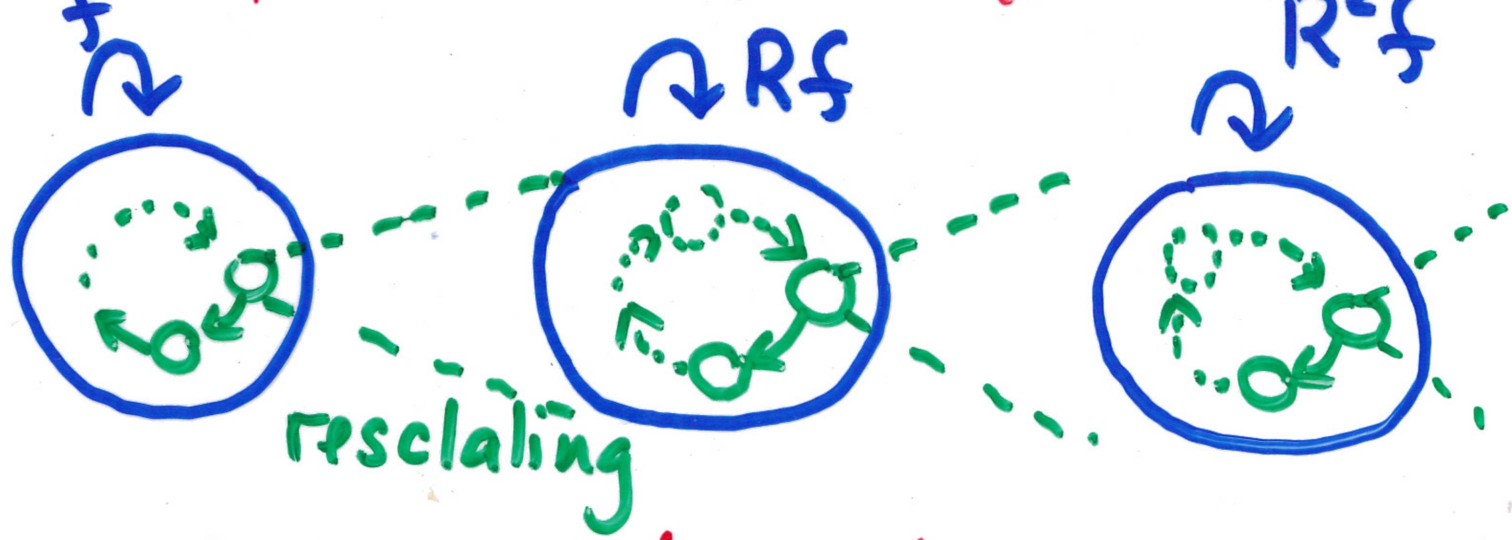
Many faces of Renormalization

Renormalization:



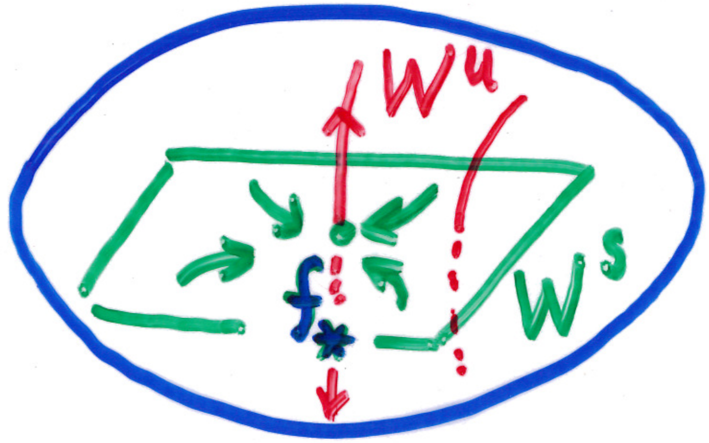
A priori bounds:

$R^n f$ form a pre-compact family

Universality:

$\exists$ a hyperbolic renorm. fixed pt f_*,

$\dim W^u < \infty$



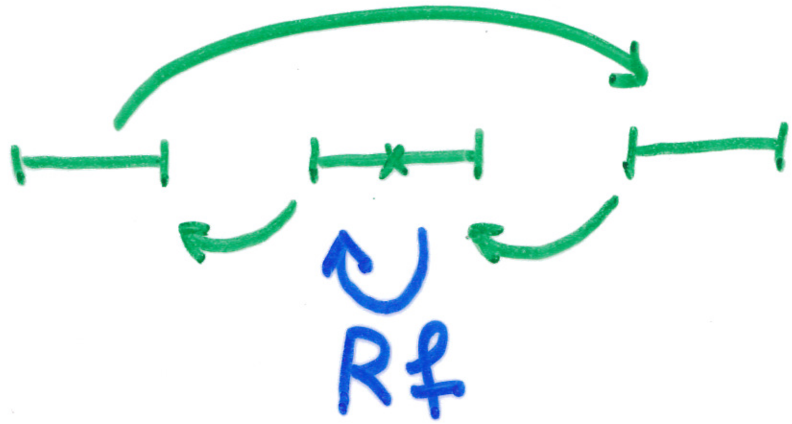
Space of maps

$W^s \rightsquigarrow$ dynamical small scale rigidity

$W^u \rightsquigarrow$ parameter rigidity

Unimodal Renormalization

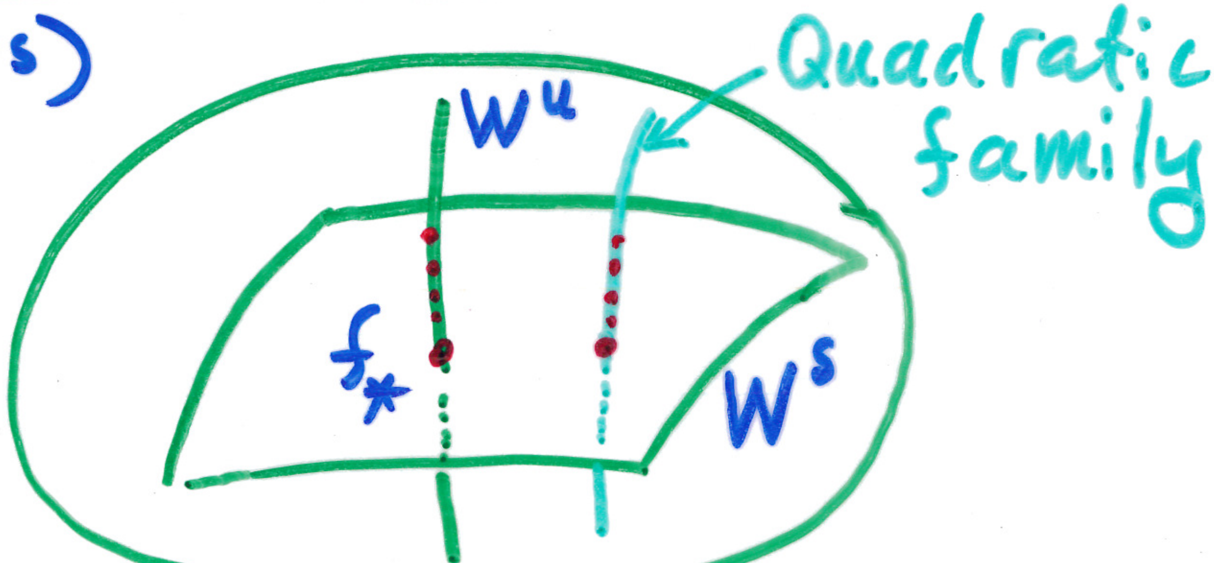
(e.g. $x \mapsto x^2 + c$)



Feigenbaum-Coullet-Tresser Conjecture

(1970s)

$R \circlearrowright$



Space of Unimodal maps

Proved by Sullivan, McMullen & L in the 1990s

Real Feigenbaum maps

(∞-renorm-le maps w. bounded comb-s)

Attractor

- $\exists$ a global physical Cantor attractor A_f
- dynamics on A_f is a group translation
- A_f is endowed w. a canonical invariant meas ν_f (Haar)

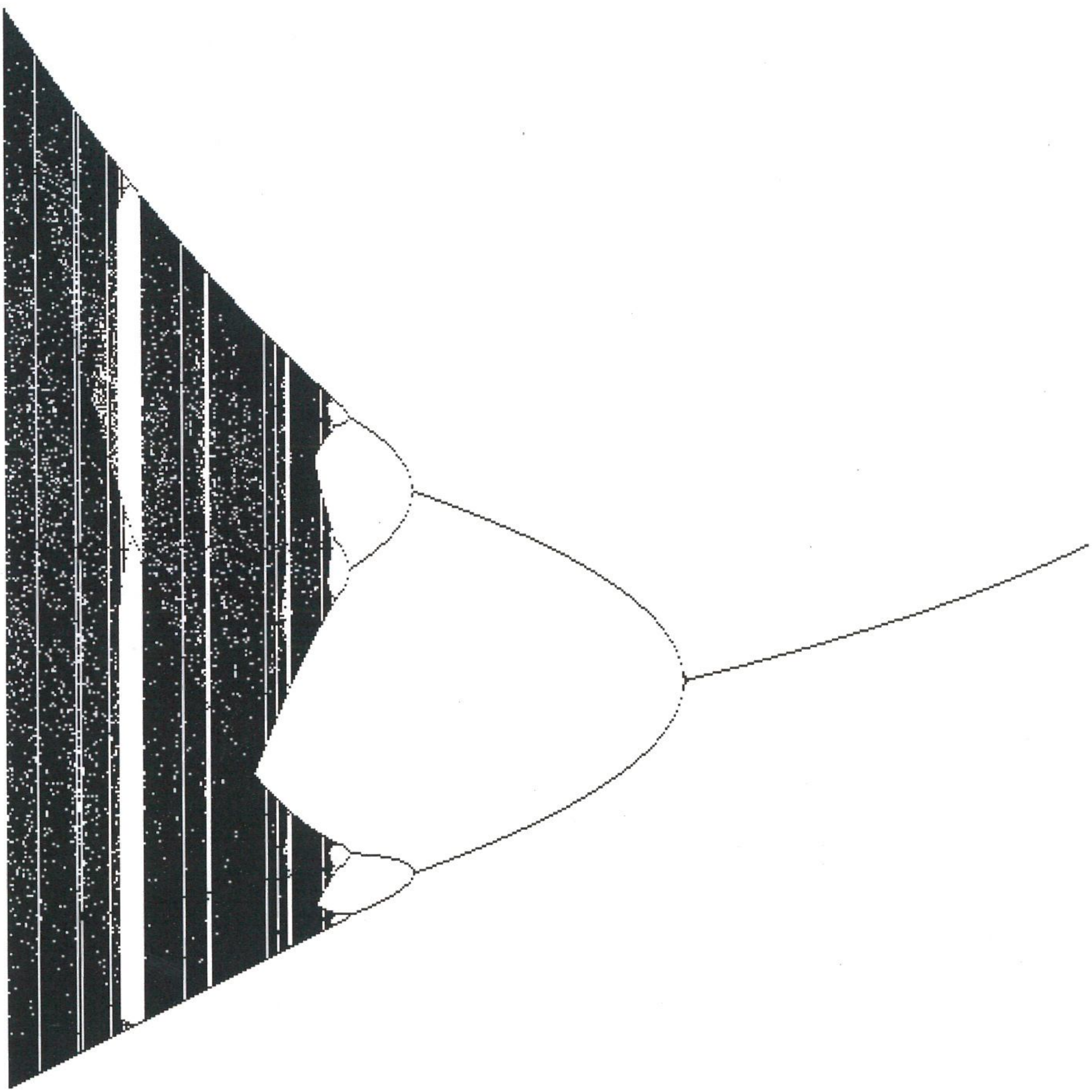
Rigidity

$$\begin{tikzcd} A_f \arrow[r, "f"] \arrow[d, "h"', "\wr"] & A_f \arrow[d, "h"', "\wr"] \\ A_g \arrow[r, "g"'] & A_g \end{tikzcd}$$

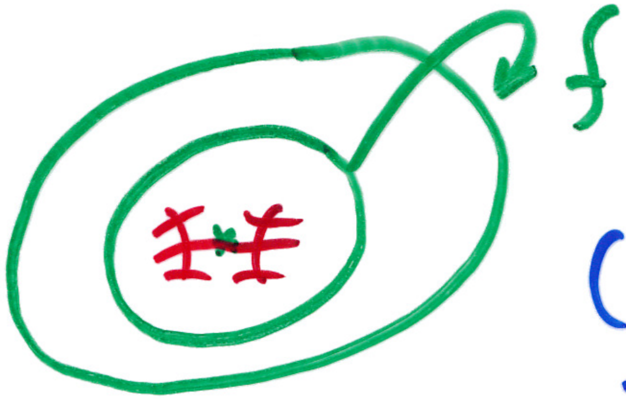
Conjugacy h is $C^{1+\alpha}$ smooth

$\Downarrow$

small scale geom of A_f is universal



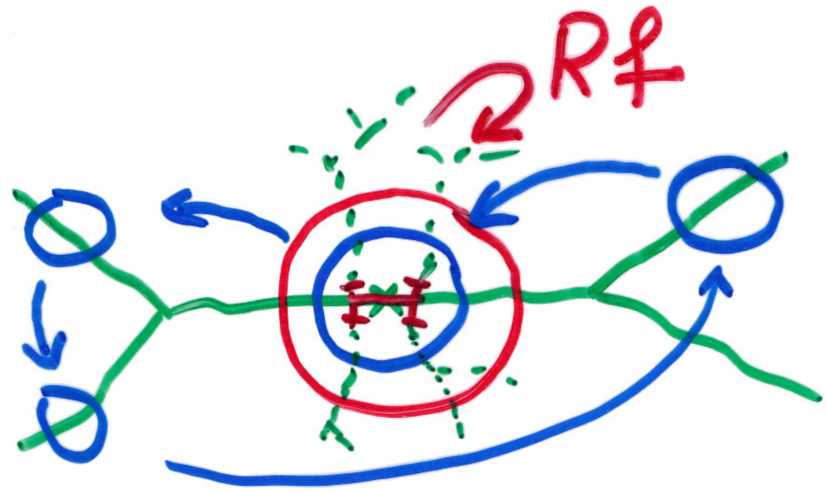
Quadratic-like Renormalization



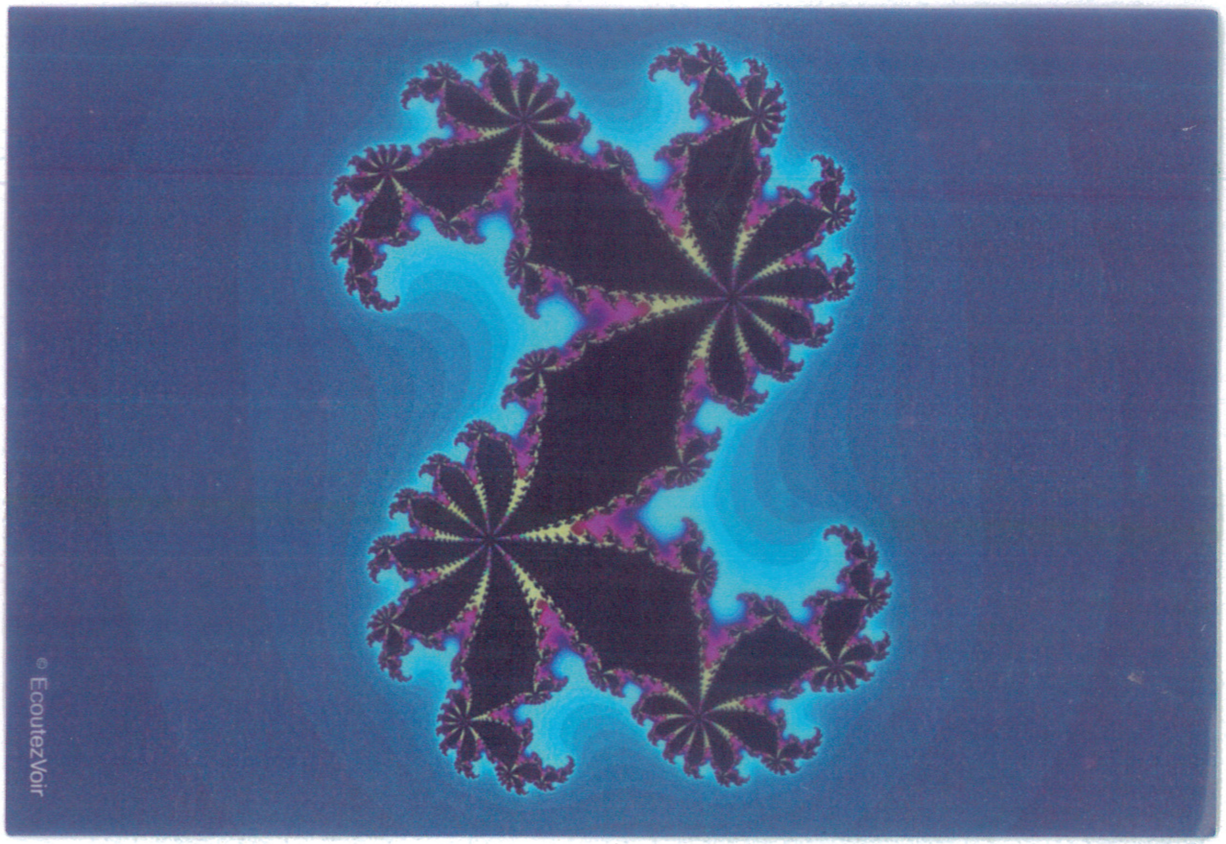
quadratic-like map
(e.g. restricted $f_c : z \mapsto z^2 + c$)

