Renormalisation of asymmetric interval maps

Kozlovski & van Strien

March 24, 2019

There is an increasing interest in understanding families of maps of the form $f_c \colon \mathbb{R} \to \mathbb{R}$, defined by

$$f_c(x) = \begin{cases} |x|^\alpha + c & \text{when } x < 0, \\ x^\beta + c & \text{when } x \geq 0 \end{cases} \tag{1}$$

where $\beta \geq \alpha \geq 1$ and their generalisations.

In the symmetric case when $\alpha = \beta = 2$ this corresponds to the family $f_c(x) = x^2 + c$.

Aim talk: to discuss the first results about this setting.

Summary of results

Partial results on:

- Period doubling,
- Renormalisation,
- Absence of wandering intervals.

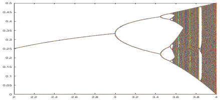
Alternative prototype family:

$$f_t(x) = \begin{cases} t - 1 - t|x|^\alpha & \text{when } x < 0, \\ t - 1 - tx^\beta & \text{when } x \geq 0 \end{cases} \tag{2}$$

Period doubling in the quadratic case

Consider the family $f_a(x) = ax(1-x)$, $x \in [0,1]$ and $a \in [0,4]$.

- For $a = 2$ it has a fixed point which attracts all points in $(0,1)$
- for $a = 4$ it contains a one-sided shift of two symbols.



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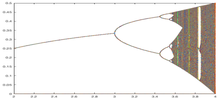

Numerical observation: Feigenbaum & Coullet-Tresser

1. Period doubling occurs as increasing parameters $a_2 = 3$, $a_4 = 3.4494897428$, $a_8 = 3.5440903596$, $a_{16} = 3.5644072661$, $a_{32} = 3.5687594195$, $a_{64} = 3.5696916098$, $a_\infty = 3.5699456$.
2. rate of converence: $(a_{2^{n-1}} - a_{2^{n-2}})/(a_{2^n} - a_{2^{n-1}}) \to 4.669201609\ldots$

I: Monotonicity of bifurcations

Theorem (Sullivan, Thurston, Milnor, Douady, Tsujii, ... (1980's))

As a increases, periodic points appear and never disappear.

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- Douady's approach is based on the fact that hyperbolic components of the Mandelbrot can be parameterised by multipliers and combinatorics of certain rays.
- Tsujii's approach considers some transfer operator.

All proofs are somewhat related and rely on complex tools and only work when $\alpha = \beta$ is an even integer.

I: Tsujii's approach for proving monotonicity

Assume that f_{c_*} has 0 as a periodic point of (minimal) period q.

- **Prove "Positive" transversality:**

$$\frac{\frac{d}{dc} f_c^q(0) \mid_{c=c_*}}{Df_{c_*}^{q-1}(f_{c_*}(0))} = \sum_{n=0}^{q-1} \frac{1}{Df_{c_*}^i(f_{c_*}(0))} > 0. \qquad (3)$$

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- By the pos. transversality inequality (3)

$$\begin{array}{ll} \frac{d}{dc} f_c^q(0)\Big|_{c=c_*} < 0 & \text{if } f_{c_*}^q \text{ has a local maximum at 0,} \\ \frac{d}{dc} f_c^q(0)\Big|_{c=c_*} > 0 & \text{if } f_{c_*}^q \text{ has a local minimum at 0.} \end{array}$$

- $\implies$ (using real arguments) periodic orbits cannot be reborn.

I: Tsujii's vs Douady-Hubbard approach

Compare with Douad-Hubbard approach:

- Douady-Hubbard: $c \mapsto \lambda(c)$ is univalent in each hyperbolic component of the family of quadratic maps.
- Tsujii's approach $\implies$ $c \mapsto \lambda(c)$ is increasing.

As mentioned, all those approaches require $\alpha = \beta$ to be an even integer.

How to overcome this?

I: Monotonicity (with Levin and Shen)

With Genadi Levin and Weixiao Shen we use a transfer operator approach to show monotonicity for many families.

