bounded*, Ann. Acad. Sci. Fenn. Ser. A I Math. **41** (2016), 561–578.

Singular values

Given a transcendental entire function f, we define the **singular set** of f by

$$S(f) := \overline{\operatorname{sing}(f^{-1})}$$

where $\operatorname{sing}(f^{-1})$ consists of the critical values and the asymptotic values of f. We will also consider the **postsingular set** of f

$$P(f) := \overline{\bigcup_{n \geqslant 0} f^n(S(f))}.$$

EL92 A. E. Eremenko and M. Yu. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020.

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Among all transcendental entire functions, functions in the following two classes exhibit nicer properties:

$$\mathcal{B} := \{f \text{ transcendental entire function } : S(f) \subseteq \mathbb{D}(0, R) \text{ for some } R > 0\},$$

$$\mathcal{S} := \{f \text{ transcendental entire function } : \#S(f) < \infty\} \subseteq \mathcal{B}.$$

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Theorem (Eremenko and Lyubich 1992)

If $f \in \mathcal{B}$, then $I(f) \subseteq J(f)$.

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Wandering domains

Suppose that U is a component of $F(f)$ and let U_n be the Fatou component that contains $f^n(U)$ for $n \in \mathbb{N}$. We say that U is a **wandering domain** if

$$U_m \cap U_n \neq \emptyset \quad \Rightarrow \quad m = n.$$

If U is a wandering domain, let $L(U) \subseteq \widehat{\mathbb{C}}$ be the set of all **limit functions** of f^n on U.

BHKMT93 W. Bergweiler, M. Haruta, H. Kriete, H.-G. Meier and N. Terglane, *On the limit functions of iterates in wandering domains*, Ann. Acad. Sci. Fenn. Ser. A I Math., **18** (1993), 369–375.

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Let U be a wandering domain. Then, $L(U) \subseteq (J(f) \cap P(f)') \cup \{\infty\}$.

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Wandering domains can be classified into the following 3 types:

- U is an **escaping wandering domain** if $L(U) = \{\infty\}$, that is, $U \subseteq I(f)$;
- U is a **bounded orbit wandering domain** if $L(U) \subseteq \mathbb{C}$, that is, $U \subseteq K(f)$;
- U is an **oscillating wandering domain** if $L(U) \supseteq \{\infty, a\}$ for some $a \in \mathbb{C}$, that is, $U \subseteq BU(f)$.

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Theorem (Eremenko and Lyubich 1992, Goldberg and Keen 1986)

If $f \in \mathcal{S}$, then f has no wandering domains.

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History of wandering domains

- Baker (1963, 1976): first example of a function with a wandering domain, which was an infinite product and the wandering domains were multiply connected;
- Herman (1984): obtained a simply connected wandering domain by modifying a 2π-periodic function with infinitely many basins of attraction;
- Eremenko and Lyubich (1987): constructed an oscillating wandering domain using approximation theory (not known if in $\mathcal{B}$);
- Kisaka and Shishikura (2008): constructed multiply connected wandering domains using quasiconformal surgery;
- Bishop (2015): first example of a function in class $\mathcal{B}$ with a wandering domain, which is oscillating.

Bishop's quasiconformal folding

We say that a planar tree T has **bounded geometry** if

- the edges of T are $\mathcal{C}^2$ with uniform bounds;
- the angles between adjacent edges are bounded uniformly away from zero;
- adjacent edges have uniformly comparable lengths;
- for non-adjacent edges e and f, $\mathrm{diam}(e)/\mathrm{dist}(e, f)$ is uniformly bounded;
- the union of edges that meet at a vertex for a uniformly bi-Lipschitz star.

Bis15 C. Bishop, *Constructing entire functions by quasiconformal folding*, Acta Math. **214** (2015), no. 1, 1–60.

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Assume for every component Ω_j of $\Omega = \mathbb{C} \setminus T$, there is a conformal map $\tau_j : \Omega_j \to \mathbb{H}_r$. Then, we define the τ-**size** of an edge $e \in T$ as the minimum length of the two images of e by τ.

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Assume for every component Ω_j of $\Omega = \mathbb{C} \setminus T$, there is a conformal map $\tau_j : \Omega_j \to \mathbb{H}_r$. Then, we define the τ-**size** of an edge $e \in T$ as the minimum length of the two images of e by τ.