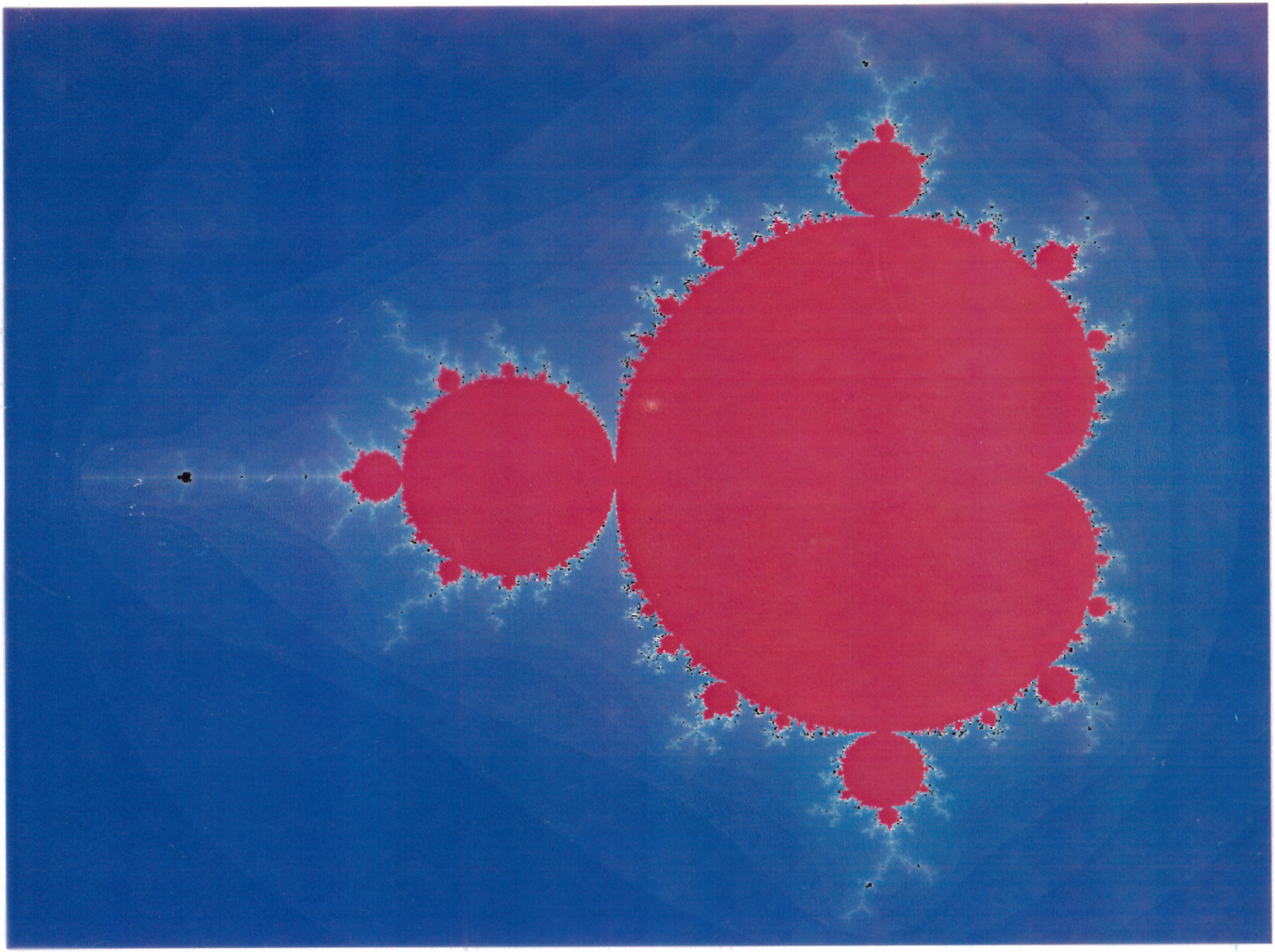
(Filled) Julia set $K(f)$ is the set of non-escaping pts

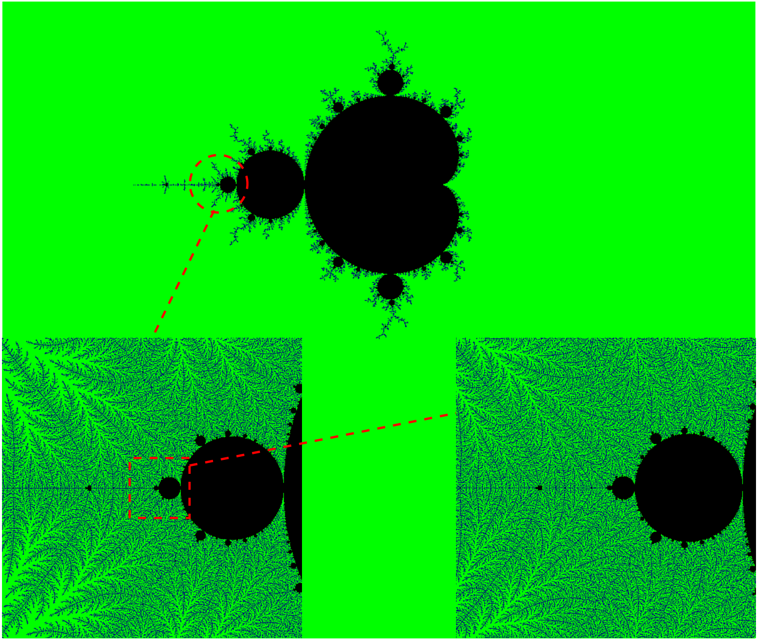
Basic Dichotomy: $K(f)$ is either connected or Cantor

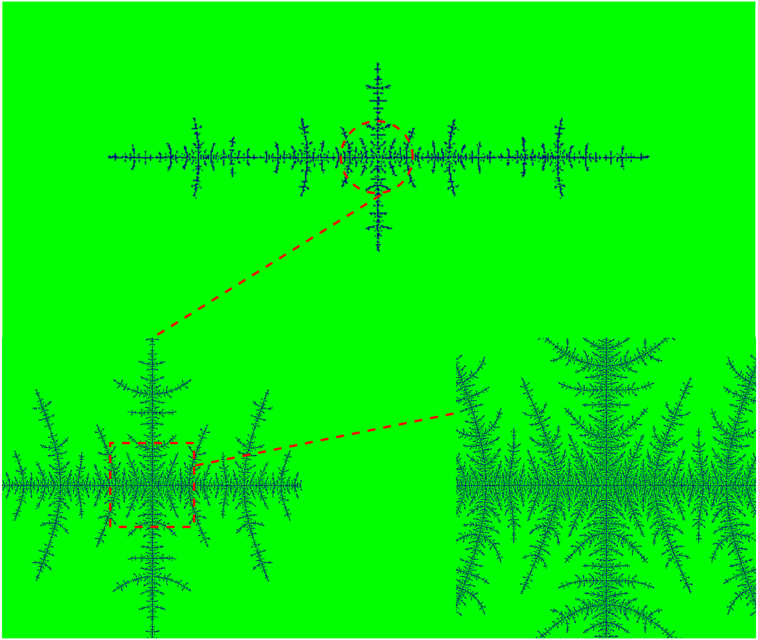


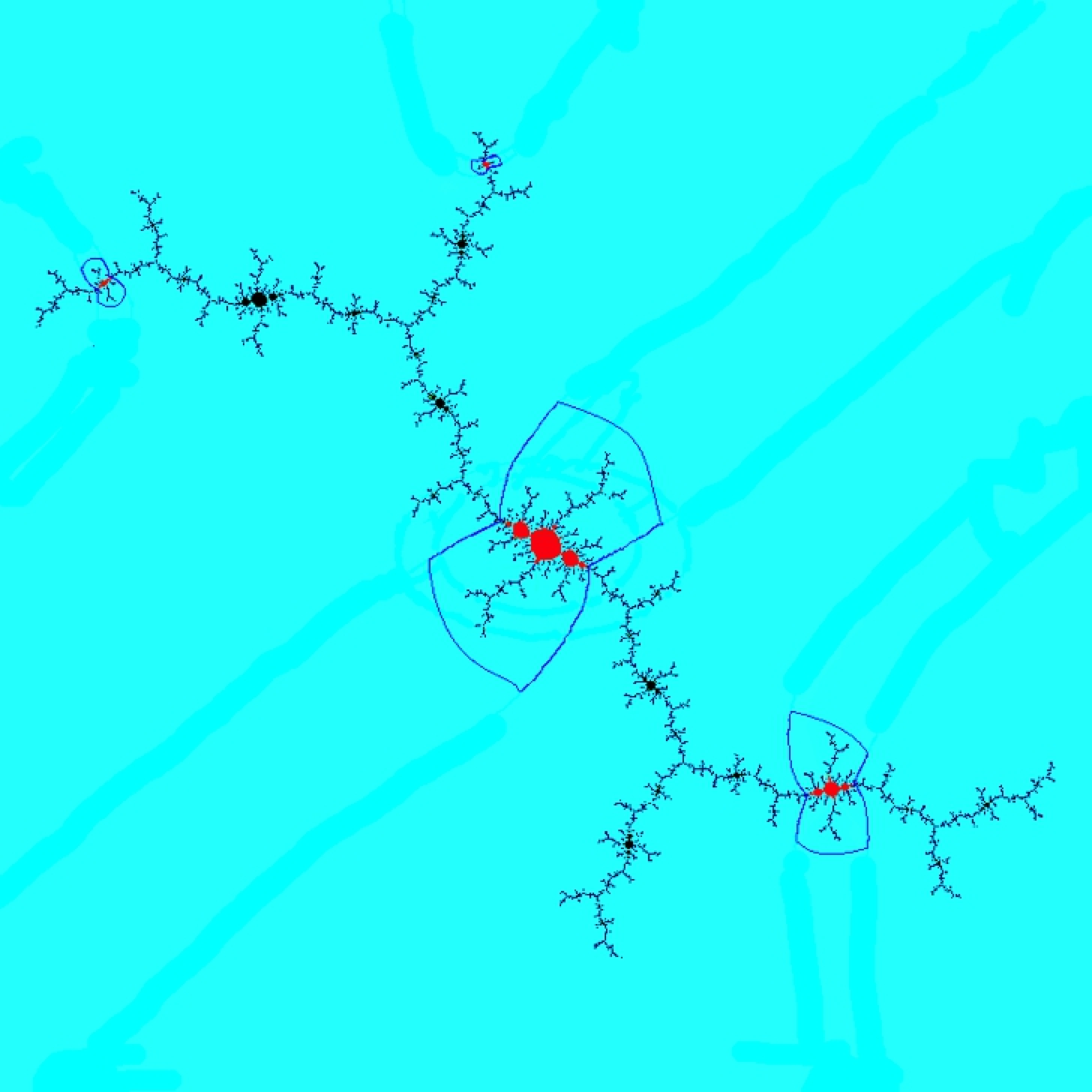
The set of quadratic maps f_c that are renormalizable with a given combinatorics form a little M-copy
(Douady-Hubbard)

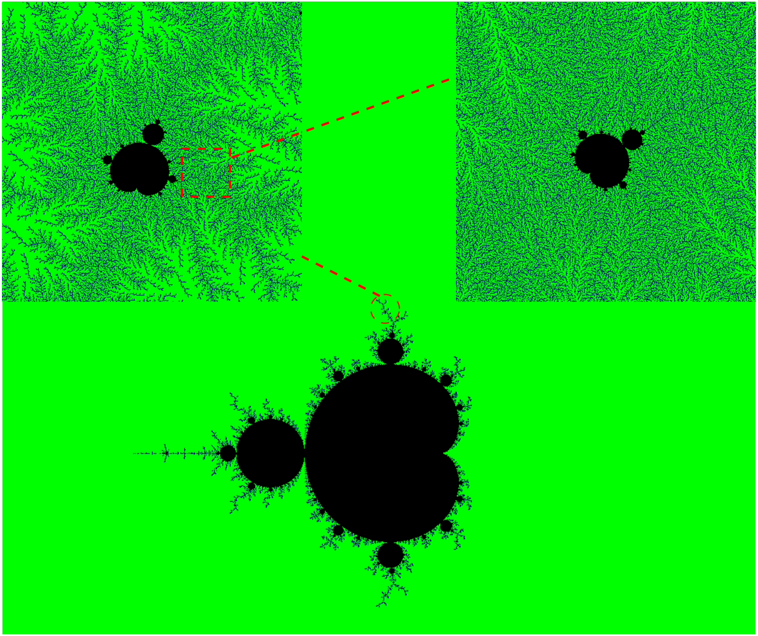

© EcoutezVoir











Feigenbaum maps:

∞-renormalizable maps with stationary (or bounded) combinatorics.