- For example, for many families of the form $f_c(x) = f(x) + c$ and $f_\lambda(x) = \lambda f(x)$; f does not need to be of finite type.
- Assume
  - f_{c_0} has a critical relation and
  - f_{c_0} has a polynomial-like extension $f\colon U \to V$ and
  - some other mild assumptions.

  Then our **Main Theorem** states:

  Some lifting propery holds $\Longrightarrow$ *either* critical relation persists *or* positive transversality.
- The above result holds for complex families.
- Also results for transversal unfolding of parabolic periodic points, see arXiv preprint Jan 2019.

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Theorem (with Levin, Shen)

Let $\ell_-, \ell_+ > 1$ and consider the family of unimodal maps

$$f_c(x) = \begin{cases} |x|^{\ell_-} + c & \text{if } x \leq 0 \\ |x|^{\ell_+} + c & \text{if } x \geq 0. \end{cases}$$

$\forall L \geq 1$ $\exists \ell_0 > 1$ *so that if* $\mathbf{i} = i_1 i_2 \cdots \in \{-1, 0, 1\}^{\mathbb{Z}^+}$ *is a* q *periodic kneading sequence (*q *arbitrary) with*

$$\#\{1 \leq j < q; i_j = -1\} \leq L,$$

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$$\#\{1 \le j < q; i_j = -1\} \le L,$$

then $\forall \ell_-, \ell_+ \ge \ell_0$ there is **at most** *one $c_* \in \mathbb{R}$ for which the kneading sequence of f_c is equal to $\mathbf{i}$.*
In fact, one has positive transversality at c_*.

II: Is there even period doubling?

So we **do not know**, when $\beta > \alpha \geq 1$ or when $\alpha = \beta \notin 2\mathbb{N}$, whether the family $f_t\colon [-1,1] \to [-1,1]$, $t \in [1,2]$ defined by

$$f_t(x) = \begin{cases} t - 1 - t|x|^\alpha & \text{when } x < 0, \\ t - 1 - tx^\beta & \text{when } x \geq 0 \end{cases} \tag{4}$$

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is 'monotone'. However, at least the family is full:

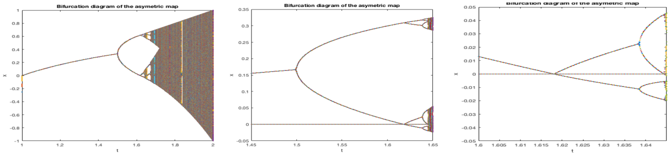
Theorem

$\exists$ $t_2 < t_4 < t_8 < \cdots < t_{2^n} < t_\infty$ and $\epsilon_n > 0$ so that for

- *$t \in (t_{2^n} - \epsilon_n, t_{2^n})$, f_t has only periodic orbits of periods $\leq 2^n$*
- *$t \in (t_{2^n}, t_{2^n} + \epsilon_n)$, f_t also has a periodic orbit of period 2^{n+1}.*

Theorem

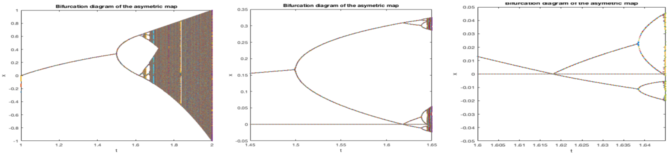
When $\alpha = 1$ and n is even, then period doubling from period 2^n to period 2^{n+1} takes place when $f^{2^n}(0) = 0$ rather than when multiplier at periodic attractor -1.



Theorem

There exists t_∞ so that f_{t_∞} has a periodic orbits of period 2^n for each n and no other periodic orbit.

II. Existence of period doubling limit



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- From the numerics (and also from the results below), it seems that the scaling of period doubling is quite different when $\alpha < \beta$ than in the quadratic case.
- $\nexists$ Feigenbaum-Coullet-Tresser-Sullivan-McMullen-Lyubich-Avila-Lyubich renormalisation theory
- $\nexists$ proofs based on rigorous numerical estimates.

