Rigidity of Newton Dynamics

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A word about general philosophy on rigidity

- **Dynamical rigidity:** a holomorphic map f is *rigid* if one can distinguish, **in combinatorial terms**, all orbits of f.
- **Parameter rigidity:** a family $\mathcal{F}$ of holomorphic maps is *rigid* if any pair of combinatorially equivalent maps in $\mathcal{F}$ are *quasiconformally conjugate* in some neighborhood of their Julia sets.

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Take-away general philosophy (Rational Rigidity Principle):

(dynamical version) a rational map is either *rigid*, or it contains an *embedded polynomial dynamics* (excluding flexible examples); (parameter space version) a family of rational maps is *rigid* provided it contains no embedded polynomial dynamics, or this dynamics is embedded in "the same way".

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A **puzzle piece of depth** n (notation P_n^i) is a closed topological disk s.t.

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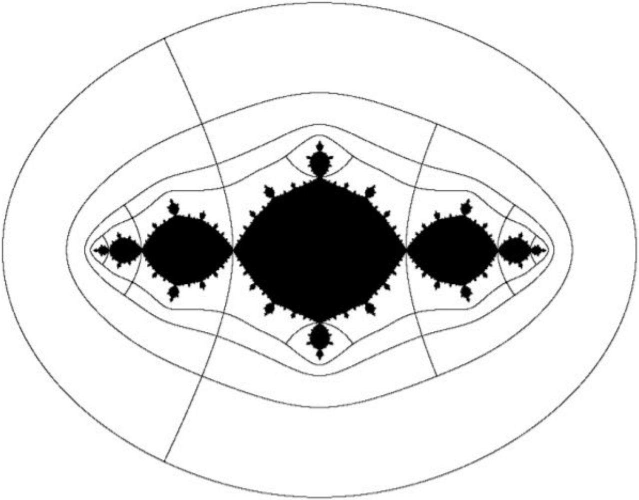
- $g(P_n^i) = P_{n-1}^j$ for some j, and $g\colon \mathring{P}_n^i \to \mathring{P}_{n-1}^j$ is a branched covering.

Yoccoz puzzles for polynomials



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Non-escaping set: $K(g) := \left\{z \in U\colon g^k(z) \in U \forall k > 0\right\}$ (the filled-in Julia set);
the Julia set: $J(g) := \partial K(g)$.

For $x \in K(g)$, let $P_n(x)$ the union of puzzle pieces containing x.

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$\to$ **fib(x)** is the set of points with the **same itinerary** w.r.t. dynamically defined puzzle partition $\to$ the fiber consists of points "traveling together" $\to$ if the fiber of x is trivial, then the orbit of x is **combinatorially distinguishable** among all other orbits.

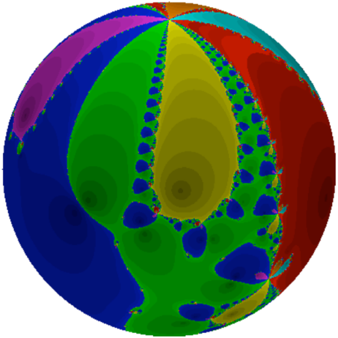
Dynamical Rigidity for Newton maps

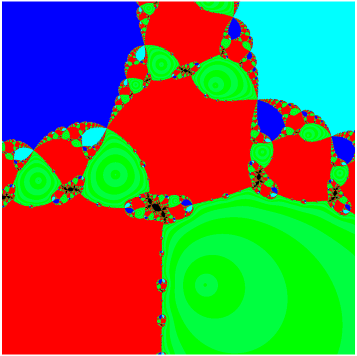
$p\colon \mathbb{C} \to \mathbb{C}$ is a complex polynomial. The **Newton map of** p is the rational map $N_p\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ s.t.

$$N_p(z) := z - \frac{p(z)}{p'(z)}.$$

Fixed points in $\widehat{\mathbb{C}}$: $N_p(z) = z \Leftrightarrow z = \infty$ (repelling) or z is a root of p (attracting) (hence each of the roots has its own basin of attraction).

Newton dynamical plane: examples





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Let N_p be a polynomial Newton map of degree $d \geqslant 3$. Then for every point $z \in \widehat{\mathbb{C}}$ exactly one of the following alternatives holds true:

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Corollary

The boundaries of the components of the basins of roots are locally connected.