A priori bounds:

L (1990s): High type primitive comb-s

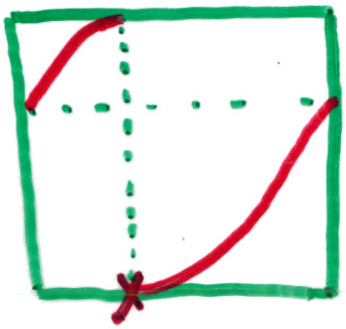
Kahn (2000s): Any primitive comb-s

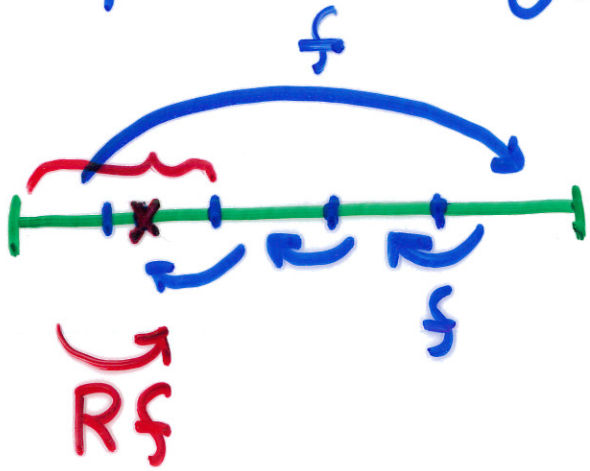
$\Downarrow$

- Self-similarity of Feigenbaum Julia sets at the critical pt
- Self-similarity of the Mandelbrot set at a Feigenbaum parameter

[Making use of ql Renormalization Theory]

Critical circle maps $f : \mathbb{T} \circlearrowleft$:
real analytic homeomorphisms with one critical pt of cubic type



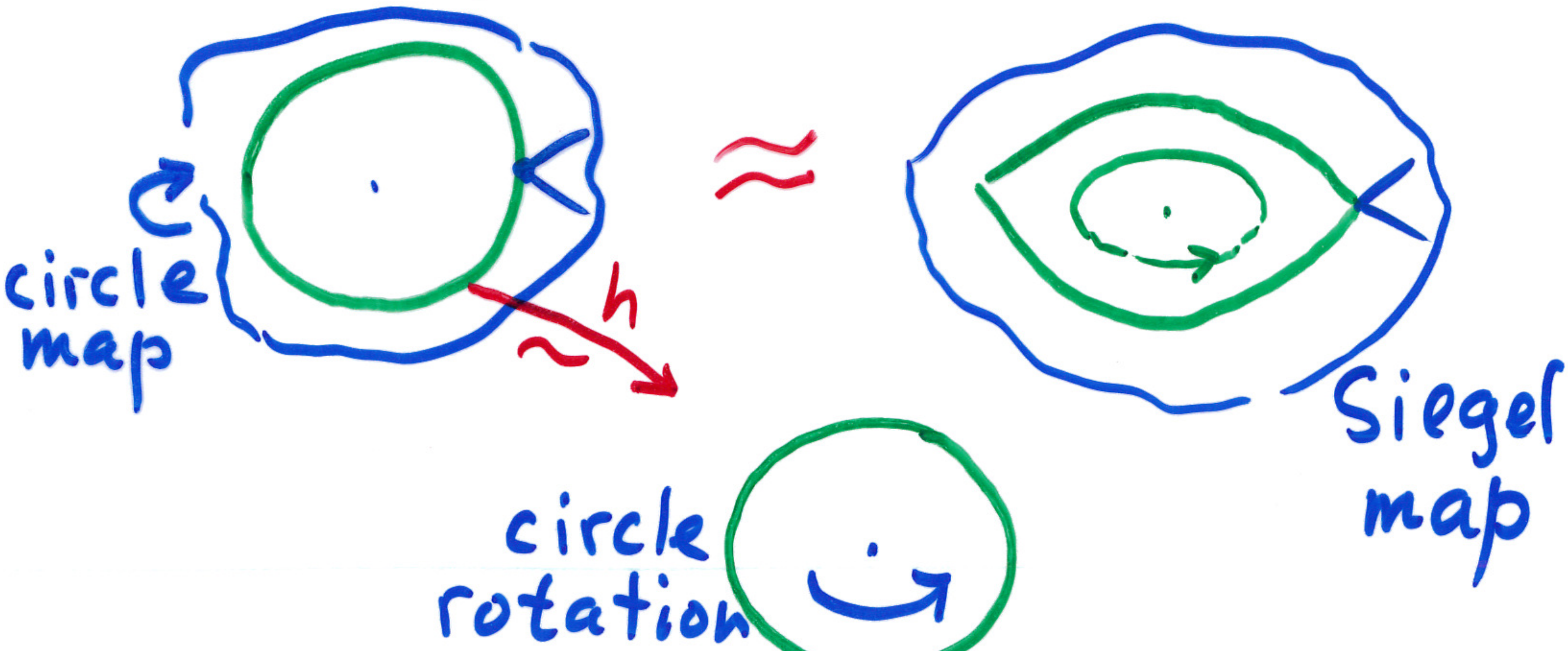


Combinatorics is encoded by the continued fraction expansion for the rotation number θ.

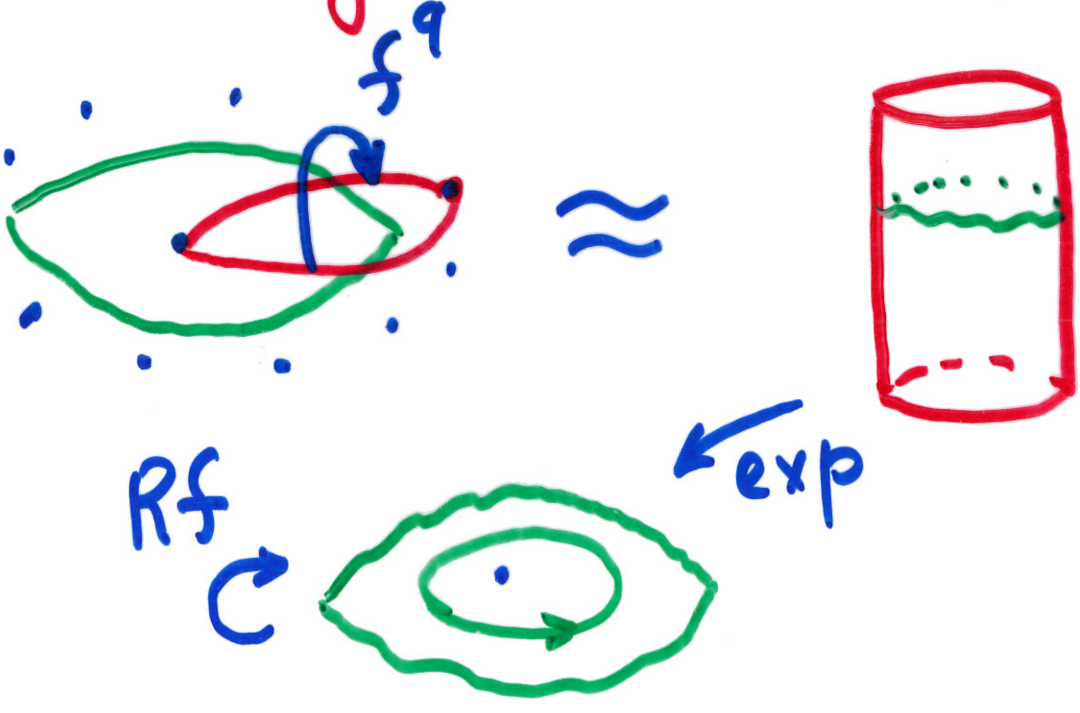
Butterfly Renormalization:

de Faria – de Melo – Yampolsky:
Renormalization Theory for critical circle / butterfly maps.