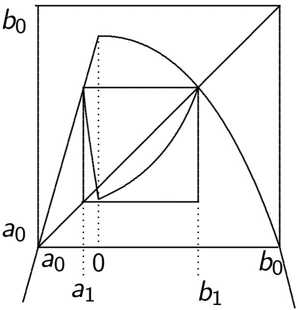

Figure: f together with it renormalisation and its semi-extension.

III. Results for the Feigenbaum map f_{t_∞}.

From now on we concentrate on $f := f_{t_\infty}$ in the case $\alpha = 1$.

- Then there exists a nested sequence $[a_k, b_k] \ni 0$, $k = 0, 1, \ldots$ so that f^{2^k} is a unimodal map from $[a_k, b_k]$ into itself.
- If we had $\alpha = \beta$ then

  $$|a_k| = b_k \sim \delta^{-n} \downarrow 0$$

  where

  $$\delta = 2.502907875095892822283902873218...$$

  (which is equal to an eigenvalue of the associated periodic doubling renormalisation operator).
- What happens when $1 = \alpha < \beta$?

III. Superexponential scaling of b_k when $1 = \alpha < \beta$

Notation: Assume $u_k, v_k > 0$, $u_k, v_k \to 0$. We write

$$\begin{aligned} u_k \sim v_k \quad &\iff u_k/v_k \to 1 \\ u_k \approx v_k \quad &\iff 0 < \liminf u_k/v_k \leq \limsup u_k/v_k < \infty. \end{aligned}$$

As before assume

$$f(x) - f(0) \sim \begin{cases} -K_-|x| & \text{for } x < 0 \\ -K_+ x^\beta & \text{for } x > 0 \end{cases}$$

and let

$$K = K_+/K_-.$$

III. Superexponential scaling of b_k when $1 = \alpha < \beta$

Theorem (Scaling laws)

The following scaling properties hold for b_k:

- *For large even values of k one has*

$$\begin{aligned} b_{k+1} &\sim \lambda b_k, \\ c_{2^k} &\sim b_k, \end{aligned} \tag{5}$$

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$$\begin{aligned} b_{k+1} &\sim \beta^{\frac{-2}{\beta-1}} K_0^{\frac{1}{\beta-1}} \lambda^{-2} b_k^2 \\ c_{2^k} &\sim -\beta^{-\frac{\beta+1}{\beta-1}} K_0^{\frac{\beta}{\beta-1}} \lambda^{-\beta-1} b_k^{\beta+1} \end{aligned} \tag{6}$$

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In particular, $\exists\ C > 0$ and $\mu \in (0,1)$ so that

$$|b_k - a_k| < C\mu^{k^{\sqrt{2}}}, k \geq 0.$$

Theorem (Renormalization limits of R^k)

For k even we have

$$f^{2^k}(x) = \begin{cases} c_{2^k} - s_k|x| + O(b_k^{\frac{3}{2}}) & \text{when } x \in [a_k, 0] \\ c_{2^k} - t_k x^{\beta} + O(b_k^{\frac{3}{2}}) & \text{when } x \in [0, b_k] \end{cases} \tag{7}$$

where

$$s_k \sim \frac{b_k^{1-\beta}}{K_0} \text{ and } t_k \sim b_k^{1-\beta}. \tag{8}$$

V. Rigidity

In fact $\exists \Theta > 0$ s.t. $1/b_{2k} \sim \beta^{\frac{-2}{\beta-1}} K_0^{\frac{1}{\beta-1}} \exp(2^k \Theta + o(1))$.

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Theorem (Complete invariants for C^1 universality)

Take two maps $f, \tilde{f} \in \mathcal{A}(2^\infty)$. If $h \colon \Lambda_f \to \Lambda_{\tilde{f}}$ is conjugacy then

- *h is Hölder at 0,*
- *h is bi-Lipschitz at 0 $\iff \Theta = \tilde{\Theta}$,*
- *h is differentiable at 0 $\iff \Theta = \tilde{\Theta}$ and $\beta = \tilde{\beta}$.*

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Relationship with other work:

- Marco Martens and Liviana Palmisano consider circle maps with plateaus and with critical points at the boundary points of the form x^β, $\beta \in (1, 2)$.
- Gorbovickis and Yampolsky obtain renormamlisation for unimodal maps with critical points $\approx f(x) = f(c) + |x - c|^\beta$ for $x \approx c$ where β almost an integer.

