Critical points of the multiplier map

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Independence of multipliers

Theorem (G. 2014): The multipliers of any $n - 1$ distinct periodic orbits, considered as algebraic maps on the space of degree n polynomials, are locally independent at a *generic* polynomial f.

The multiplier map on the space $\mathrm{Poly}_2 = \{z^2 + c \mid c \in \mathbb{C}\}$

For any $k \in \mathbb{N}$,

- let Poly_2^k be the set of all pairs $(f_c, \mathcal{O})$, such that $f_c \in \mathrm{Poly}_2$ and $\mathcal{O}$ is a periodic orbit of f_c of period k.
- The multiplier map $\rho_k \colon \mathrm{Poly}_2^k \to \mathbb{C}$ is defined by

$$\rho_k(f_c, \mathcal{O}) := \text{the multiplier of the periodic orbit } \mathcal{O}.$$

Question: What can we say about the critical points of the maps ρ_k?

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When $c = 0$,

$$\frac{d\rho_k}{dc}(0, \langle z_0 \rangle) = -2^k \sum_{j=0}^{k-1} z_0^{-2^{j+1}}.$$

k	6	12	18	20	21	24	30
z_0	$e^{2\pi i/9}$	$e^{2\pi i/45}$	$e^{2\pi i/27}$	$e^{2\pi i/25}$	$e^{2\pi i/49}$	$e^{2\pi i/153}$	$e^{2\pi i/99}$

Table: The list of all $k \leq 30$, for which ρ_k has a critical point at $c = 0$. (z_0 is a corresponding periodic point.)

Critical points of the multiplier maps ρ_k

For any $k \in \mathbb{N}$, define

- $\sigma_k(f_c, \mathcal{O}) := \dfrac{d\rho_k}{dc}(f_c, \mathcal{O})$;
- $X_k := \{c \in \mathbb{C} \mid \sigma_k(f_c, \mathcal{O}) = 0,\ \text{for some periodic orbit } \mathcal{O}\}$. (Points in X_k are counted with multiplicity.)

$$\nu_k := \frac{1}{\# X_k} \sum_{c \in X_k} \delta_c.$$

Theorem (Firsova, G.): The sequence of measures $\{\nu_k\}_{k\in\mathbb{N}}$ converges to μ_{bif} in the weak sense of measures on $\mathbb{C}$.
Theorem (Firsova, G.): For every $k_0 \in \mathbb{N}$ and $c \in X_{k_0} \setminus \mathbb{M}$, there exists a sequence $\{c_k\}_{k=3}^{\infty}$, such that each $c_k \in X_k$ and

$$\lim_{k\to\infty} c_k = c.$$

Related results for quadratic polynomials

$$\mu_{\mathrm{bif}} = \Delta G_{\mathbb{M}},$$

where $G_{\mathbb{M}} \colon \mathbb{C} \to [0, +\infty)$ is the Green's function of the Mandelbrot set and Δ is the generalized Laplacian.

$$G_c(z) = \lim_{n \to +\infty} \max\{2^{-n} \log |f_c^{\circ n}(z)|,\ 0\},$$

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Theorem (Brolin 1965): For any $z_0 \in \mathbb{C}$ (possibly avoiding two exceptional values), the points $f_c^{-k}(z_0)$ (counted with multiplicity) equidistribute on the Julia set J_c, as $k \to \infty$.

Theorem (Levin 1989, Bassanelli-Berteloot 2011, Buff-Gauthier 2015): For any $\rho_0 \in \mathbb{C}$, the set of parameters c (counted with multiplicity), such that $\rho_k(f_c, \mathcal{O}) = \rho_0$, for some $(f_c, \mathcal{O}) \in P_k$, equidistributes on the boundary of $\mathbb{M}$, as $k \to \infty$.

Critical points of the multiplier maps ρ_k

For any $s \in \mathbb{C}$ and any $k \in \mathbb{N}$,

- define
  $X_{s,k} := \{c \in \mathbb{C} \mid \sigma(f_c, \mathcal{O}) = s, \text{ for some periodic orbit } \mathcal{O}\}$.
  (Points in $X_{s,k}$ are counted with multiplicity.)

$$\nu_{s,k} := \frac{1}{\# X_{s,k}} \sum_{c \in X_{s,k}} \delta_c.$$

Theorem (Firsova, G.): For every sequence of complex numbers $\{s_k\}_{k \in \mathbb{N}}$, such that

$$\limsup_{k \to +\infty} \frac{1}{k} \log |s_k| \le \log 2,$$

the sequence of measures $\{\nu_{s_k,k}\}_{k \in \mathbb{N}}$ converges to μ_{bif} in the weak sense of measures on $\mathbb{C}$.

Idea of the proof: Potentials!

Step 1: For each measure ν_k, construct a potential (a subharmonic function) $u_k\colon \mathbb{C} \to [-\infty, +\infty)$, such that

$$\Delta u_k = \nu_k.$$

Step 2: Then convergence $u_k \to G_{\mathbb{M}}$ in L^1_{loc} as $k \to \infty$ implies weak convergence of measures $\nu_k \to \mu_{\mathrm{bif}}$.

Step 1: Potentials

$$\tilde{S}_k(c,s) := \prod_{\mathcal{O}|(c,\mathcal{O})\in P_k} (s - \sigma_k(f_c, \mathcal{O}))$$

$\tilde{S}_k$ is a rational map in c with simple poles at primitive parabolic c.

$$C_k(c) := \prod_{\tilde{c}\in\tilde{\mathcal{P}}_k} (c - \tilde{c}).$$

$$S_k(c,s) = C_k(c)\tilde{S}_k(c,s) \quad \text{– polynomials in } c \text{ and } s.$$

Lemma: $S_k(c,0) = 0$, iff c is a critical point of the multiplier map ρ_k.