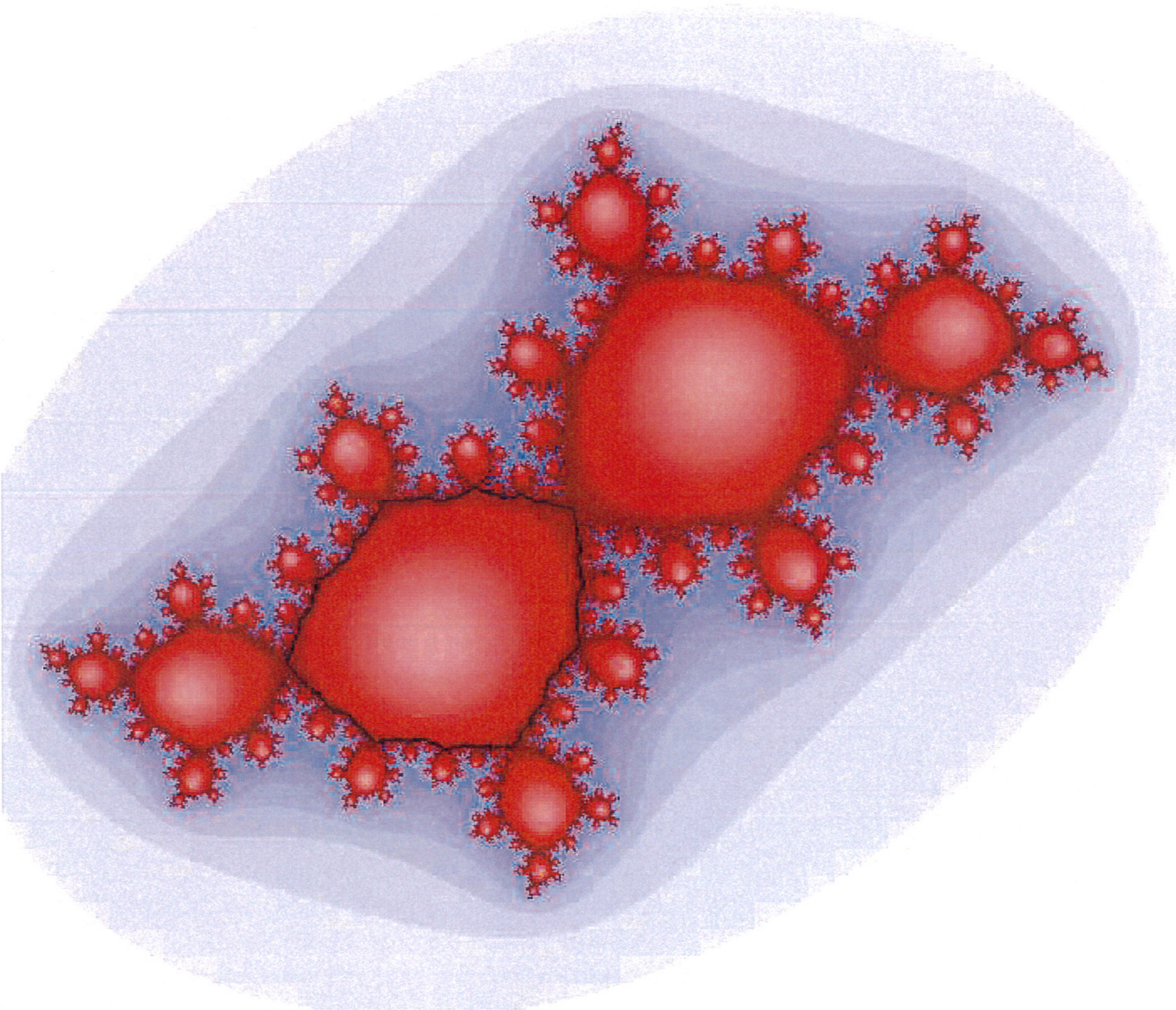
Douady-Ghys surgery

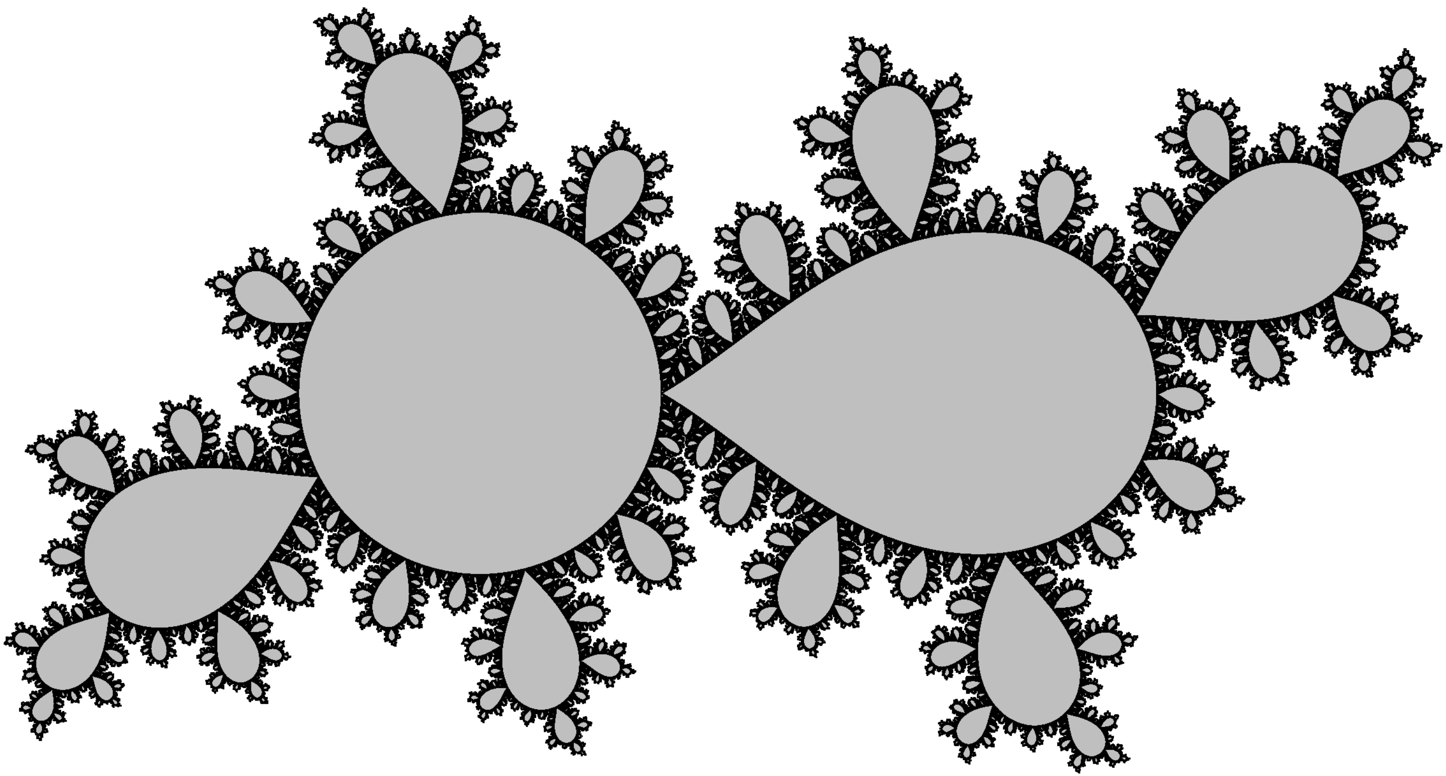


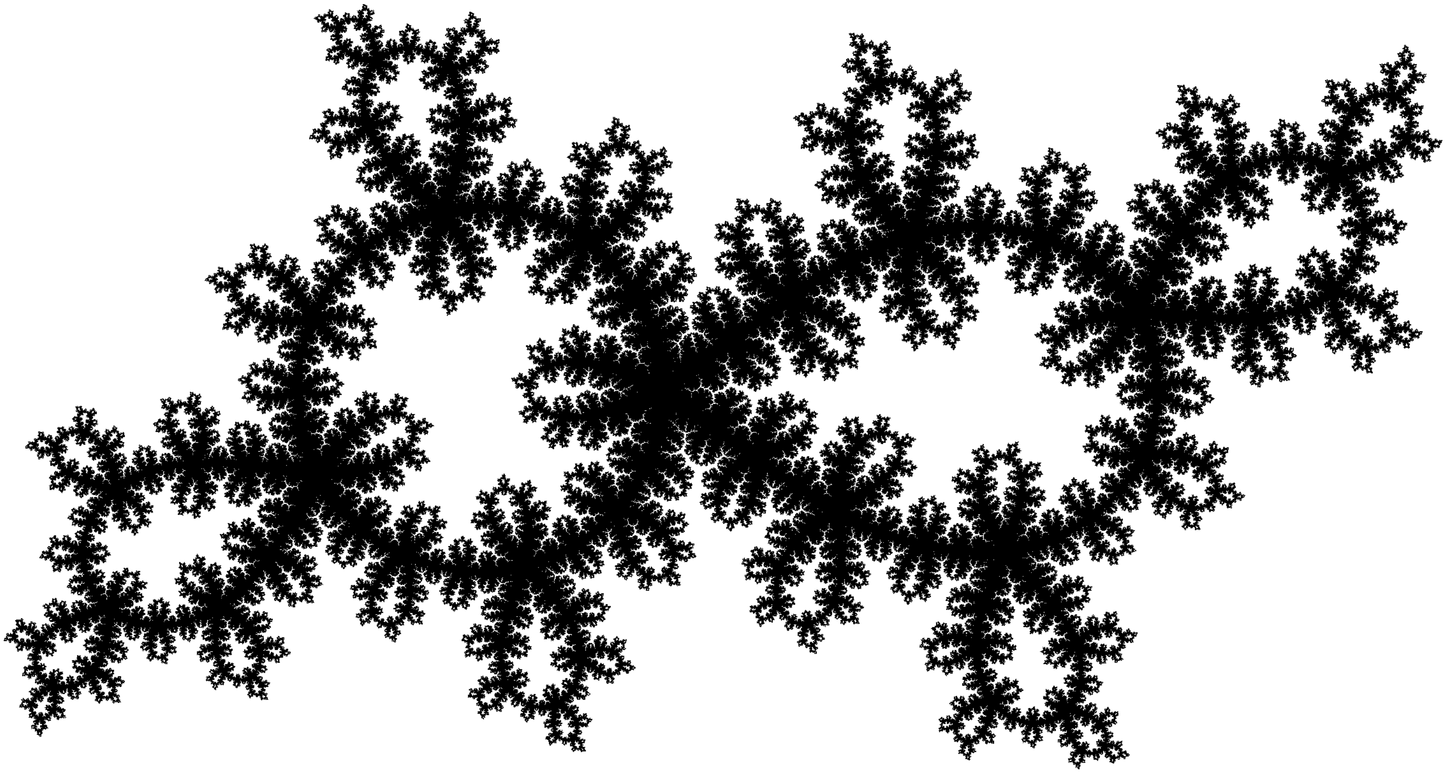
Siegel Renormalization



$$\theta' = \left\{ -\frac{1}{\theta} \right\}$$







McMullen (1990s):

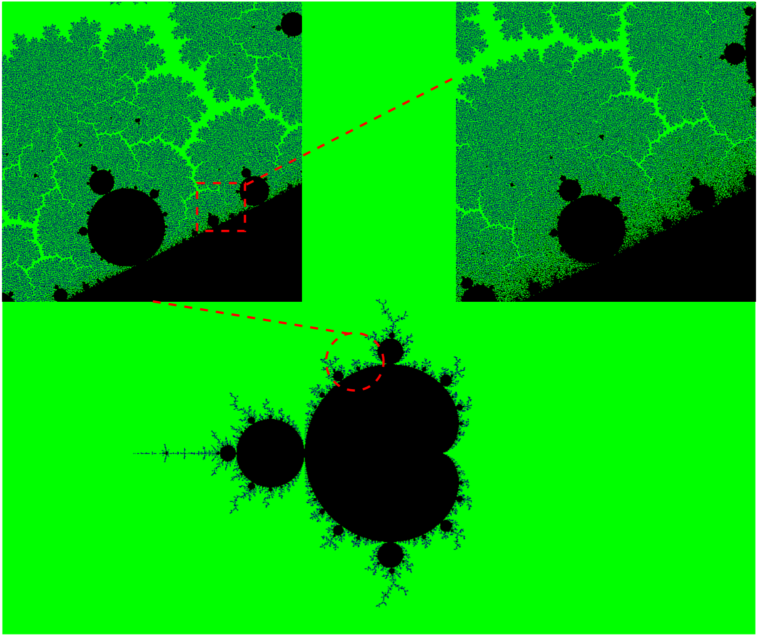
For any rot number Θ of stationary type, Siegel Renormalization has a fixed pt f_*

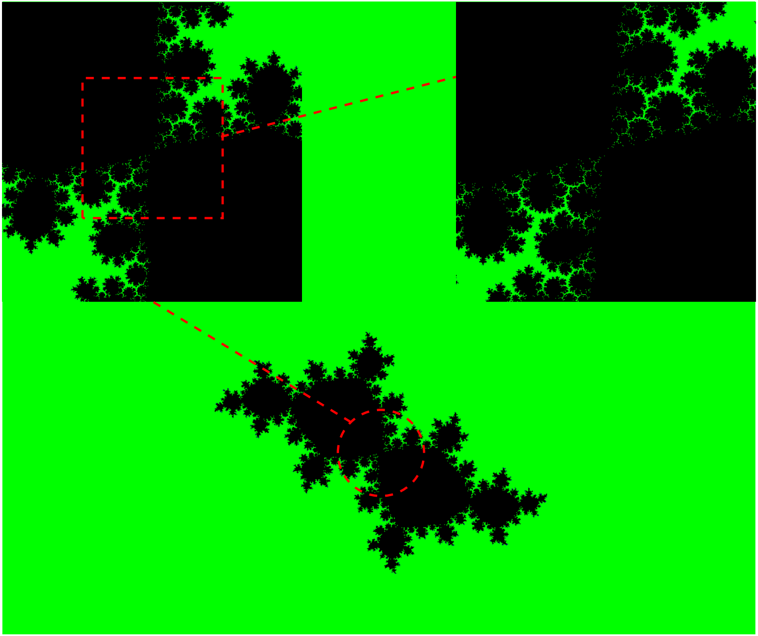
$$\Downarrow$$

For the corresponding quadratic polynomial $f: z \mapsto e^{2\pi i \Theta} z + z^2$, the Siegel disk is asymptotically self-similar at the crit pt c

---

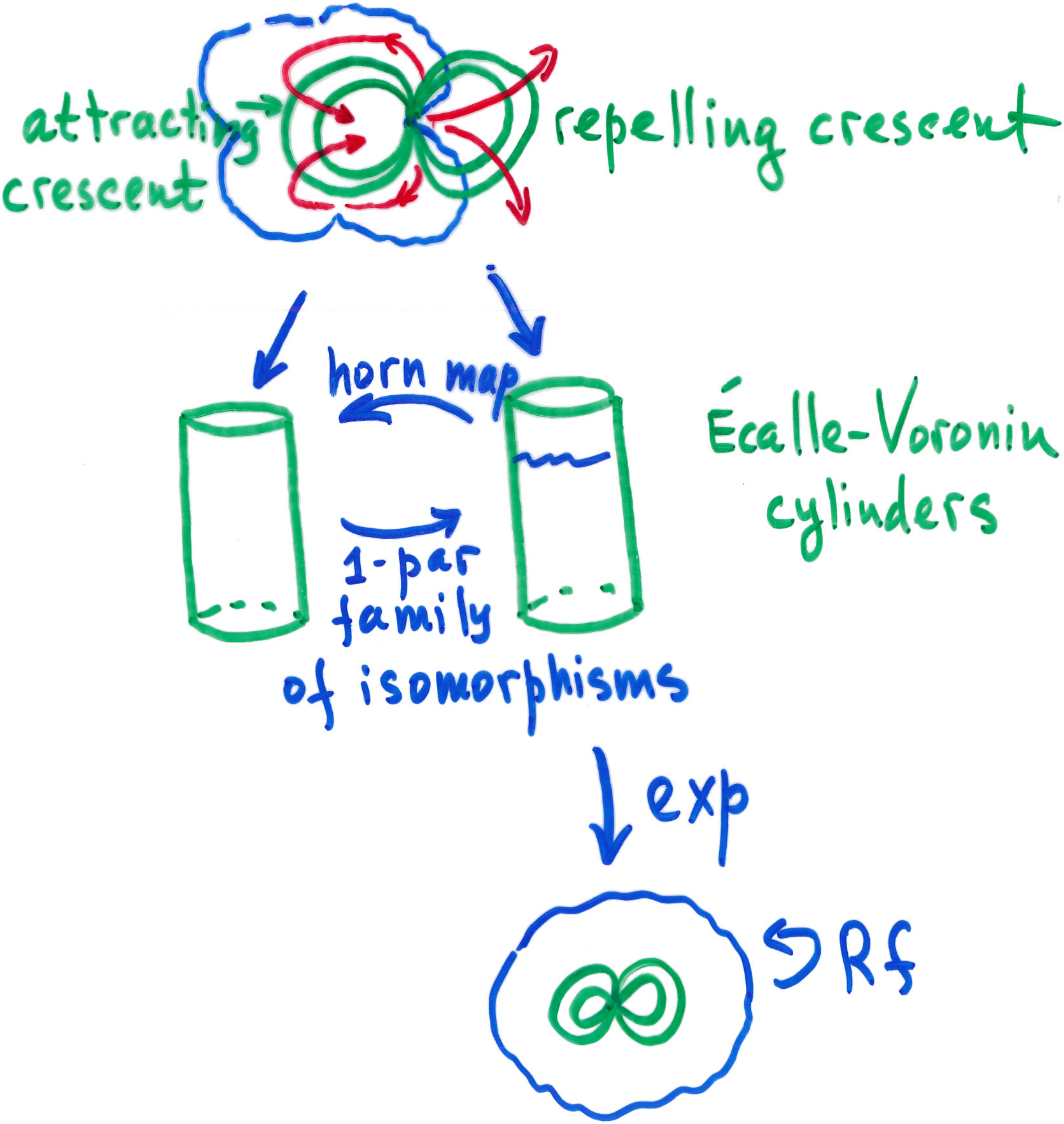
Important step of the argument: c is the Leb density pt for $K(f)$