VI. Diffeomorphic extensions / Non-existence of Koebe space

Theorem

The first return map to f^{2^k} to $[a_k, b_k]$ is a composition of f and the map f^{2^k-1} from a neighbourhood of $f(0)$ which is almost linear.

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Theorem (Absence of Koebe space)

For each $\tau > 0$ there exists $x \in \mathbb{R}$ and k so that the maximal semi-extension of the first entry map from x into $[a_k, b_k]$ does **not** *contain a τ-scaled neighbourhood of $[a_k, b_k]$.*

Theorem

The map f does not have wandering intervals.

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Remarks:

- We have not yet been able to prove absence of wandering intervals for the general case when $1 \leq \alpha < \beta$. Our current proof requires the scaling results from the earlier theorems.

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Theorem

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Remarks:

- We have not yet been able to prove absence of wandering intervals for the general case when $1 \leq \alpha < \beta$. Our current proof requires the scaling results from the earlier theorems.
- Absence of wandering intervals also unknown for circle homeomorphisms which are local diffeomorphisms except at two points x_0, x_1, where of the form

$$x \mapsto f(x_0) + (x - x_0)^3 \text{ for } x \approx x_0,$$

$$x \mapsto f(x_1) + (x - x_1)^{1/3} \text{ for } x \approx x_1.$$

VIII. Renormalization limit of return map

What does a rescaled version of $f^{2^k} \colon [a_k, b_k] \to [a_k, b_k]$ look like?

It is degenerate: By definition $f(a_k) = f(b_k)$ and therefore

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Nevertheless it is very good:

Theorem

$f^{2^k} : [a_k, b_k] \to [a_k, b_k]$ is a composition of

- *f and*
- *a diffeomorphism $\phi_k : J_k \to [a_k, b_k]$ so that ϕ_k tends to a linear map in the C^1 topology.*

Remark: In the quadratic case the analogue of ϕ_k converges to a nonlinear map.

VIII. Koebe space

In one-dimensional dynamics, usually one obtains non-linearity bounds from Koebe space in the range:

Theorem (Koebe Theorem)

Let $g\colon T \to g(T)$ be a diffeomorphism with $Sg < 0$. Assume that $J \subset T$ is an interval so that

$$g(T) \text{ contains a } \tau\text{-scaled neighbourhood, i.e.}$$

$$g(T) \supset (1+\tau)g(J).$$

Then for all $x, y \in J$,

$$\frac{\tau^2}{(1+\tau)^2} \leq \frac{Dg(x)}{Dg(y)} \leq \frac{(1+\tau)^2}{\tau^2}.$$

VIII. Bounding non-linearity due to semi-extensions

It turns out that ϕ_k does **not** have big Koebe space in the range. So how to get almost linearity?

Since $\alpha = 1$,

- $f|[a_0, 0]$ has a diffeomorphic extension to a map $f_1 : [a_0, \epsilon] \to \mathbb{R}$.
- Let $f_2 = f|[0, b_0]$
- Can assume $Sf_i \leq 0$.

Definition (Semi-extensions)

Let J be an interval and $f^n|J$ be monotone. Then $F : T \to \mathbb{R}$ is called *monotonic semi-extension* of $f^n|J$ if

- $J \subset T$ and $F|J = f^n|J$;
- $F = f_{i_1} \circ \cdots \circ f_{i_n}$, where $i_k \in \{1, 2\}$ for $k = 1, ..., n$.