Theorem (Bishop 2015)

Suppose that T has bounded geometry and every edge has τ-size $\geqslant \pi$. Then there is an entire function f and a K-quasiconformal map ϕ so that

$$f \circ \phi = \cosh \circ \tau, \qquad \text{outside a nbhd } T(r_0) \text{ of } T.$$

K only depends on the bounded-geometry constants of T. The only critical values of f are ± 1 and f has no asymptotic values.

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Bishop's quasiconformal folding

There is a more general version of the construction that involves 3 types of components:

- R-components: $\tau : \Omega \to \mathbb{H}_r$ and $\sigma = \cosh$, as before;
- L-components: $\tau : \Omega \to \mathbb{H}_l$ and $\sigma = \rho_w \exp z)$;
- D-components: $\tau : \Omega \to$ and $\sigma = \rho_w(z^d)$.

Theorem (Bishop 2015)

Let T be a bounded-geometry graph and suppose τ is a conformal map from each complementary component to its standard version. Assume that D-components and L-components only share edges with R-components. Assume that on a D-componen with n edgest, τ maps the vertices to the nth roots of unity and on L components τ maps the edges to intervales of length 2π on $\partial\mathbb{H}_l$ with endpoints in $2\pi i\mathbb{Z}$. On R-components assume that the τ-sizes fo all edges are $\geqslant 2\pi$. Then, there is an entire function f and a K-quasiconformal map ϕ so that

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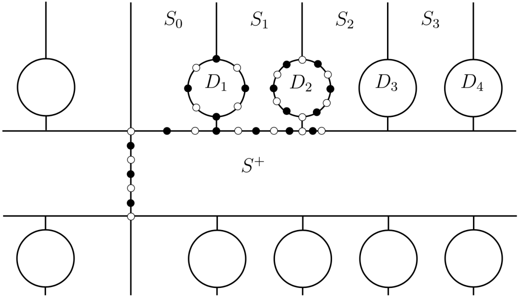
Bishop's construction of a function in the class B with a wandering domain

Theorem (Bishop 2015, see also Fagella, Godillon and Jarque 2015)

There exists a function in the class $\mathcal{B}$ with a wandering domain.



(picture borrowed from [FGJ15])

This function equals $f(z) = \cosh(\lambda \sinh(\phi(z)))$ for $z \in \mathbb{R}_+$.

Bis15 C. Bishop, *Constructing entire functions by quasiconformal folding*, Acta Math. **214** (2015), no. 1, 1–60.

FGJ15 N. Fagella, S. Godillon and X. Jarque, *Wandering domains for composition of entire functions*, J. Math. Anal. Appl. **429** (2015), no. 1, 478–496.

Functions of finite order

Let f be a transcendental entire function. We define the **order** and **lower order** of f as

$$\rho(f) := \limsup_{r\to+\infty} \frac{\log\log M(r)}{\log r} \quad \text{and} \quad \lambda(f) := \liminf_{r\to+\infty} \frac{\log\log M(r)}{\log r}$$

respectively, where $M(r) := \max_{|z|=r} |f(z)|$.

For example, $\rho(e^{z^k}) = k$ for $k \in \mathbb{N}$, and $\rho(e^{e^z}) = +\infty$.

Hei48 M. Heins, *Entire functions with bounded minimum modulus; subharmonic function analogues*, Ann. of Math. (2) **49** (1948), 200–213.
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Theorem (Heins 1948)

If $f \in \mathcal{B}$, then $\lambda(f) \geqslant 1/2$.

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Theorem (Rottenfusser, Rückert, Rempe and Schleicher 2011)

Let $f \in \mathcal{B}$ be a function of finite order or, more generally, a finite composition of such functions. Then, every point of $I(f)$ can be joined to ∞ by a curve in which points escape uniformly.

Hei48 M. Heins, *Entire functions with bounded minimum modulus; subharmonic function analogues*, Ann. of Math. (2) **49** (1948), 200–213.
RRRS11 G. Rottenfusser, J. Rückert, L. Rempe and D. Schleicher, *Dynamic rays of bounded-type entire functions*, Ann. of Math. (2) **173** (2011), 200–213.