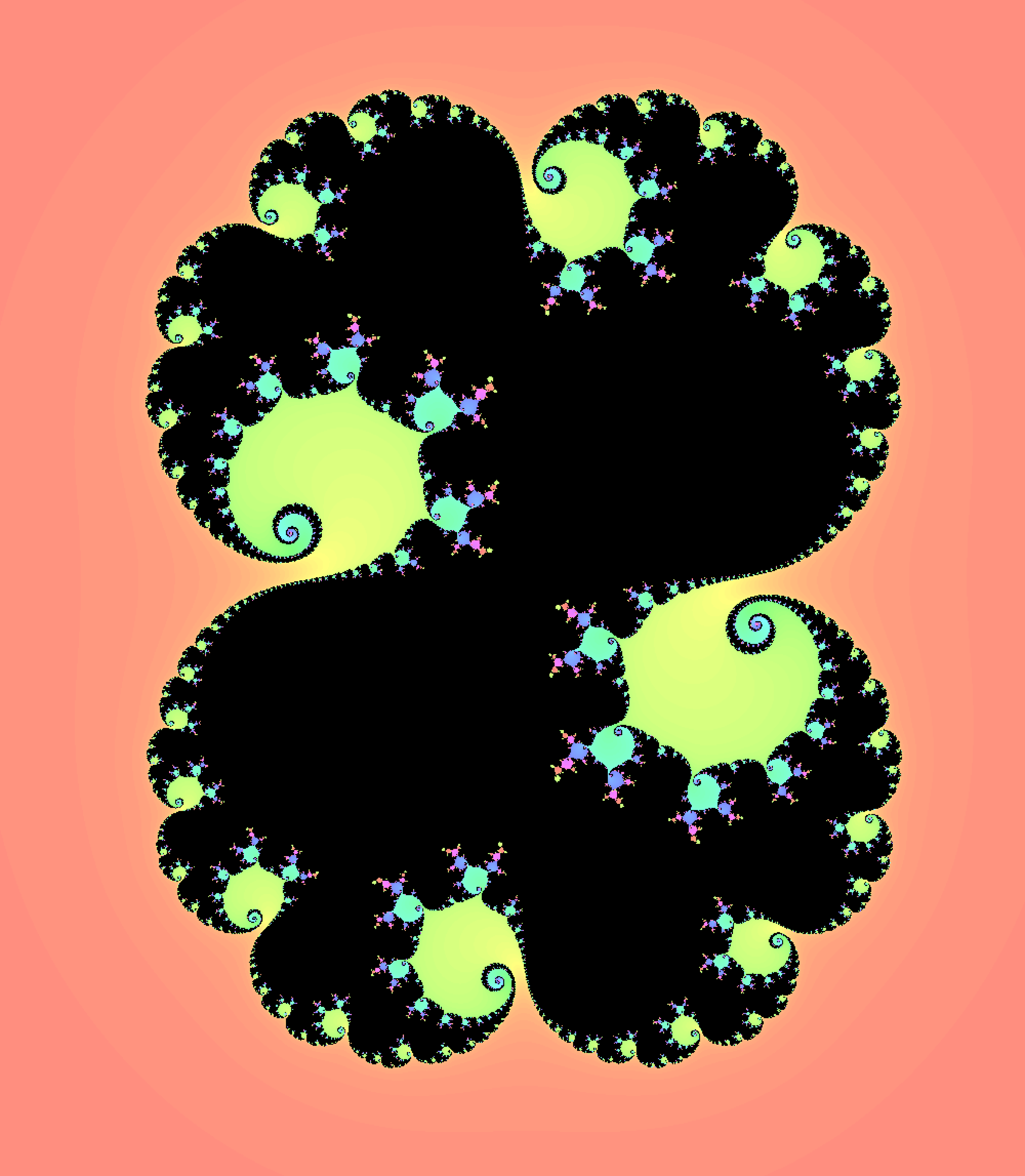


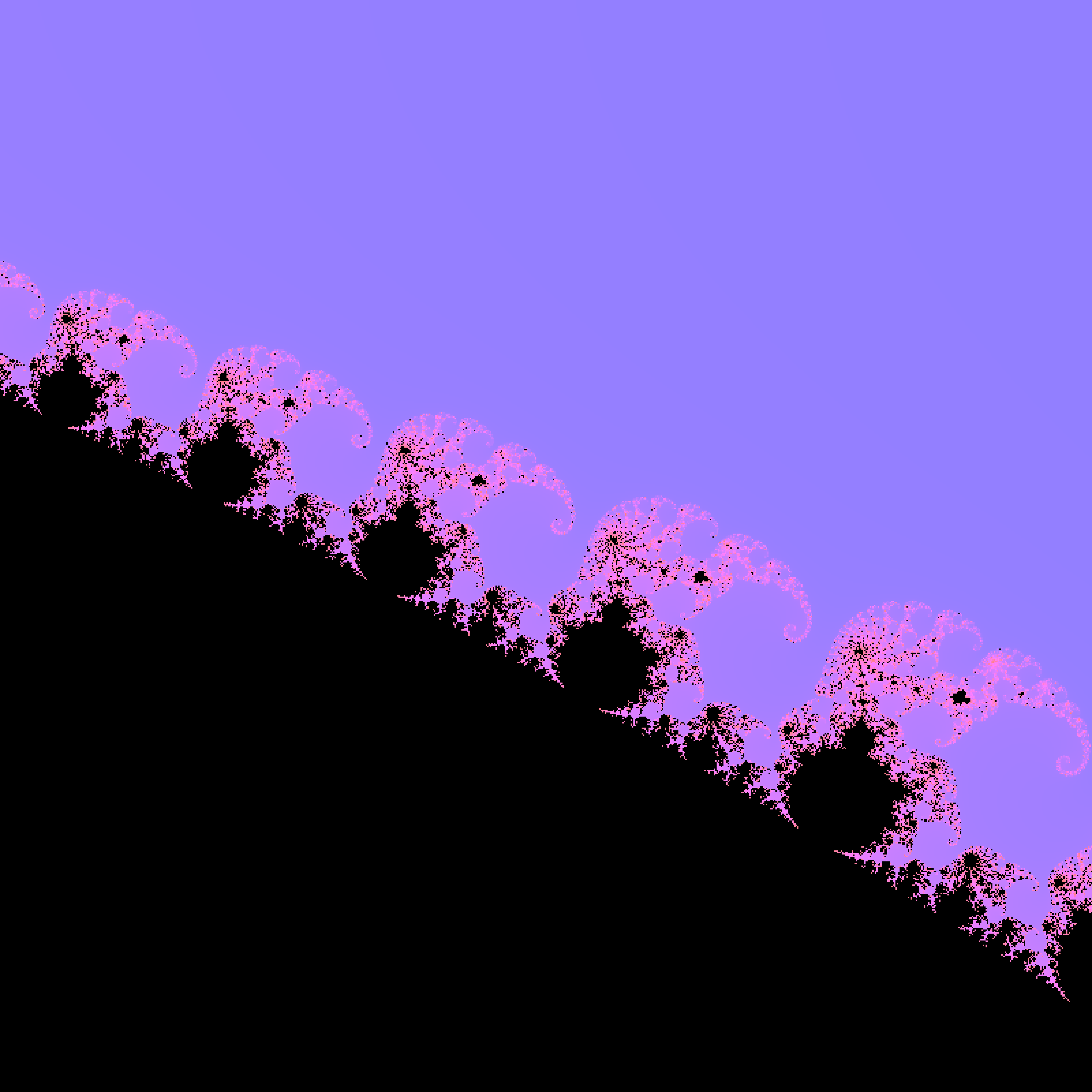


Parabolic Renormalization

$$f: z \mapsto z + z^2 + \dots$$

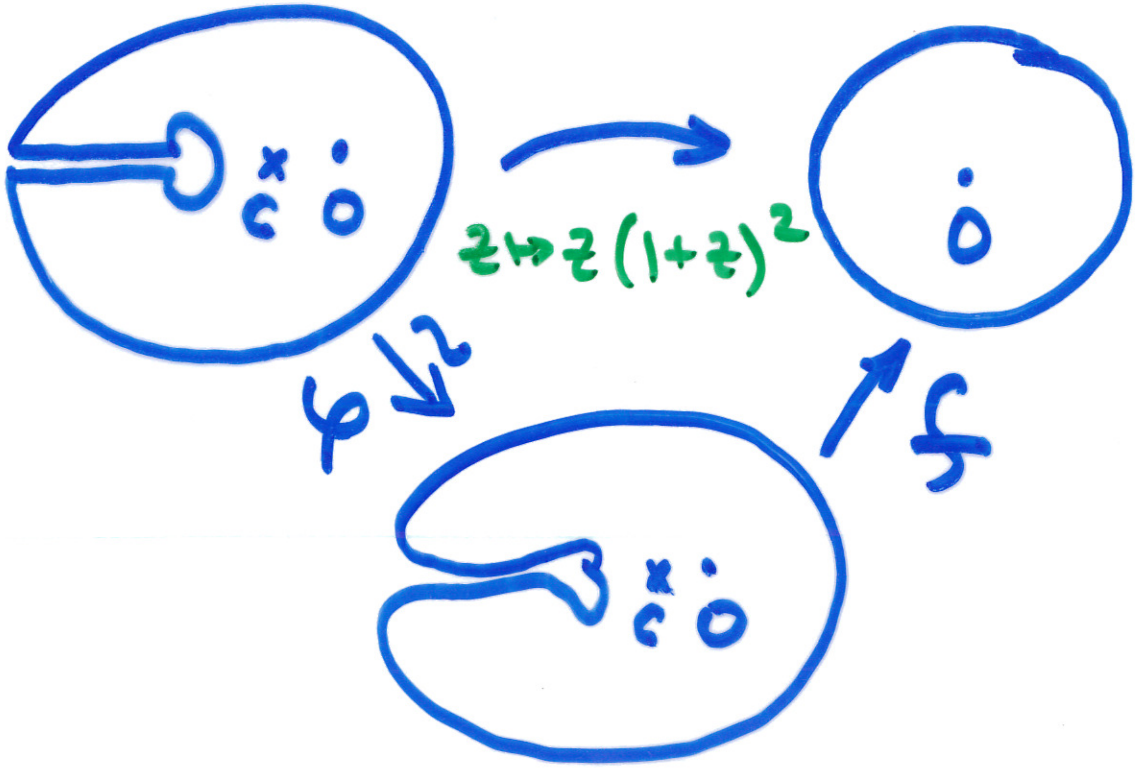






Inou-Shishikura class:

(2000s)

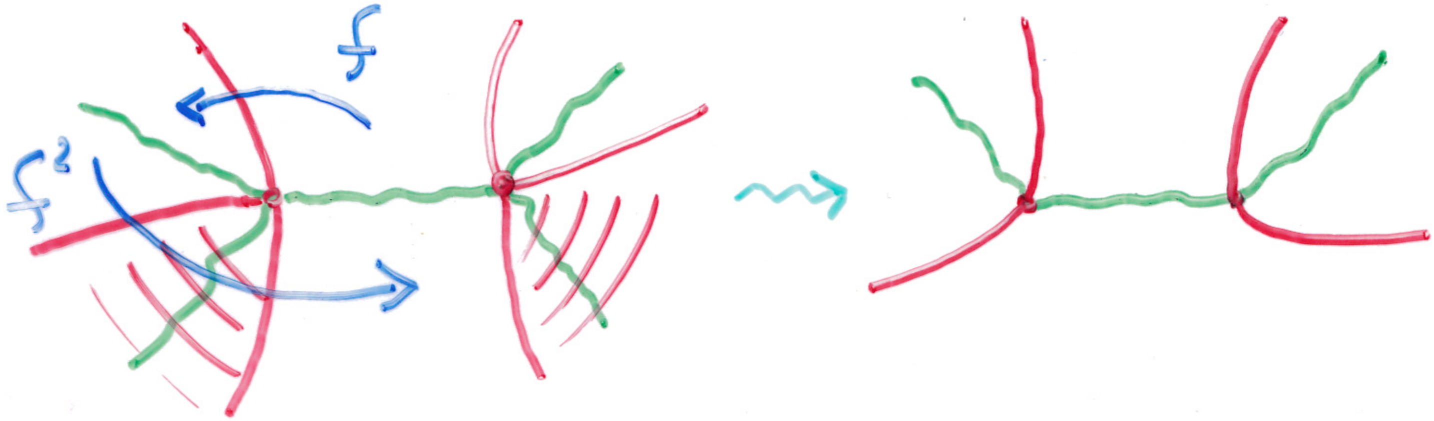


The IS class is invariant under the parabolic renormalization

$\Downarrow$

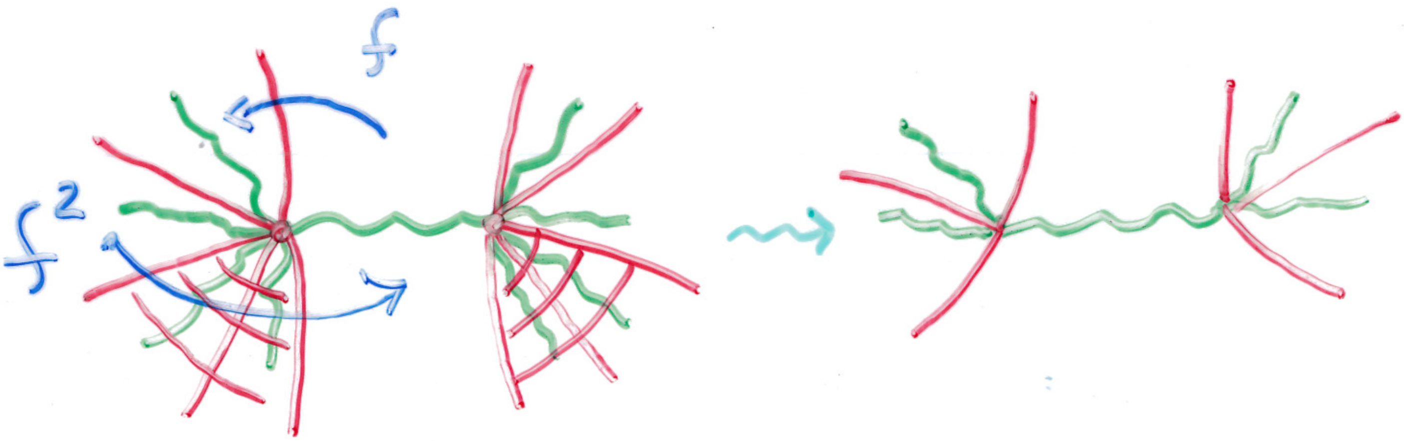
For a sufficiently big type $\Theta = [N N \ldots]$, the Siegel fixed point f_* is hyperbolic under Siegel Renormalization

Branner - Douady Surgery



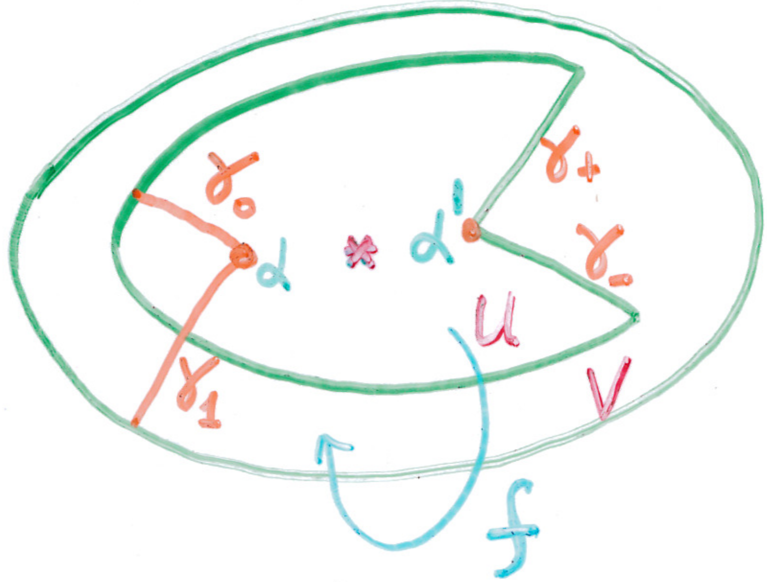
Part of $\mathcal{L}_{1/3} \xrightarrow{\simeq} \mathcal{L}_{1/2}$

More generally:



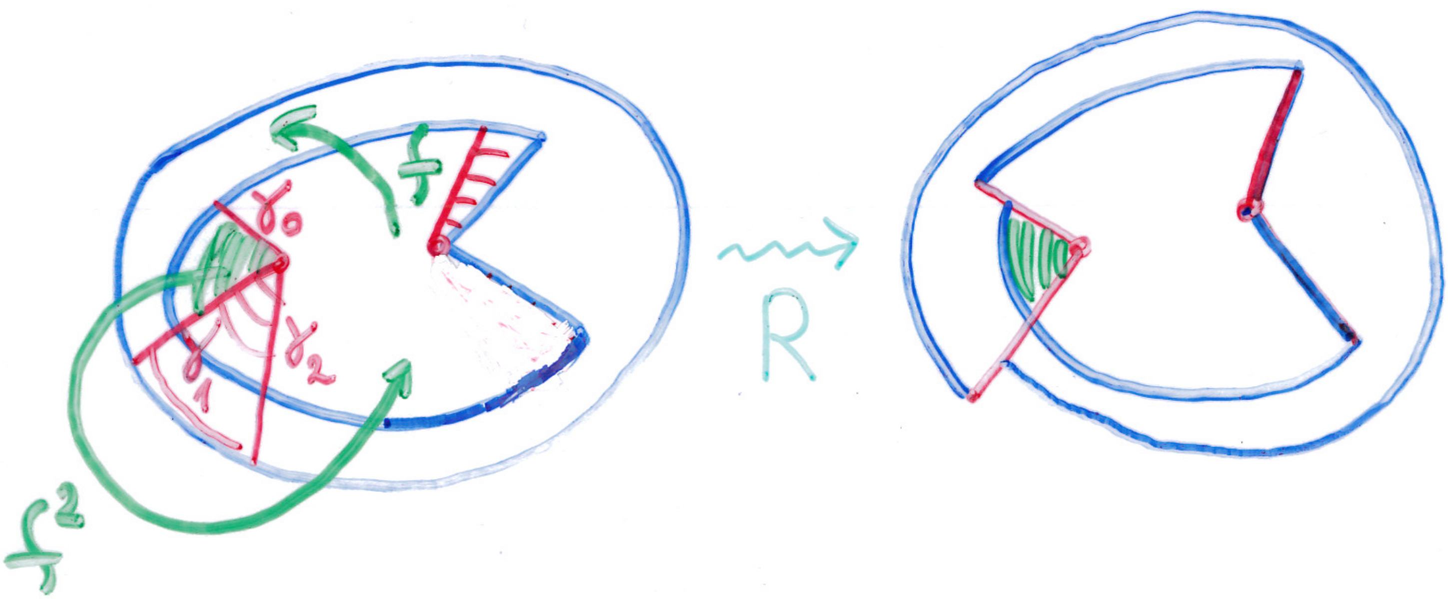
Part of $\mathcal{L}_{p/q} \xrightarrow{\simeq} \mathcal{L}_{p/q-p}$ $\left(0 < p/q < \frac{1}{2}\right)$

Pacmen



$f : U \setminus \gamma_0 \to V \setminus \gamma_1$ is 2-to-1 branched covering

Pacman Renormalization



Hyperbolicity Theorem
(Dudko – L – Selinger)

- The Pacman Renormalization can be locally realized as a holomorph op-r.
- For any periodic quadratic irrational θ, $\exists$ a renormalization per pt f_* which is a Siegel pacman with rotation number θ.
- f_* is hyperbolic.
- The unstable man-d W^u is 1D and is parametrized by rotation numbers near θ.
- The stable man-d W^s consists of Siegel pacmen with rot # θ; all of them are hybrid equivalent.

