Dynamics of Generalized Tangent maps

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Asymptotic values and tracts

1. A value $v \in \hat{\mathbb{C}}$ an **asymptotic value** of a holomorphic function f if there is a path $\gamma : [0, 1) \to \mathbb{C}$ such that

$$\lim_{t \to 1^-} r(t) = \infty \text{ and } \lim_{t \to 1^-} f(\gamma(t)) = v.$$

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2. Let v be asymptotic value of f. If there exist a neighborhood V of v and a simply connected set U such that $f : U \to V \setminus \{v\}$ is a universal covering, then U is an **asymptotic tract** of v.

Exponential Maps

The most well-known family of exponential maps

$$f_k(z) = e^z + k \quad \text{or } f_\lambda(z) = \lambda e^z$$

Two asymptotic values are k and ∞.

Bifurcation of Exponential Maps

1. The bifurcation locus is connected.
2. Each hyperbolic component is unbounded and its boundary is a unbounded Jordan arc tending to ∞ in both directions.

L. Rempe-Gillen and D. Schleicher
Bifurcations in the Space of Exponential Maps
Invent. Math. 175 (2009), No. 1, 103 - 135.

Tangent maps

The family of tangent maps

$$f_\lambda = \lambda \tan z$$

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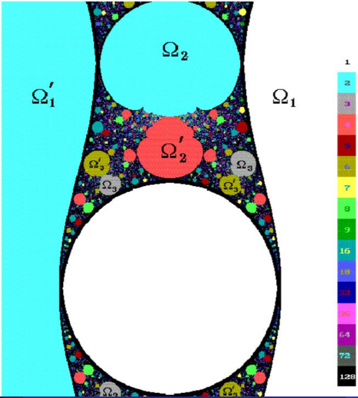
Note that if $\{z_1, z_2, \cdots, z_p\}$ is a cycle, then $\{-z_1, -z_2, \cdots, -z_p\}$ is also a cycle with the same multiplier.
Let $\Omega_k = \{\lambda : f_\lambda \text{ has two attracting cycles of period } k\}$
$\Omega'_k = \{\lambda : f_\lambda \text{ has one attracting cycle of period } 2k\}$

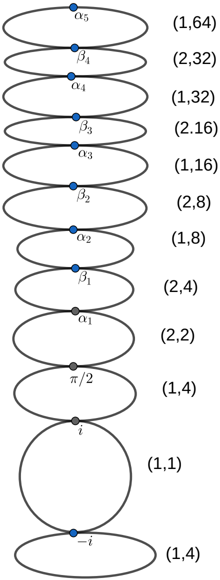
Parameter Space of $\lambda \tan z$

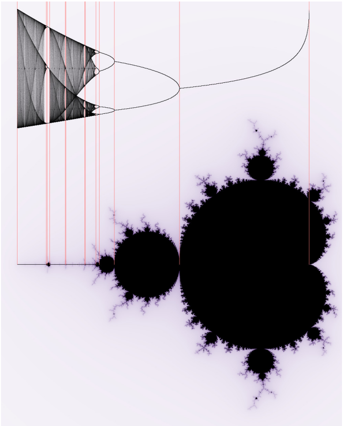
Theorem (L. Keen-J. Kotus, 97)

1. *Ω_1 and Ω_1' are connected and unbounded; other components are bounded.*
2. *Every Ω_k meets Ω_k' at one solution of $f_\lambda^{k-1}(\lambda i) = \infty$.*



Path to Chaos





Path to Chaos

Theorem (C.-Keen-Jiang, 2018)

For the family of the map $f_t = it\tan z$, $t \in [\pi/2, \pi]$, there are two sequences interleaved of parameters $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ for the tangent family

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \cdots < \beta_n < \alpha_n < \cdots < \pi,$$

such that

1. *If $t \in (\alpha_n, \beta_n)$, f_t has two attracting cycles of period 2^{n+1}, denoted as $(2, 2^{n+1})$.*
2. *If $t \in (\beta_n, \alpha_{n+1})$, f_t has one attracting cycle of period 2^{n+2}, denoted as $(1, 2^{n+2})$.*

Path to Chaos

Theorem (C.-Keen-Jiang, 2018)

The map f_{t_∞} has no attracting or parabolic cycle where

$$t_\infty = \lim_{t\to\infty} \alpha_n = \lim_{t\to\infty} \beta_n$$

and it has an attractor $\mathcal{C}$ contained in the real and imaginary lines and it attracts almost all points on the these lines.

Genadi Levin, Weixiao Shen, Sebastian van Strien
Monotonicity of entropy and positively oriented transversality for families of interval maps. (*Positive Transversality via transfer operators and holomorphic motions with applications to monotonicity for interval maps*)
arXiv:1611.10056.

Meromorphic functions of finite type

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Meromorphic functions of finite type

1. Fagella and Keen in [FK] considered $\mathcal{M}_\infty$, a family of meromorphic transcendental maps of finite type for which infinity is not an asymptotic value.
2. **Dynamically natural slice**–the dynamics of singular values is fixed expect one asymptotic value, denoted by v_λ.
3. A component of which the free asymptotic value v_λ is attracted by a **new** attracting cycle is called a **shell component**. Let Ω_k denote components where the period of the cycle is k.