IX. The semi-extensions

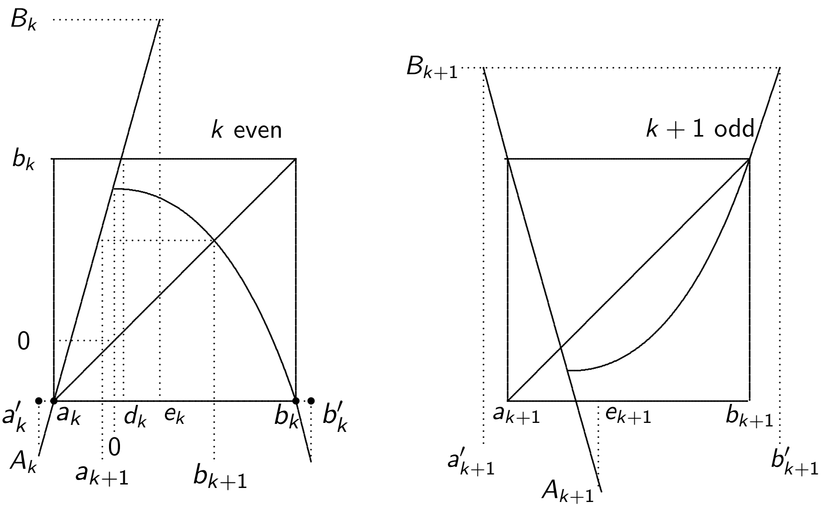

Figure: $f^{2^k}|I_k$ and $f^{2^{k+1}}|I_{k+1}$ when k is even and their semi-extensions. Note that the points d_k, e_k, a'_k, b'_k are defined using the semi-extension rather than dynamically.

IX. $\phi_k : J_k \to [a_k, b_k]$ has semi-extensions with huge Koebe space

Theorem (Exponentially growing Koebe space for semi-extensions)

For any $k \geq 0$ there exists τ_k with the following property. Let $\phi_k := f^{2^k - 1} : J_k \to [a_k, b_k]$ be the first entry map when $J_k \ni f(0)$. Then

- *$\phi_k : J_k \to [a_k, b_k]$ has a monotonic semi-extension $F : T \to \mathbb{R}$ such that $F(T)$ is τ_k-scaled neighbourhood of $[a_k, b_k]$.*
- *$\tau_k \to \infty$ as $k \to \infty$.*
- *τ_{2k} grows superexponentially with k, i.e. $\log \tau_{2k}$ grows exponentially.*

Proof: rather non-trivial bootstrap argument.

Corollary: ϕ_{2k} tends to an affine map and so the previous theorem follows.

IX. Other first entry maps are **not** almost linear

Suppose that W is an interval which under some iterate

- first visits $[0, b_k]$ for some k odd;
- under the first return to $[a_k, b_k]$ this interval visits $[0, b_k] \setminus [0, b_{k+1}]$ a number of times;
- then the interval makes a first visit into $[0, b_{k+2}]$ and then the process repeats (replacing $k \to k+2$).

The resulting map f^n is extremely non-linear and $|f^n(W)| << |W|$.

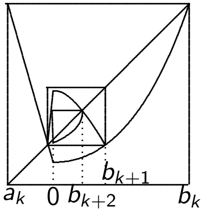

Figure: $f^{2^k}|[0, b_k]$ and $f^{2^{k+2}}|[0, b_{k+2}]$ when k is odd.

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- $\nexists$ definite Koebe space, even when $1 = \alpha < \beta$.
- When $1 < \alpha < \beta$ semi-extensions do not make sense. Nevertheless we think that b_n decays super-exponentially.
- Presumably, as in the work of Martens-Palmisano, the set $\Theta = const$ defines a codimension-one submanifold of the space of ∞-renormalizable period doubling maps.

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- When $1 < \alpha < \beta$ semi-extensions do not make sense. Nevertheless we think that b_n decays super-exponentially.
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$$f_c(x) = \begin{cases} |x|^\alpha + c & \text{when } x < 0, \\ x^\beta + c & \text{when } x \geq 0 \end{cases} \tag{10}$$

is an ∞-renormalizable period doubling map.