Strategy

Step 1: Promotion of the Siegel renorm fixed pt to a pacman f_*

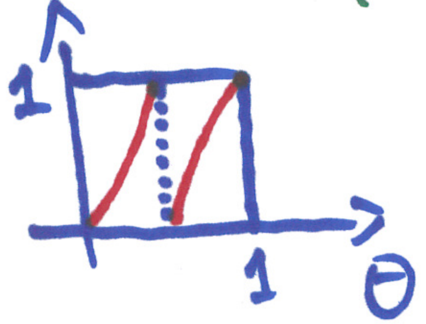
Step 2: $W^s(f_*)$ is a hybrid class

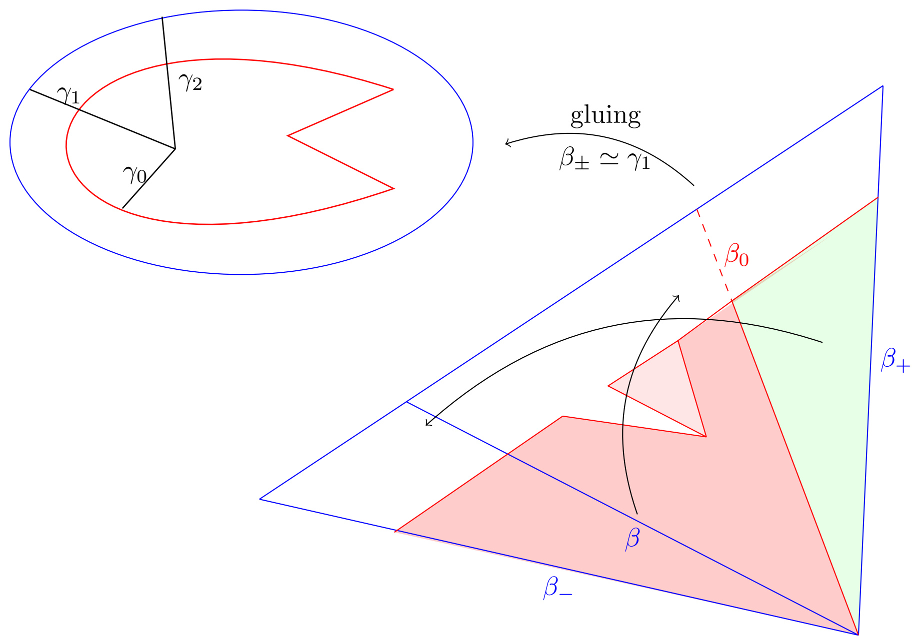
[Pullback Argument]

Step 3: $\dim W^u(f_*) \geq 1$

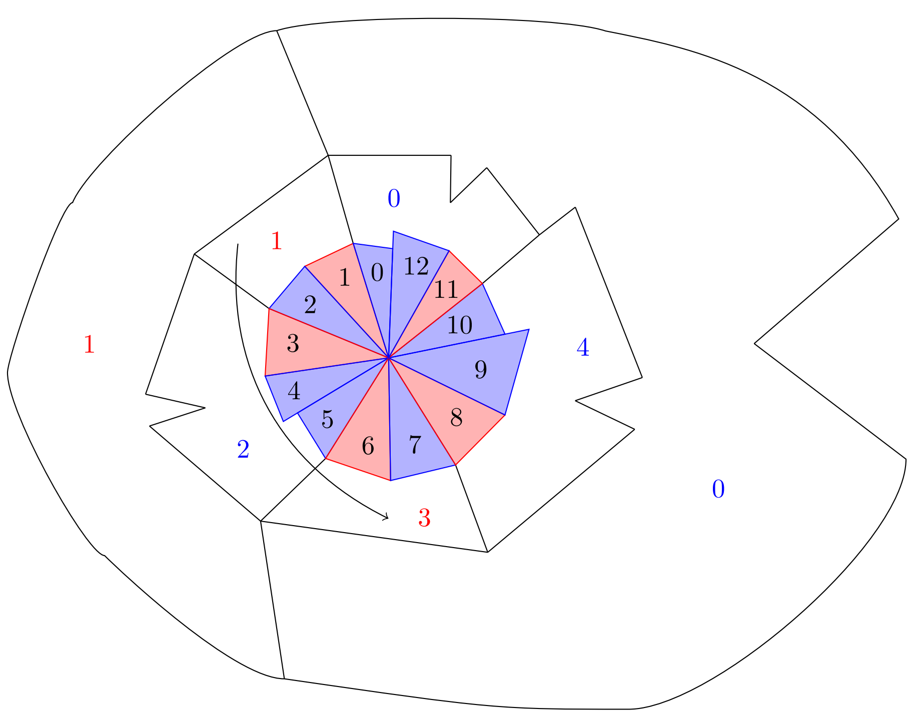
[expanding action on rot numbers]

$$\theta(Rf) = \begin{cases} \dfrac{\theta}{1-\theta}, & 0 \leq \theta \leq 1/2 \\[2ex] \dfrac{2\theta - 1}{\theta}, & \dfrac{1}{2} \leq \theta \leq 1 \end{cases}$$



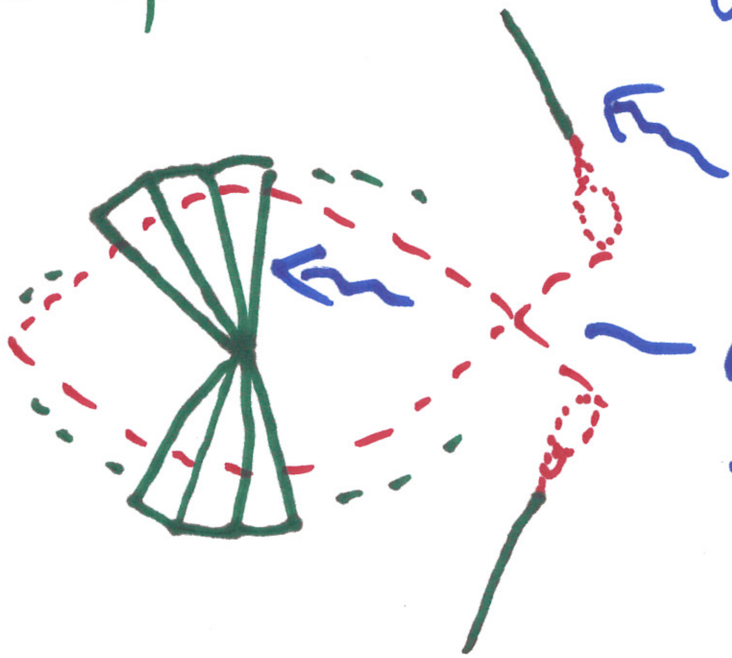


γ_1
γ_2
γ_0
gluing
$\beta_\pm \simeq \gamma_1$
β_0
β_+
β
β_-



0
1
1
0
12
11
10
9
8
7
6
5
4
3
2
1
4
2
3
0

Step 4: Robust geometric control of f near f_*.



Step 5: Maximal pre-pacmen.

For $f \in W^u(f_*)$, $\exists$ a σ-proper holomorphic extension $f: \mathbf{W} \to \mathbb{C}$.

Corollary. The critical orbit is captured in $\mathbf{W}$.

Step 6: $\dim W^u(f_*) = 1$

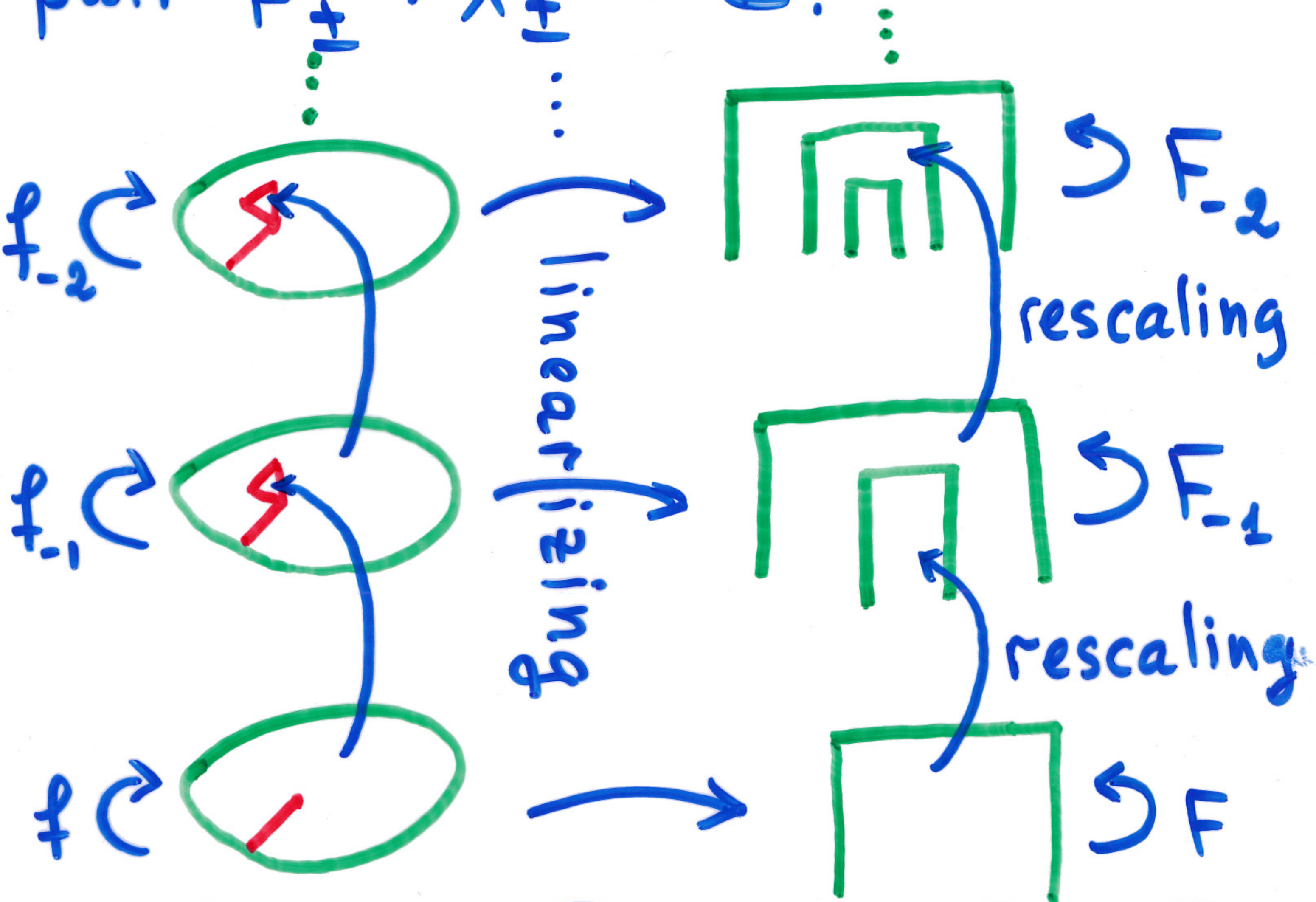
[λ-lemma argument]

Step 7: No neutral directions

[Small Orbits Lemma]

Global Unstable Manifold

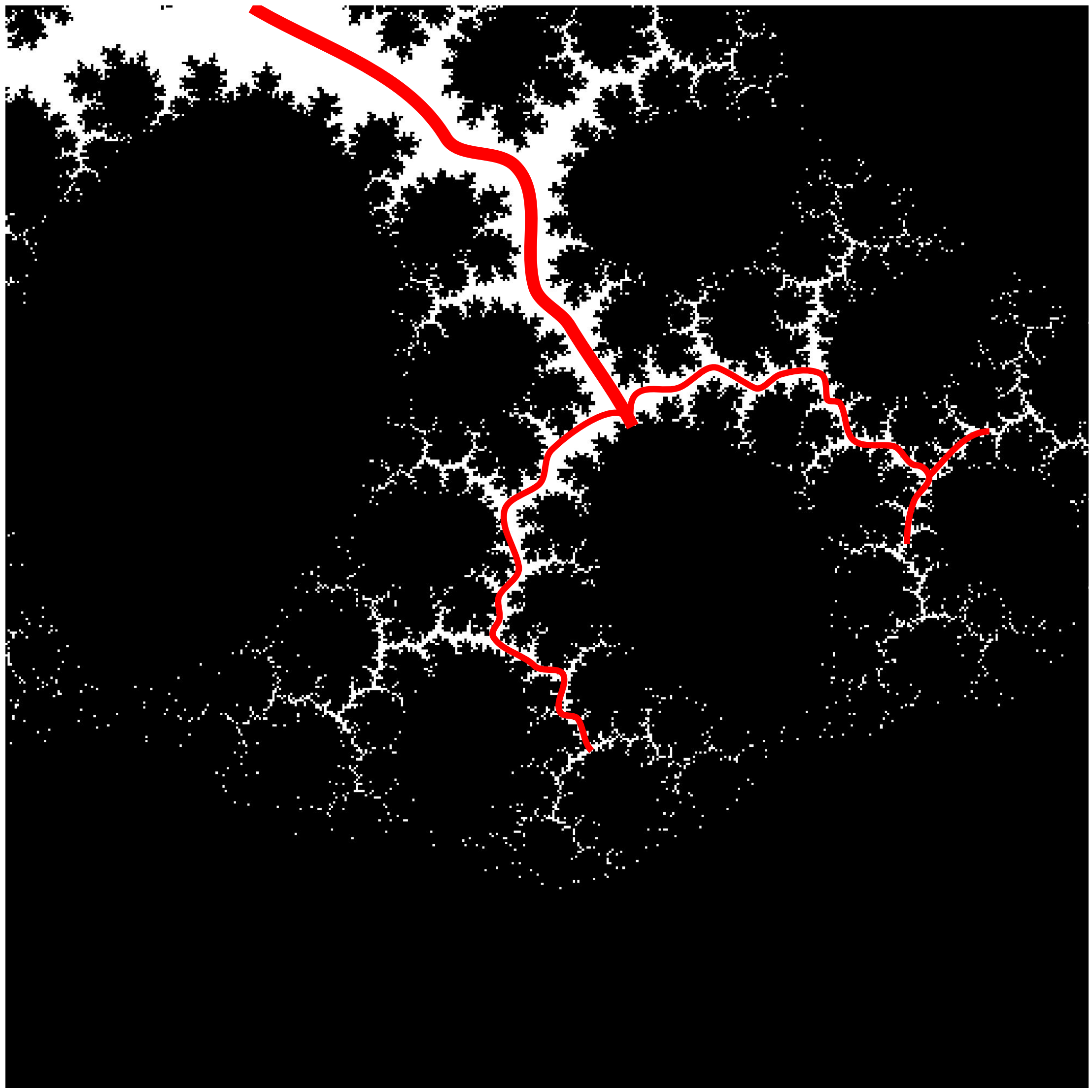
Any prepacman $f \in W^u_{loc}(f_*)$ extends to a maximal prepacman which is a σ-proper transcendental pair $F_\pm : X_\pm \to \mathbb{C}$.