L. Keen, N. Fagella
Stable components in the parameter plane of meromorphic functions of finite type
arXiv:1702.06563 .

Meromorphic functions of finite type

Theorem (Fagella-Keen)

1. *Each shell component is simply connected.*
2. *Each component in Ω_1 is unbounded.*
3. *On the boundary, there exists a λ, such that $f_\lambda^{k-1}(v_\lambda) = \infty$, called* **virtual center parameter**

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Questions/Conjecture

1. Conjecture [Fagella-Keen] Each component of Ω_k, $k \geq 2$, is bounded.
2. For each virtual center parameter λ, is it on the boundary of a shell component?

Generalized Nevanlinna Family

Consider the family $\mathcal{N}_{p,q,r}$ consisting of $f = P \circ g \circ Q$, where P, Q are polynomials of degree p, q respectively, and $g \in \mathcal{N}_r$ is meromorphic function with no critical values and r asymptotic values counted with multiplicity of asymptotic tracts.

Note that $f(z) = e^{e^z}$ has 3 asymptotic values $\{0, 1, \infty\}$. However, $f \notin \mathcal{N}_3$ since both 0 and ∞ have infinitely many asymptotic tracts.

Enumerate shell components

Theorem (C-Keen)

For any dynamical natural slice of $\mathcal{N}_{p,q,r}$, at every virtual center parameter λ, there exists a shell component Ω, such that $\lambda \in \partial\Omega$.

Generalized Tangent maps

The family $\mathcal{F}_\lambda = \{\lambda \tan^p z^q\}$, $\lambda \in \mathbb{C}^*$ $p, q \in \mathbb{N}$.

- Critical points: solutions of $\tan^{p-1} z^q = 0$ and 0;
- Critical Value: 0, super-attracting.
  attracting basin: $A(0) = \{z : \; f_\lambda^n \to 0 \text{ as } n \to \infty\}$
  immediate basin: $A^*(0)$ the component of $A(0)$ containing 0.
- Asymptotic value: $(\pm i)^p \lambda$.
  Denote $v_\lambda = (i)^p \lambda$

Dichotomy

Theorem

The Julia set of f_λ is connected if and only if $v_\lambda \notin A^*(0)$.

Capture and Shell components

1. Capture components $\mathcal{C}$ consist of λ such that $f_\lambda^n(v_\lambda) \to 0$.
   $\mathcal{C}_k = \{\lambda : k \text{ is the smallest integer such that } f^k(v_\lambda) \in A^*(0)\}$
2. Shell components $\mathcal{S}$ consist of λ such that v_λ is attracted to a nonzero attracting cycle.
   - pq is even, $\mathcal{S}_k$ consists of λ such that f_λ has an attracting cycle of period k
   - pq is odd, $\mathcal{S}_k$ consists of λ such that f_λ has an attracting cycle of period $2k$ or two attracting cycles of period k.

An example

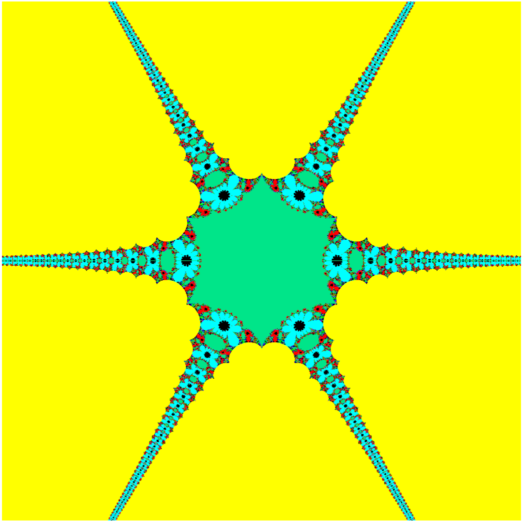

Figure: Parameter space of $\lambda \tan^2 z^3$. Thanks to a program of N. Fagella.

Enumerate capture components

Theorem (C.-Keen)

1. *$\mathcal{C}_0 \cup \{0\}$ is simply connected.*
2. *Each component of $\mathcal{C}_k$ for $k \geq 1$ is simply connected and contains a unique solution of*

$$f_\lambda^k(v_\lambda) = 0.$$

N Fagella, A Garijo
The parameter planes of $\lambda z^m e^z$ for $m \geq 2$
Communications in mathematical physics 273 (3), 2007, 755-783.

Enumerate shell components

Theorem (C.-Keen)

1. *The set* $\mathcal{S}_1$ *consists of* $2q$ *unbounded simply connected components.*
2. *For* $k \geq 2$*, at each solution of*

$$f_\lambda^{k-1}(v_\lambda) = \infty$$

there are $2pq$ *components of* $\mathcal{S}_k$*.*

3. *All components of* $\mathcal{C}_k$*,* $k \geq 0$*, and* $\mathcal{S}_k$*,* $k \geq 2$*, is bounded.*

Thank you!!