Main theorem

The function $f \in \mathcal{B}$ from Bishop's construction has infinite order as

$$f(x) = \cosh(\lambda \sinh(\phi(x))) \geqslant \cosh(\lambda \sinh(10x/\lambda)), \quad \text{for } x \in \mathbb{R}_+,$$

where $\lambda \in \pi\mathbb{N}^*$.

Theorem (Martí-Pete and Shishikura 2018)

For every $p \in \mathbb{N}$, there exists a transcendental entire function $f_p \in \mathcal{B}$ of order $p/2$ with an oscillating wandering domain.

Fagella, Godillon and Jarque proved that the function from Bishop's example has exactly two grand orbits of wandering domains. We can also modify our construction to obtain the following result.

Theorem (Martí-Pete and Shishikura 2018)

There exists a function $f \in \mathcal{B}$ of finite order with infinitely many grand orbits of wandering domains.

FGJ15 N. Fagella, S. Godillon and X. Jarque, *Wandering domains for composition of entire functions*, J. Math. Anal. Appl. **429** (2015), no. 1, 478–496.
MS18 D. Martí-Pete and M. Shishikura, *Oscillating wandering domains for functions in the Eremenko-Lyubich class*, in preparation.

The base map $g(z) = 2\cosh z$

The function $g(z) := 2\cosh z = e^z + e^{-z}$ has critical points at $i\pi\mathbb{Z}$, critical values ± 2 and no finite asymptotic value.

Define the **reference orbit**

$$x_0 := \tfrac{1}{2}, \quad \text{and} \quad x_n := g^n(x_0), \quad \text{for } n \in \mathbb{N},$$

which escapes to ∞ exponentially fast. Then, for $n \in \mathbb{N}$, define the quantities

$$d_n := \left\lfloor \frac{x_{n+1}}{x_n} \right\rfloor, \quad R_n := \left(d_n - \tfrac{1}{3}\right)\pi, \quad h_n := 2\pi \left\lfloor \frac{x_{n+1} + \pi}{2\pi} \right\rfloor.$$

Consider the sets

$$S_+ := \{z \in \mathbb{C} \ : \ \operatorname{Re} z > 0, \ |\operatorname{Im} z| < \pi\}.$$

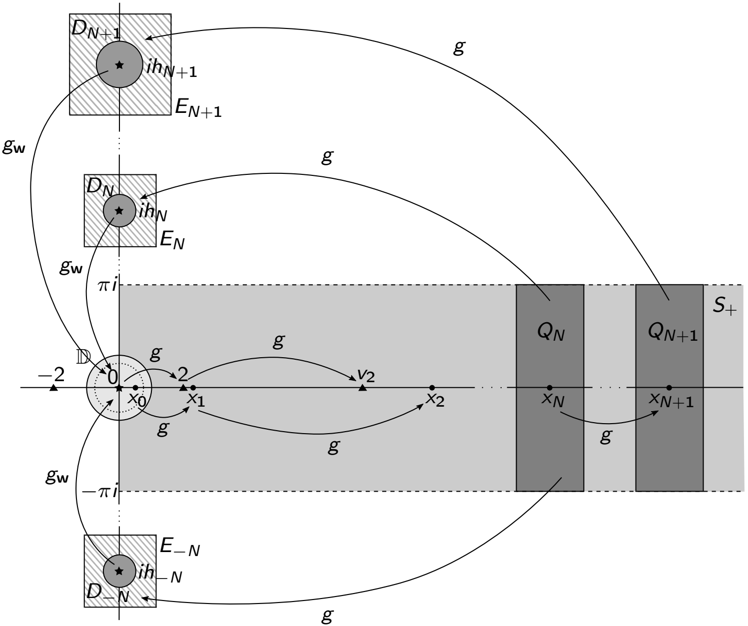
and, for $n \geqslant 3$,

$$Q_n := Q(x_n) = \{z \in \mathbb{C} \ : \ |\operatorname{Re} z - x_n| < 1, \ |\operatorname{Im} z| < \pi\} \subseteq S_+,$$

$$E_{\pm n} := \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| < 2d_n\pi, \quad |\operatorname{Im} z \mp h_n| < 2d_n\pi\} \subseteq \mathbb{C} \setminus S_+,$$