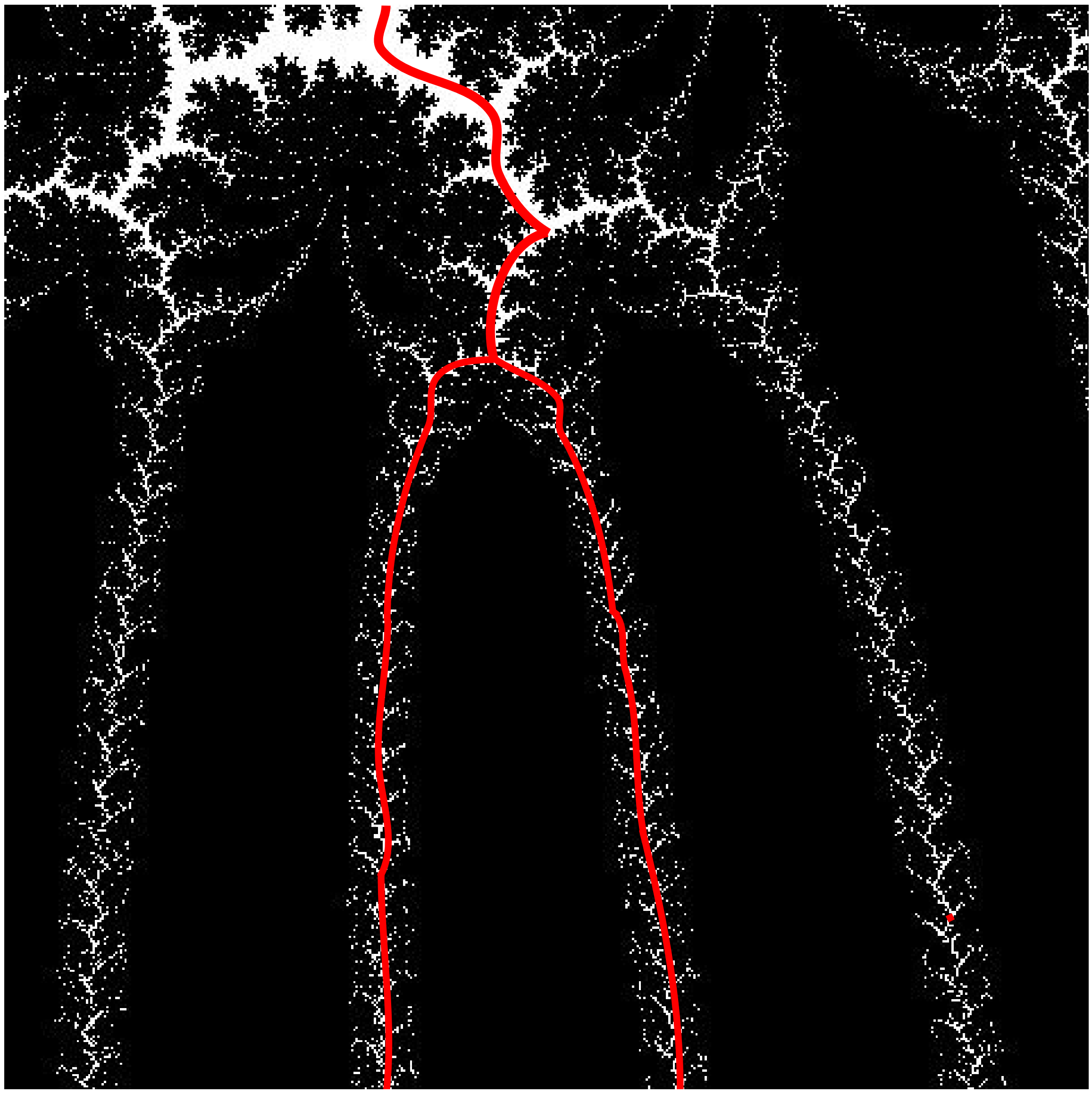


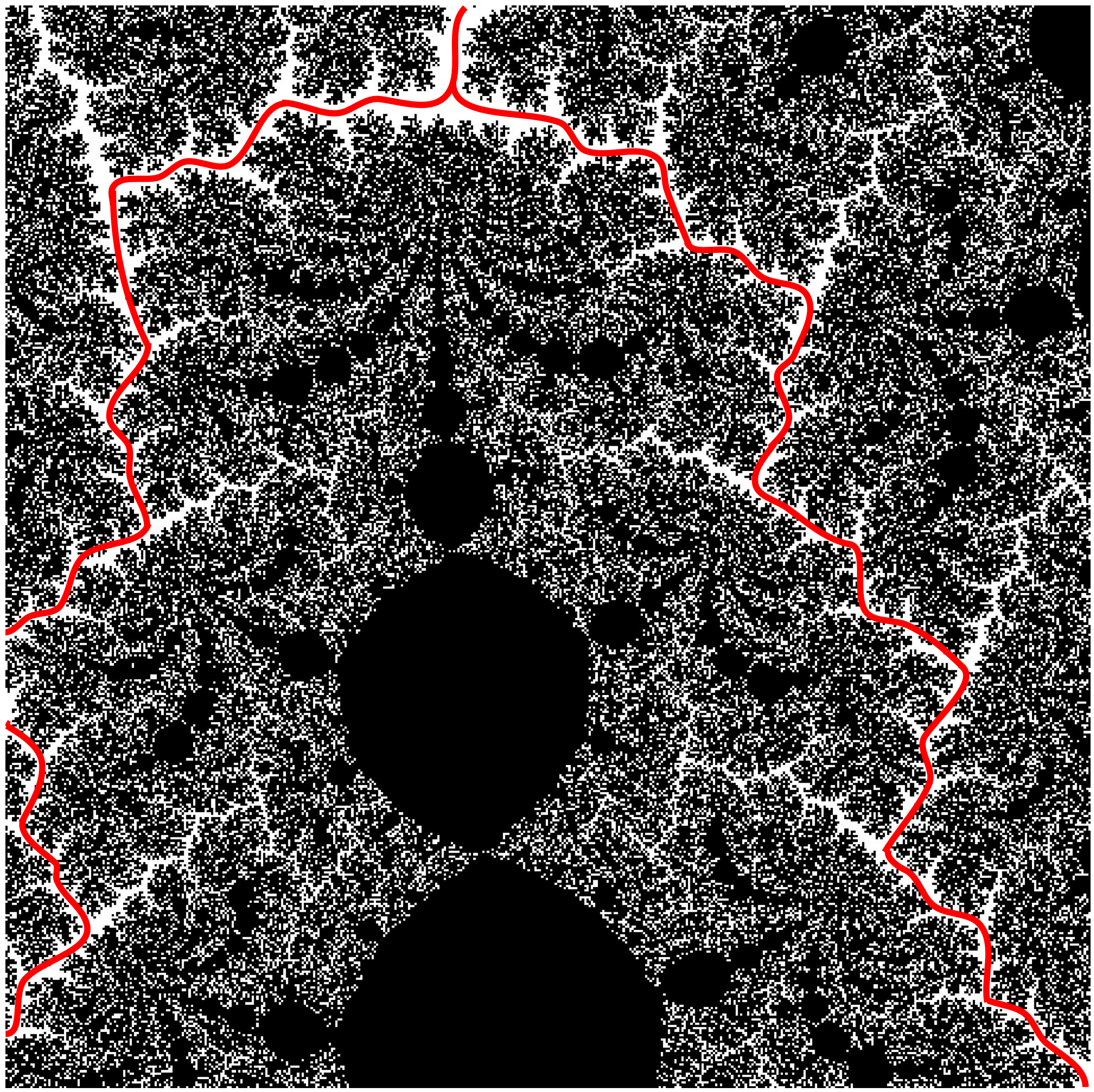
The local unstable manifold $W^u_{loc}(f_*)$ can be globalized to a family $W^u(f_*) \approx \mathbb{C}$ of transc-1 pairs (F_λ).

Structure of the maximal Siegel Prepacman F_*

- $\mathcal{K}(F) = \cup K(F_{-n})$ - "filled J-set"
  $\mathcal{E}(F)$ - escaping set
- $\mathcal{K}_*$ is the union of the Siegel disk Z_* and limbs attached to precritical pts
- Each limb is a bounded set with "tips" at pre-α pts
- $\mathcal{E}_*$ has a tree-like structure branched at pre-α pts
- "External rays" are embedded into $\mathcal{E}_*$

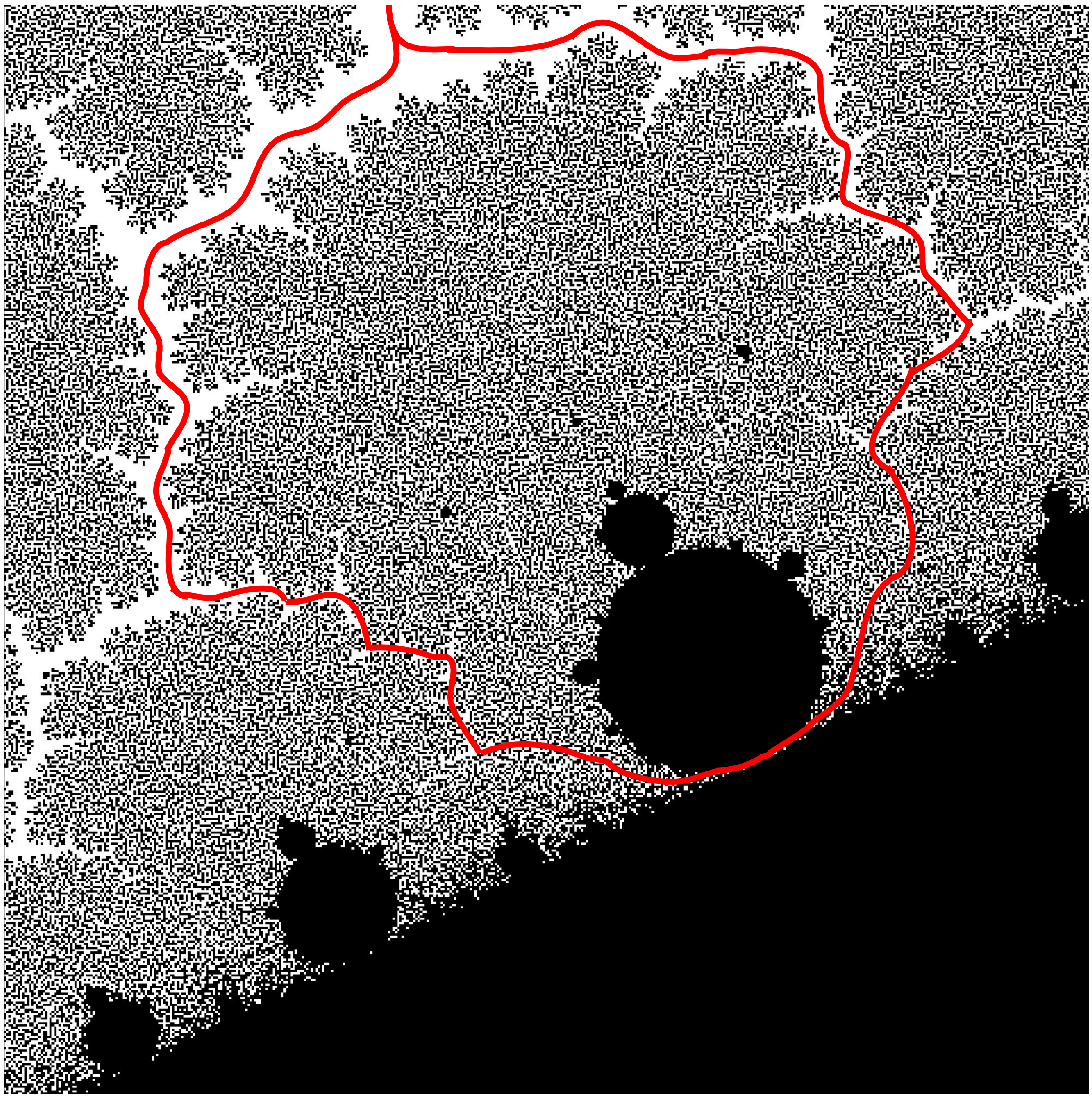


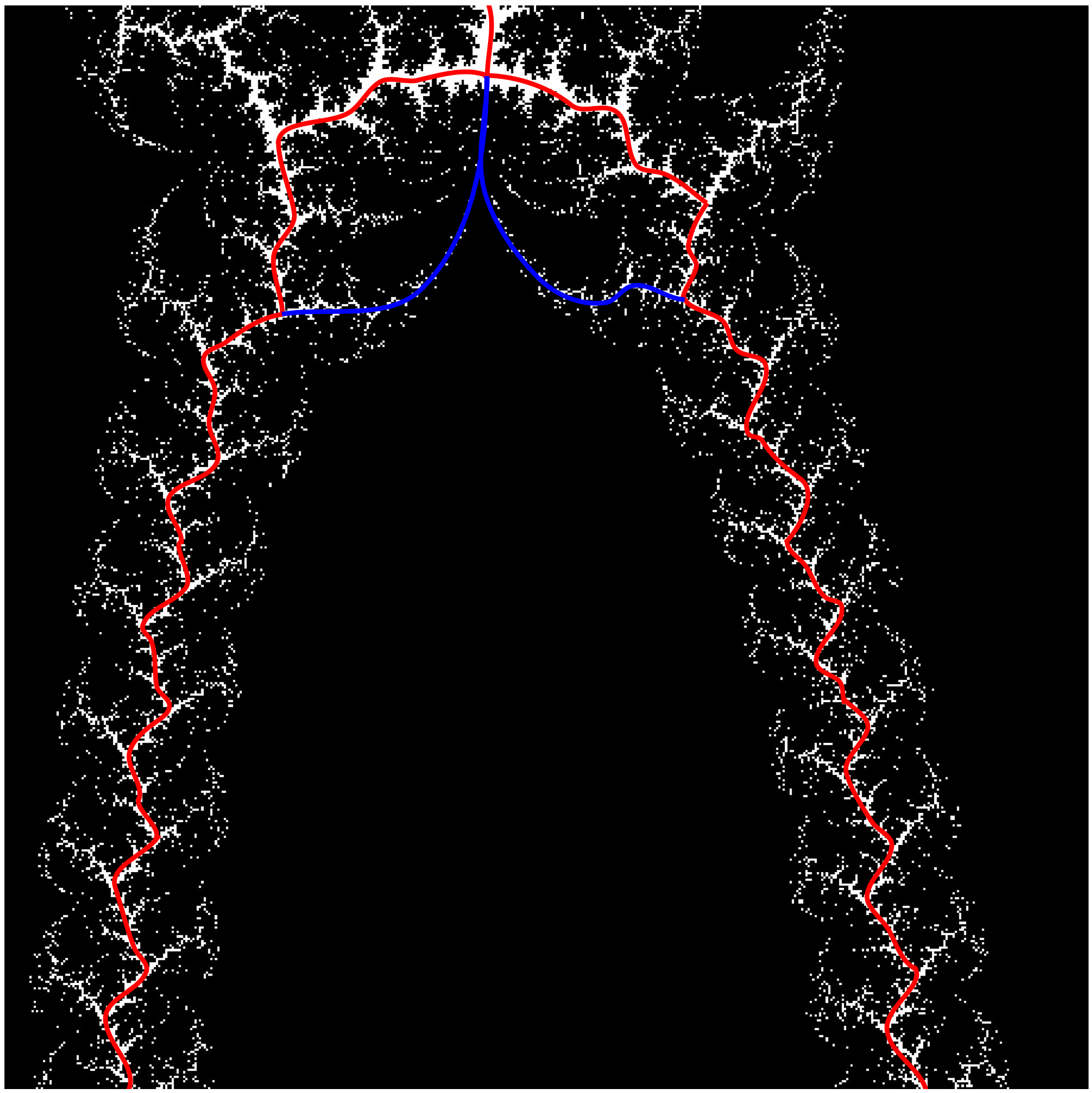




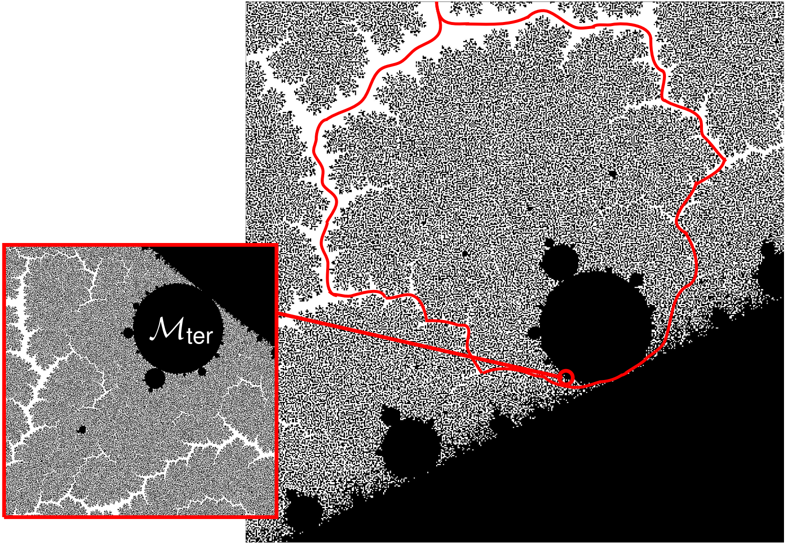
Structure of the Parameter Plane $W^u(F_*)$