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Related work: **Roesch**[2](cubic Newton maps), **Wang–Yin–Zeng**[3](local connectivity of the boundaries of the basins of roots, done in parallel).

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Newton puzzles

(Up to a quasiconformal deformation in the basins of roots)
The channel diagram Δ := a finite invariant graph connecting all roots to ∞.
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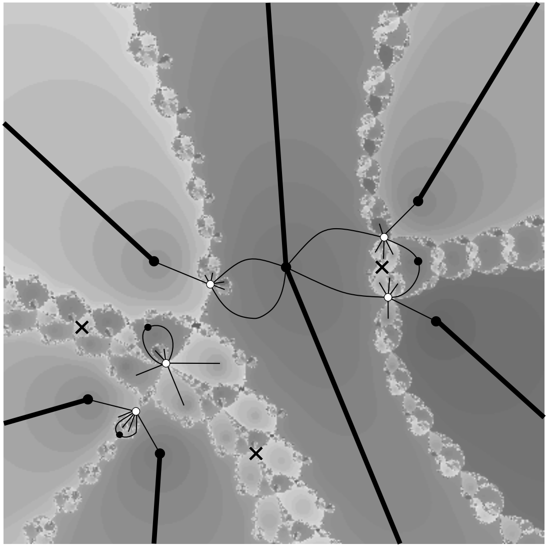
Newton graph Δ_1



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Lemma (All poles are in, D–Mikulich–Rückert–Schleicher[4])

There exists $N > 0$ so that Δ_N contains all the poles of N_p.

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Lemma (Circle separation property, D–Lodge–Schleicher–Sowinski[5])

There exists a least integer $K > N$ so that for every component V of $\widehat{\mathbb{C}} \setminus \Delta$ there exists a topological circle $X_V \subset \Delta_K \cap \overline{V}$ that passes through all finite fixed points in ∂V and separates ∞ from all critical values of N_p in V.

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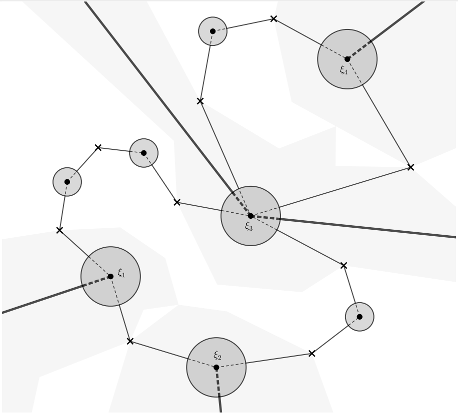

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$\Delta_n^+ :=$ the component containing ∞ of $N_p^{-n}(\Delta \cup \bigcup_V X_V)$.
Components of $\widehat{\mathbb{C}} \setminus \Delta_n^+$ (suitably truncated) are **Newton puzzle pieces**.

[6] Puzzles and the Fatou–Shishikura injection for rational Newton maps. arXiv:1805.10746 (28 May 2018)

Parameter Rigidity: combinatorially equivalent maps

N_p is **renormalizable around a critical point** $c \Leftrightarrow \exists$ puzzle piece W containing c and $\exists$ minimal $s > 1$ (the *period of the renormalization*) such that $N_p^{sk}(c') \in \mathring{W}$ for every critical point $c' \in W$ and $k \geqslant 0$.

Triviality of fibers at ∞ (D–L–S–S[7], D–Mikulich–Rückert–Schleicher[8])

If $\infty \in \operatorname{orb}(z)$, *then* $\operatorname{fib}(z) = \{z\}$.

Triviality of fibers at $\infty \implies$ if a Newton map is renormalizable around a critical point c, we can extract a polynomial-like map $\varrho \colon U \to V$ with $K(\varrho) = \operatorname{fib}(c)$.

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Definition (Combinatorially equivalent Newton maps)

Two (suitably normalized) Newton maps are **combinatorially equivalent** if their Newton graphs coincide $\Leftrightarrow$ all the components of the basins of roots are connected to each other in the same way.

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Parameter Rigidity: the statement

Newton Parameter Rigidity (D–Schleicher[9])

If N_p and $N_{\widetilde{p}}$ are combinatorially equivalent Newton maps, then they are quasiconformally conjugate in a neighborhood of the Julia set provided

[9] Rigidity of Newton dynamics. arXiv:1812.11919 (31 Dec 2018).