$$D_{\pm n} := (\pm i h_n, R_n) \subseteq E_{\pm n}.$$

Sketch of the function $g_{\mathbf{w}}$



Cosh-power interpolation lemma

Lemma

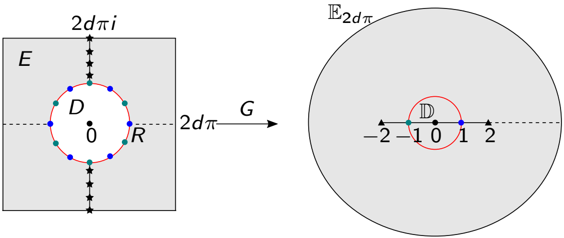
Let $d \in \mathbb{N}$ and define $R := (d - \frac{1}{3})\pi$. Consider the sets

$$E := \{z \in \mathbb{C} \;:\; |\mathrm{Re}\, z| \leqslant 2d\pi,\; |\mathrm{Im}\, z| \leqslant 2d\pi\} \quad \text{and} \quad D := D(0, R).$$

There exists $K_1 \geqslant 1$ independent of d and a K_1-quasiregular map $G : E \to \overline{\mathbb{E}_{2d\pi}}$ with $\operatorname{supp} \mu_G \subseteq E \setminus D$ satisfying that $G(-z) = G(z)$, $G(\overline{z}) = \overline{G(z)}$ and

$$G(z) = \begin{cases} 2\cosh z, & \text{if } z \in \partial E \cup ((E \cap i\mathbb{R}) \setminus D), \\ \left(\dfrac{z}{R}\right)^{2d}, & \text{if } z \in D, \end{cases}$$

where $\overline{\mathbb{E}_{2d\pi}} = 2\cosh(E)$.



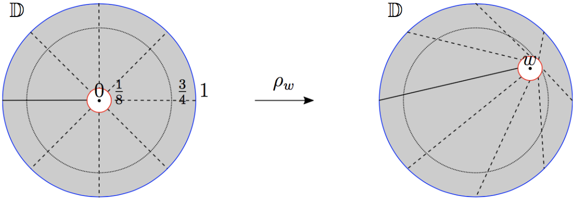
The map ρ_w

Lemma

There exists $K_2 > 1$ such that for all $w \in \overline{\mathbb{D}_{3/4}}$, there exists a K_2-quasiconformal mapping $\rho_w : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that

$$\rho_w(z) = \begin{cases} z, & \text{if } z \in \partial\mathbb{D}, \\ z + w, & \text{if } z \in \overline{\mathbb{D}_{1/8}}. \end{cases}$$

and $\operatorname{supp} \rho_w \subseteq \overline{\mathbb{D}} \setminus \mathbb{D}_{1/8}$. Moreover the Beltrami coefficient μ_{ρ_w} depends holomorphically on $w \in \mathbb{D}_{3/4}$.



Definition of $f_{\mathbf{w}}$

Let $G_n : E \to \overline{\mathbb{E}_{2d\pi}}$ be the quasiregular mapping G as before with $d = d_n$ and $R = R_n$, so that $E = E_n - ih_n$ and $D = D_n - ih_n$. Define $K := \max\{K_1, K_2\}$.

For every sequence $\mathbf{w} = (w_N, w_{N+1}, w_{N+2}, \dots) \in \mathbb{D}_{3/4}^{\mathbb{N}_N}$, define the function $g_{\mathbf{w}} : \mathbb{C} \to \mathbb{C}$ as follows:

$$g_{\mathbf{w}}(z) := \begin{cases} G_n(z \mp ih_n), & \text{if } z \in E_{\pm n} \setminus D_{\pm n} \text{ with } n \geqslant N, \\ \rho_{w_n} \circ G_n(z - ih_n), & \text{if } z \in D_{\pm n} \text{ with } n \geqslant N, \\ 2\cosh z, & \text{otherwise.} \end{cases}$$

Then $g_{\mathbf{w}}$ is a K-quasiregular map such that

$$\operatorname{supp} \mu_{g_{\mathbf{w}}} \subseteq \bigcup_{n \in \mathbb{Z}_N} E_n \setminus \left(ih_n, \left(1 - \left(\tfrac{1}{8}\right)^{1/(2d_n)}\right) R_n \right),$$

and $g_{\mathbf{w}}(z) = g(z) = 2\cosh z$ for all $z \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}_N} E_n$.