- Escaping locus
  $\mathcal{E}_{par} = \{\lambda : v_\lambda \in \mathcal{E}(F_\lambda)\}$
- Phase-Parameter Relation
  $\mathcal{E}_{par} \xrightarrow{\sim} \mathcal{E}_*, \quad \lambda \mapsto h_\lambda^{-1}(v_\lambda)$
- "Parameter rays" correspond to the dynamical rays under PPR
- Wake $W_{p/q}$: a domain attached to the parab pt $\lambda_{p/q}$ and bounded by two external rays;
- $\exists$ a pinched q-1 family over $W_{p/q}$
- Satellite copy $\mathcal{M}_{p/q} \subset W_{p/q}$

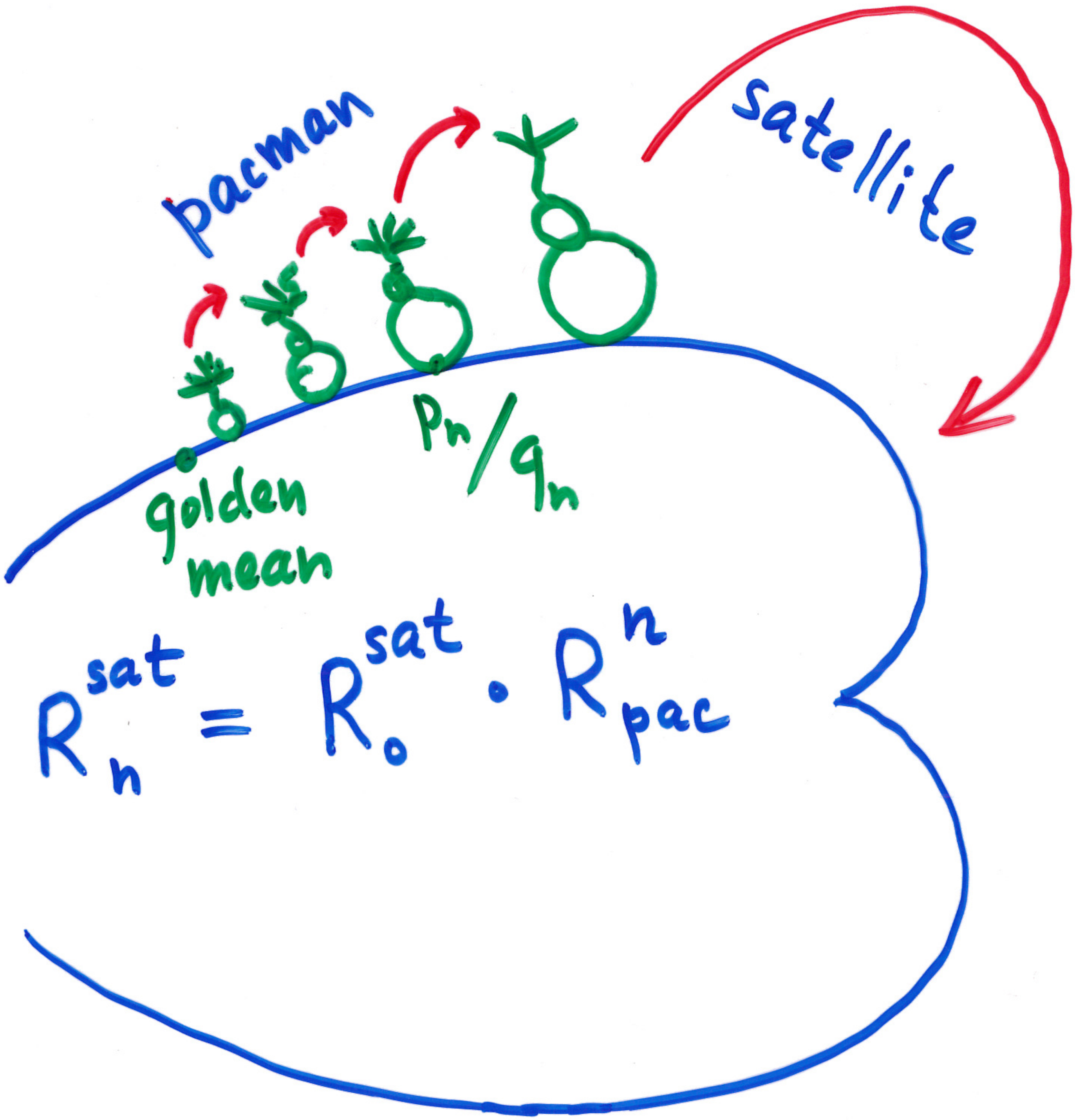




Thm (Dudko & L) There is a ternary copy $\mathcal{M}_{\text{ter}}$ such that MLC holds at ∞-renormalizable parameters with combinatorics $\mathcal{M}_{\text{ter}}$

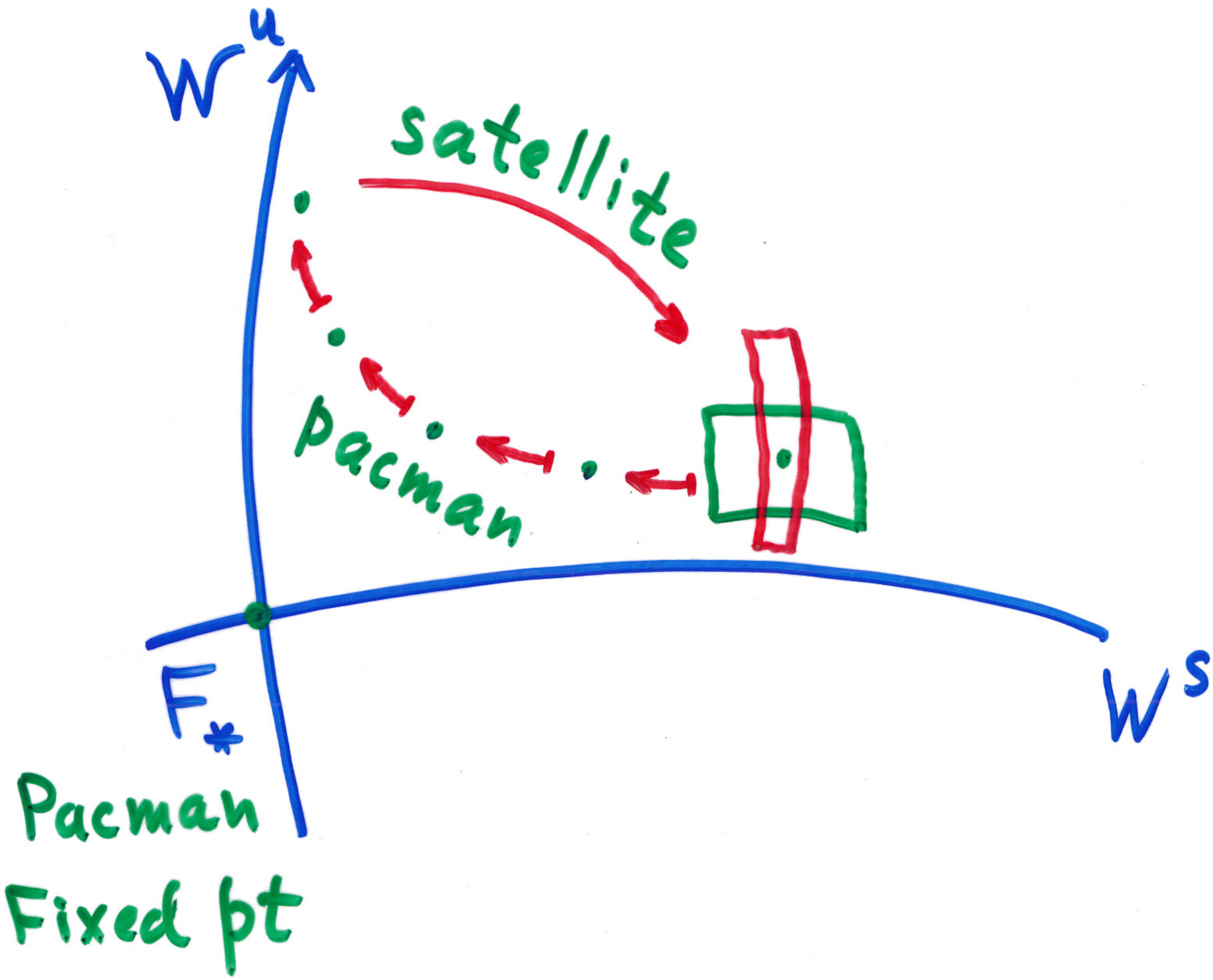


Factoring satellite renormalization through the pacman one

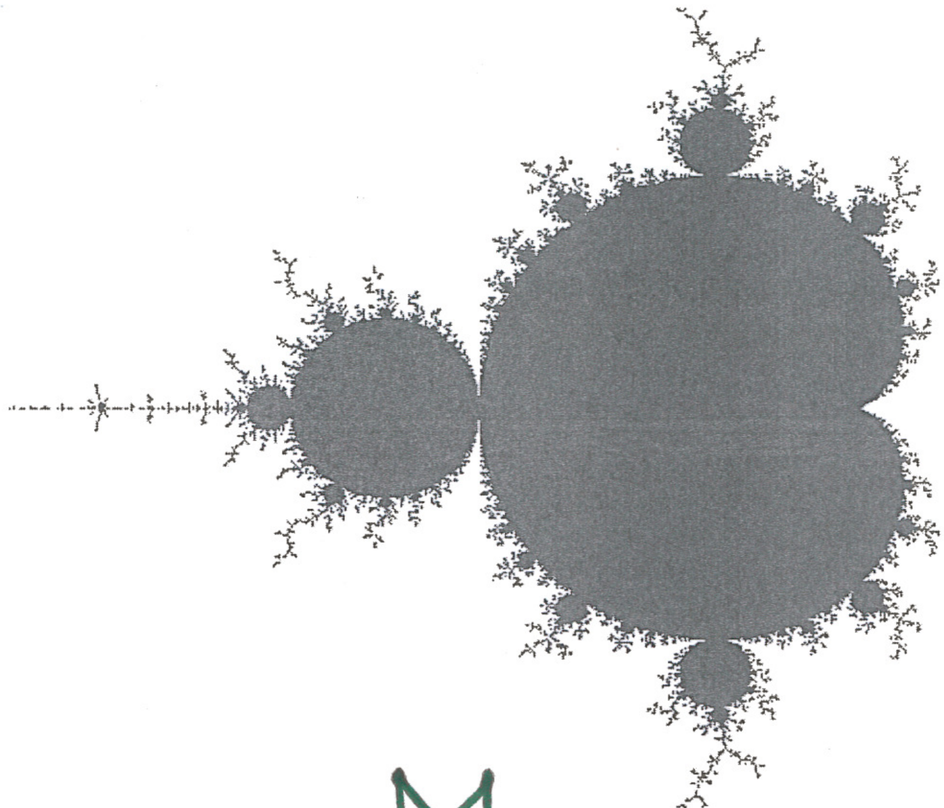


$$R_n^{sat} = R_0^{sat} \circ R_{pac}^n$$

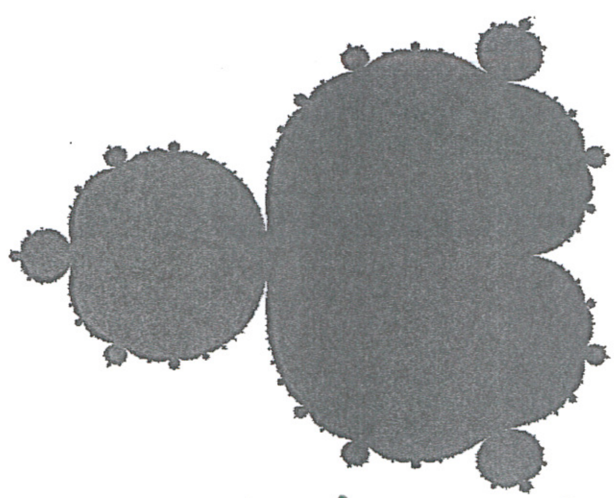
Renormalization Homoclinic Picture



Conjecture: Full Renormalization Horseshoe



M



Cubic model for Molecule

$z \mapsto z(z+1)^2$

Julia sets of positive area

$[\ J(f) = \partial K(f)\ ]$

Buff & Cheritat (2000s):

$\exists$ a Cremer map $f: z \mapsto e^{2\pi i \theta} z + z^2$

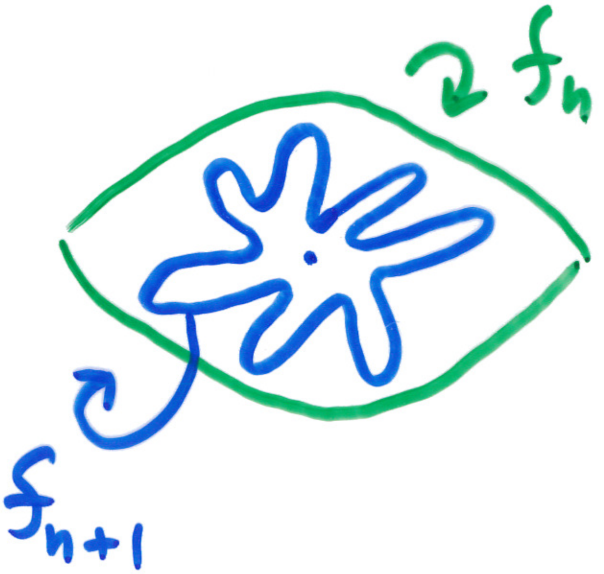
with area $J(f) > 0$

Strategy (Douady): Construct

Siegel maps $f_n: z \mapsto e^{2\pi i \theta_n} z + z^2$

s.t. area $K(f_{n+1}) \geq (1-\varepsilon_n)$ area $K(f_n)$

$\sum \varepsilon_n < \infty$, and $f_n \to f$



area $S_{n+1} \geq$

$\frac{1}{2}$ area S_n

New examples

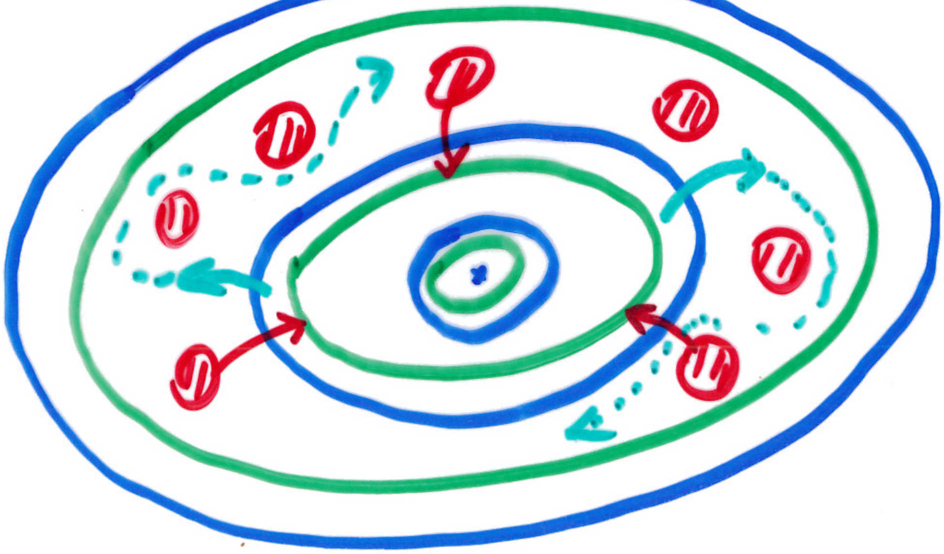
Avila–L: $\exists$ Feigenbaum Julia sets of positive area ("Black Holes")

Some new features:

- Tameness: admit a top model
- Parameter visibility:

  HD (parameter set) $\geq 1/2$

Random walk argument:



Prob of landing $\gg$ Prob of escape

Construction