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Lemma: $S_k(c,0) = 0$, iff c is a critical point of the multiplier map ρ_k.

For all $c \in \mathbb{C}$, define

$$u_k(c) := \frac{1}{\deg_c S_k} \log |S_k(c,0)| = \frac{1}{\deg_c S_k} \left[\log |\tilde{S}_k(c,0)| + \log |C_k(c)| \right].$$

Then

$$\nu_k = \Delta u_k.$$

Step 2: L^1_{loc} convergence of potentials

Lemma (Buff, Gauthier): Any subharmonic function $u\colon \mathbb{C} \to [-\infty, +\infty)$ which coincides with $G_{\mathbb{M}}$ outside $\mathbb{M}$, coincides with $G_{\mathbb{M}}$ everywhere.

Lemma (Buff, Gauthier): Let $K \subset \mathbb{C}$ be a compact set such that $\mathbb{C} \setminus K$ is connected. Let v be a subharmonic function on $\mathbb{C}$ such that Δv is supported on ∂K and does not charge the boundary of the connected components of the interior of K. Then, any subharmonic function u on $\mathbb{C}$ which coincides with v outside K, coincides with v everywhere.

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Corollary: We need to prove $u_k \to G_{\mathbb{M}}$ only outside $\mathbb{M}$.

Roots of the multiplier maps

$\mathfrak{c}\colon \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\mathbb{M}$ – conformal double covering, $\mathfrak{c}(\lambda) := \phi_{\mathbb{M}}^{-1}(\lambda^2)$

$\Omega := \{0, 1\}^{\mathbb{N}}$, $\sigma\colon \Omega \to \Omega$ is the left shift.

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For any $\lambda \in \mathbb{C}\setminus\overline{\mathbb{D}}$, the map $\psi_\lambda\colon \Omega \to \mathbb{C}$ is

- a homeomorphism between Ω and $J_{\mathfrak{c}(\lambda)}$, conjugating σ to $f_{\mathfrak{c}(\lambda)}$:

$$\psi_\lambda \circ \sigma = f_{\mathfrak{c}(\lambda)} \circ \psi_\lambda; \qquad (1)$$

- $\psi_\lambda(\mathbf{w})$ depends analytically on $\lambda \in \mathbb{C}\setminus\overline{\mathbb{D}}$;

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For $\lambda \in \mathbb{C}\setminus\overline{\mathbb{D}}$, define

$$g_{k,\mathbf{w}}(\lambda) := \left(2^k \prod_{j=0}^{k-1} \psi_\lambda(\sigma^j \mathbf{w})\right)^{1/k}.$$

Motivation: If $\mathbf{w}$ is k-periodic, then $g_{k,\mathbf{w}}(\lambda)$ is the k-th degree root of the multiplier.

Roots of the multiplier maps

Ergodic Theorem: For a.e. $\mathbf{w} \in \Omega$, the sequence of maps $\{g_{k,\mathbf{w}}\}_{k\in\mathbb{N}}$ converges to 2λ on compact subsets of $\mathbb{C} \setminus \overline{\mathbb{D}}$, as $k \to \infty$.

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$$\Omega_k \subset \Omega \qquad \text{– periodic itineraries of period } k.$$

For any $\mathbf{w} \in \Omega_k$, define

$$g_{\mathbf{w}}(\lambda) := g_{k,\mathbf{w}}(\lambda) \qquad \text{– the } k\text{-th degree root of the multiplier.}$$

Theorem: For any $\varepsilon, \delta > 0$ and a compact subset $K \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, there exists $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, the following holds:

$$\frac{\#\{\mathbf{w} \in \Omega_k : \|g_{\mathbf{w}} - 2 \cdot \mathrm{id}\,\|_K < \delta\}}{\#\Omega_k} > 1 - \varepsilon.$$

Potentials outside of $\mathbb{M}$

$$\text{Recall:} \qquad u_k(c) := \frac{1}{\deg_c S_k}\left[\log|\tilde{S}_k(c,0)| + \log|C_k(c)|\right].$$

According to Buff-Gauthier, for any $c \in \mathbb{C} \setminus \mathbb{M}$,

$$\frac{1}{\deg_c S_k}\log|C_k(c)| \to \log|\lambda(c)|, \quad \text{pointwise as } k \to \infty.$$

Next

$$\frac{1}{\deg_c S_k}\log|\tilde{S}_k(c,0)| \sim \frac{1}{2^k}\sum_{\mathbf{w}\in\Omega_k}\frac{1}{k}\cdot\log\left|\frac{d}{dc}([g_{\mathbf{w}}(\lambda)]^k)\right| =$$

$$\frac{1}{2^k}\sum_{\mathbf{w}\in\Omega_k}\frac{1}{k}\left[\log k + (k-1)\log|g_{\mathbf{w}}(\lambda)| + \log|g'_{\mathbf{w}}(\lambda)| + \log\left|\frac{d\lambda}{dc}\right|\right] \to$$

$$\to \log|g_{\mathbf{w}}(\lambda)| \quad (\text{for ``nice'' } \mathbf{w}) = \log|2\lambda| = \log|\lambda(c)| + \log 2.$$