Apply the Measurable Riemann Mapping Theorem to obtain an entire function $f_{\mathbf{w}} \in \mathcal{B}$ and a K-quasiconformal map $\phi_{\mathbf{w}}$ such that

$$f_{\mathbf{w}} = g_{\mathbf{w}} \circ \phi_{\mathbf{w}}^{-1}.$$

Key Inequality

Theorem (Shishikura 2018)

Given $K > 1$*, there exist* $0 < \delta_1 < 1$ *and* $C > 0$ *such that if* $\phi : \mathbb{C} \to \mathbb{C}$ *is a* K*-quasiconformal map with* $\phi(0) = 0$ *and* $0 < |z_2| \leq \delta_1 |z_1|$*, then*

$$\left|\log \frac{\phi(z_1)}{z_1} - \log \frac{\phi(z_2)}{z_2}\right| \leqslant 2C\left(\left|\iint_{\mathbb{C}} \frac{\mu_\phi(z)\varphi_{z_1,z_2}(z)}{1-|\mu_\phi(z)|^2}\,dxdy\right| + \iint_{\mathbb{C}} \frac{|\mu_\phi(z)|^2|\varphi_{z_1,z_2}(z)|}{1-|\mu_\phi(z)|^2}\,dxdy\right)$$

where $\varphi_{z_1,z_2}(z) := \frac{z_1}{z(z-z_1)(z-z_2)}$.

Shi18 M. Shishikura, *Conformality of quasiconformal mappings at a point, revisited*, Ann. Acad. Sci. Fenn. **43** (2018), 981–990.

Key Inequality

Theorem (Shishikura 2018)

Given $K > 1$, there exist $0 < \delta_1 < 1$ and $C > 0$ such that if $\phi : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal map with $\phi(0) = 0$ and $0 < |z_2| \leq \delta_1 |z_1|$, then

$$\left|\log \frac{\phi(z_1)}{z_1} - \log \frac{\phi(z_2)}{z_2}\right| \leqslant 2C\left(\left|\iint_{\mathbb{C}} \frac{\mu_\phi(z)\varphi_{z_1,z_2}(z)}{1-|\mu_\phi(z)|^2}\, dxdy\right| + \iint_{\mathbb{C}} \frac{|\mu_\phi(z)|^2 |\varphi_{z_1,z_2}(z)|}{1-|\mu_\phi(z)|^2}\, dxdy\right)$$

where $\varphi_{z_1,z_2}(z) := \frac{z_1}{z(z-z_1)(z-z_2)}$.

Corollary

Let the constants $K>1$, $0<\delta_1<1$ and $C>0$ be as in the previous theorem. If $\phi : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal map and $\alpha, \beta, \gamma \in \mathbb{C}$ are distinct points with

$$0 < |\gamma - \alpha| \leq \delta_1 |\beta - \alpha|,$$

then

$$\left|\log \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} - \log \frac{\phi(\gamma)-\phi(\alpha)}{\gamma-\alpha}\right| \leq C(K-1) \iint_{\operatorname{supp} \mu_\phi} \frac{|\beta-\alpha|\, dxdy}{|(z-\alpha)(z-\beta)(z-\gamma)|}$$

where $\operatorname{supp} \mu_\phi = \{z \in \mathbb{C} : \mu_\phi(z) \neq 0\}$.

Shi18 M. Shishikura, *Conformality of quasiconformal mappings at a point, revisited*, Ann. Acad. Sci. Fenn. **43** (2018), 981–990.