[10] Rigidity of non-renormalizable Newton maps. arXiv:1811.09978 (25 Nov 2018).

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If N_p and $N_{\widetilde{p}}$ are combinatorially equivalent Newton maps, then they are quasiconformally conjugate in a neighborhood of the Julia set provided

- *either they are both non-renormalizable,*

[9] Rigidity of Newton dynamics. arXiv:1812.11919 (31 Dec 2018).

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The domain of this qc conjugation, say ψ, can be chosen to include all Fatou components not in the basin of the roots, and $\overline{\partial}\psi = 0$ on those Fatou components as well as on the entire Julia set.

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(f, g hybrid equivalent $\Leftrightarrow \exists$ quasiconformal conjugacy ψ between f and g defined on a neighborhood of their filled Julia sets with $\bar{\partial}\psi\,\big|_{K(f)} = 0$.)

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Parallel work: **Roesch–Yin–Zeng**[10](parameter rigidity for non-renormalizable Newton maps).

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Ingredient of the proof: complex box mappings

Definition (Complex box mapping; Kozlovski–Shen–van Strien)

A holomorphic map $F \colon \mathcal{U} \to \mathcal{V}$ between two open sets $\mathcal{U} \subset \mathcal{V} \subset \widehat{\mathbb{C}}$ is a **complex box mapping** if the following holds:

1. F has *finitely many* critical points;
2. $\mathcal{V}$ is the union of finitely many open Jordan disks with disjoint closures;
3. for every component U of $\mathcal{U}$ the image $F(U)$ is a component of $\mathcal{V}$, and the restriction $F \colon U \to F(U)$ is a proper map;
4. every component V of $\mathcal{V}$ is *either* a component of $\mathcal{U}$, *or* $V \cap \mathcal{U}$ is a union of Jordan disks with pairwise disjoint closures compactly contained in V.

A puzzle piece P_n (of depth n) is the closure of a component of $F^{-n}(\mathcal{V})$.

$\mathcal{U}$ can have ∞-many connected components.

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$\mathcal{U}$ can have ∞-many connected components. A complex box mapping arises *the first return map* to the union of critical puzzle pieces.

Rigidity of non-renormalizable box mappings

Rigidity for complex box mappings (Kozlovski–van Strien[11])

If $F\colon \mathcal{U} \to \mathcal{V}$ is a non-renormalizable complex box mapping whose periodic points are all repelling, and there exists $\delta > 0$ s.t. $\operatorname{mod}(V \setminus \overline{U}) \geqslant \delta$ for every component U of $\mathcal{U}$ and V the component of $\mathcal{V}$ with $V \supset U$, then

1. $\operatorname{fib}(z) = \{z\}$ *for each* $z \in J(F)$;
2. *F carries no measurable invariant linefields on $J(F)$;*
3. *if $\widetilde{F}\colon \widetilde{\mathcal{U}} \to \widetilde{\mathcal{V}}$ is another complex box mapping for which there exists a quasiconformal homeomorphism $H\colon \mathcal{V} \to \widetilde{\mathcal{V}}$ so that $H(\mathcal{U}) = \widetilde{\mathcal{U}}$, $\widetilde{F} \circ H = H \circ F$ on $\partial\mathcal{U}$, and $\widetilde{F}$ is combinatorially equivalent to F w.r.t. H. Then F and $\widetilde{F}$ are quasiconformally conjugate, and this conjugation agrees with H on the boundary of $\mathcal{V}$.*

The proof uses the enhanced nest construction due to **Kozlovski–Shen–van Strien** (2007), the covering lemma due to **Kahn–Lyubich** (2009).

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D–S[1] $\rightsquigarrow$ upgrade to this result to include the renormalizable dynamics (**Generalized Rigidity for box mappings**).

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Newton Rigidity: general outline of the proof

Using Newton puzzles we can extract a box mapping (as the first return map to a collection of puzzle pieces)

$\Longrightarrow$ we can apply the Generalized Rigidity of complex box mappings + triviality of fibers at ∞

$\Longrightarrow$ rigidity for Newton dynamics.

Thank you for your attention!