Standing assumption for the QC estimates

Assumption

Suppose that $K > 1$ is a fixed constant and that there exists a sequence of discs

$$B_m := \mathbb{D}(\zeta_m, r_m), \quad \text{for } m \in \mathbb{N},$$

satisfying that

(i) $|\zeta_m| \geqslant 4$ *and* $r_m/|\zeta_m| \leqslant \min\{\frac{1}{4}, \delta_1\}$ *for* $m \in \mathbb{N}$*, where* $0 < \delta_1 < 1$ *is the constant from the Key Inequality*

(ii) $\displaystyle\sum_{m=1}^{\infty} \frac{r_m}{|\zeta_m|} < +\infty$

(iii) $\phi : \mathbb{C} \to \mathbb{C}$ *is a K-quasiconformal map normalised so that* $\phi(0) = 0$, $\phi(1) = 1$ *and*

$$\operatorname{supp} \mu_\phi \subseteq \bigcup_{m=1}^{\infty} B_m$$

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Later on we will apply them with

$$\zeta_{2k} = -\zeta_{2k+1} = ih_{L+k} \quad \text{and} \quad r_{2k} = r_{2k+1} = 3R_{L+k}, \quad \text{for } k \in \mathbb{N},$$

with $L \geqslant N$ sufficiently large, so

$$B_m \supseteq E_{L+m}, \qquad \text{for } m \in \mathbb{N},$$

and $\phi = \phi_{\mathbf{w}}$ the K-quasiconformal map in the definition of $f_{\mathbf{w}}$ with $N \geqslant L$.

QC estimate 1

Lemma

Suppose that Assumption holds. For every $\varepsilon > 0$, there exists $M_1 = M_1() \in \mathbb{N}$ such that if $\operatorname{supp} \mu_\phi \subseteq \bigcup_{m=M_1}^{\infty} B_m$, then

$$\left| \log \frac{\phi(\zeta)}{\zeta} \right|_{\mathcal{C}} < \varepsilon \quad \text{for } \zeta \in \mathbb{C} \setminus \{0\},$$

and, in particular,

$$e^{-\varepsilon}|\zeta| < |\phi(\zeta)| < e^{\varepsilon}|\zeta| \quad \text{and} \quad |\arg \phi(\zeta) - \arg \zeta \pmod{2\pi}| < \varepsilon$$

for all $\zeta \in \mathbb{C} \setminus \{0\}$.

Here, $\mathcal{C} := \mathbb{C}/2\pi i\mathbb{Z}$ and, for $w \in \mathcal{C}$,

$$|w|_{\mathcal{C}} := \inf_{n \in \mathbb{Z}} |w + 2\pi n i|$$

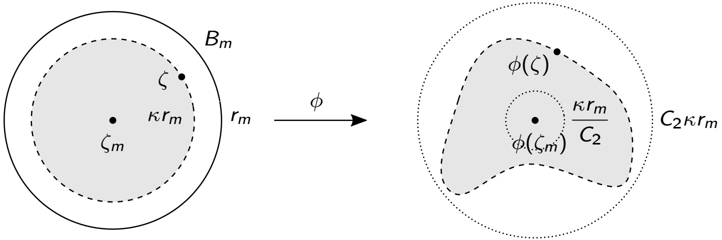
defines a distance on the cylinder $\mathcal{C}$.

QC estimate 2

Lemma

Suppose that Assumption holds and suppose also that there exists $C_1 > 0$ such that if $z \in B_m$ and $z' \in B_{m'}$ with $m \neq m'$, then $|z - z'| \geq C_1 \sqrt{|zz'|}$.
For any $0 < \kappa < 1$, there exists $C_2 > 1$ such that for any $m \in \mathbb{N}$, if $|\zeta - \zeta_m| = \kappa r_m$, then

$$\frac{1}{C_2} \kappa r_m \leq |\phi(\zeta) - \phi(\zeta_m)| \leq C_2 \kappa r_m.$$



QC estimate 3

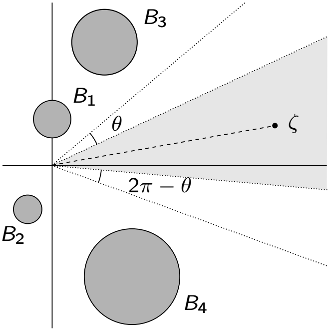
Lemma

Suppose that Assumption holds. For every $0 < \theta < 2\pi$, there exists $C_3 > 1$ such that if $\zeta \in \mathbb{C}$ satisfies that

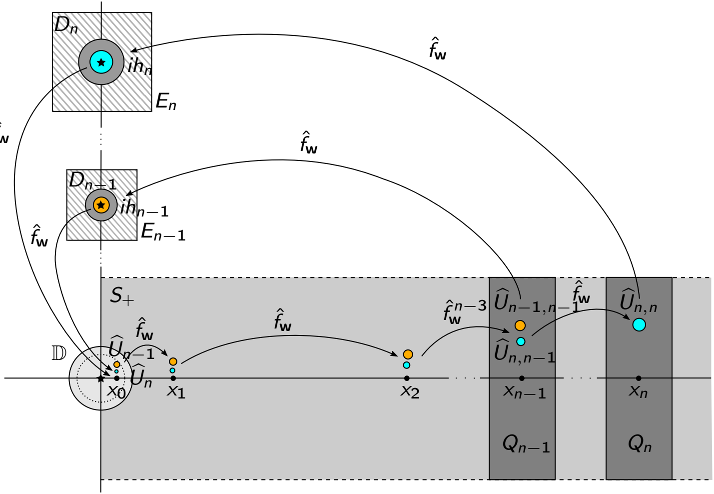
$$B_m \subseteq \{z \in \mathbb{C} \ : \ \arg \zeta + \theta < \arg z < \arg \zeta + 2\pi - \theta\} \quad \text{for all } m \in \mathbb{N},$$

then

$$\frac{1}{C_3} \leq |\phi'(\zeta)| \leq C_3.$$



Sketch of the function $g_{\mathbf{w}}$



The iterates of the domains $\widehat{U}_n$ by the function $\hat{f}_{\mathbf{w}} = \phi_{\mathbf{w}}^{-1} \circ g_{\mathbf{w}}$ which is conjugated to $f_{\mathbf{w}}$ by the map $\phi_{\mathbf{w}}$.

Domains $\{U_n\}_n$ and centers $\{c_n\}_n$

For $n \geqslant 3$ define $\widehat{U}_{n,n} := g^{-1}(\mathbb{D}(\phi_{\mathbf{w}}(ih_n), CR_n)) \subseteq Q_n$, and for $M < j \leqslant n$, define

$$U_{n,j} := \phi(\widehat{U}_{n,j}), \quad \widehat{U}_{n,j-1} := g^{-1}(U_{n,j}) \subseteq Q_{j-1},$$

and finally $U_n := (\phi \circ g^{-1})^M \circ \phi(\widehat{U}_{n,M})$ so that we have the diagram:

$$\begin{array}{ccccccccccccccc}
\mathbb{D}(\frac{1}{2},\frac{1}{8}) & \xleftarrow{f^{-M}\circ\phi} & Q_M & \xrightarrow{g} & g(Q_M) & \xleftarrow{\phi} & \cdots & \xrightarrow{g} & g(Q_{j-1}) & \xleftarrow{\phi} & Q_j & \xrightarrow{g} & g(Q_j) & \xleftarrow{\phi} & \cdots \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \\
U_n & \xrightarrow{\phi^{-1}\circ f^M} & \widehat{U}_{n,M} & \xrightarrow{g} & U_{n,M+1} & \xleftarrow{\phi} & \cdots & \xrightarrow{g} & U_{n,j} & \xleftarrow{\phi} & \widehat{U}_{n,j} & \xrightarrow{g} & U_{n,j+1} & \xleftarrow{\phi} & \cdots \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\
c_n & \longmapsto & \hat{c}_{n,M} & \longmapsto & c_{n,M+1} & \longmapsto & \cdots & \longmapsto & c_{n,j} & \longmapsto & \hat{c}_{n,j} & \longmapsto & c_{n,j+1} & \longmapsto & \cdots
\end{array}$$

$$\begin{array}{ccccccccccccc}
\cdots & \xleftarrow{\phi} & Q_{n-1} & \xrightarrow{g} & g(Q_{n-1}) & \xleftarrow{\phi} & Q_n & \xrightarrow{g} & g(Q_n) \supseteq \phi(\frac{1}{2}D_n) & \xleftarrow{\phi} & \frac{1}{2}D_n & \xrightarrow{g} & \mathbb{D}\left(w_n, \left(\frac{1}{2}\right)^{2d_n}\right) \\
& & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & & & \| \\
\cdots & \xleftarrow{\phi} & \widehat{U}_{n,n-1} & \xrightarrow{g} & U_{n,n} & \xleftarrow{\phi} & \widehat{U}_{n,n} & \xrightarrow{g} & \mathbb{D}(\phi(ih_n), R'_n) & & \xrightarrow{f} & & \mathbb{D}\left(w_n, \left(\frac{1}{2}\right)^{2d_n}\right) \\
& & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\
\cdots & \longmapsto & \hat{c}_{n,n-1} & \longmapsto & c_{n,n} & \longmapsto & \hat{c}_{n,n} & \longmapsto & \phi(ih_n) & & \longmapsto & & w_n
\end{array}$$

Estimate the inner radius ρ_n

There exists $C > 0$ such that if we define

$$\rho_n := \exp\left(-nC - \sum_{j=0}^{n-1} x_j - x_{n-1}\right), \quad \text{for } n \geqslant N,$$

then

$$\mathbb{D}(c_n(\mathbf{w}), \rho_n) \subseteq U_n, \quad \text{for all } n \geqslant N.$$

One can check that with our definitions there exists $N_1 \geqslant N$ such that

$$\left(\frac{1}{2}\right)^{2d_n} < \rho_{n+1}, \quad \text{for } n \geqslant N_1.$$

Infinite shooting problem

It just remains to find $\mathbf{w} = (w_N, w_{N+1}, \dots) \in \mathbb{D}(\frac{1}{2}, \frac{1}{8})^{\mathbb{N}_N}$ such that

$$w_n = c_{n+1}(\mathbf{w}), \qquad \text{for } n \geqslant N.$$

First, we write $\mathbf{w} = (\mathbf{w}', \mathbf{w}'')$, where $\mathbf{w}' = (w_N, w_{N+1}, \dots, w_T)$ for some $T > N$. Then, since the function

$$\begin{array}{ccc} \overline{\mathbb{D}}\left(\frac{1}{2}, \frac{1}{8}\right)^{\mathbb{N}_N \setminus \mathbb{N}_{T+1}} & \longrightarrow & \overline{\mathbb{D}}\left(\frac{1}{2}, \frac{1}{8}\right)^{\mathbb{N}_N \setminus \mathbb{N}_{T+1}} \\ \mathbf{w}' & \mapsto & (c_{N+1}(\mathbf{w}), \dots, c_{T+1}(\mathbf{w})) \end{array}$$

is continuous, we can use Brouwer's Fix Point Theorem to solve the finite shooting problem for any T with $\mathbf{w}''$ being the constant sequence $w_n = 1/2$ for $n > T$. Let $\mathbf{w}_T$ be such solution. Finally, since $\overline{\mathbb{D}}\left(\frac{1}{2}, \frac{1}{8}\right)^{\mathbb{N}_N}$ is compact, we can take a subsequence $\{\mathbf{w}_{T_k}\}_k$ that converges to some $\mathbf{w}_*$ that solves the infinite shooting problem.

Summary

- We started with a base function $g(z) = 2\cosh z$, which has order 1.
- Using the reference orbit $\{f^n(1/2)\}_n$,, we defined sequences $\{h_n\}_n, \{d_n\}_n, \{R_n\}_n$ and sets $\{E_n\}_n, \{D_n\}_n, \{Q_n\}_n$.
- For $N \in \mathbb{N}$ and for every sequence $\mathbf{w} \in \mathbb{D}(1/2, 1/8)^{\mathbb{N}_N}$, we can define a function $g_{\mathbf{w}}$ and integrate to obtain a function $f_{\mathbf{w}} = g_{\mathbf{w}} \circ \phi_{\mathbf{w}}^{-1}$.
- Find $N \in \mathbb{N}$ sufficiently large so that, using the 3 estimates on quasiconformal maps, we can control the function $\phi_{\mathbf{w}}^{-1}$ on the sets $\{D_n\}_n$ and $\{Q_n\}_n$.
- Check that the size of the domains U_n and the powers d_n are correct, and solve the shooting problem to find $\mathbf{w}_*$.
- We have $f^{n+2}(U_n) \subseteq U_{n+1}$ for all n sufficiently large, and hence are contained in the grand orbit of an oscillating wandering domain.
- The singular values of f are $\{-2, 2\}$ and $\overline{\{w_n\}_n} \subseteq \mathbb{D}$ and hence $f \in \mathcal{B}$, and it has order 1.

Thank you for your attention!