Dynamics of Chebyshev Polynomials

Christian Henriksen

Department of Applied Mathematics and Computer Science
Technical University of Denmark

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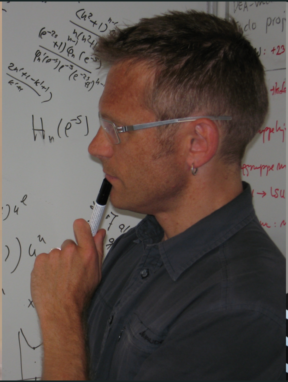
Joint work


Jacob Stordahl Christiansen, LTH

Henrik Laurberg Pedersen, KU

Carsten Petersen, RUC

Plan

Dynamics of polynomials
Chebyshev polynomials
Dynamics of Chebyshev polynomials
Pictures
Mathematics and conjectures behind the pictures

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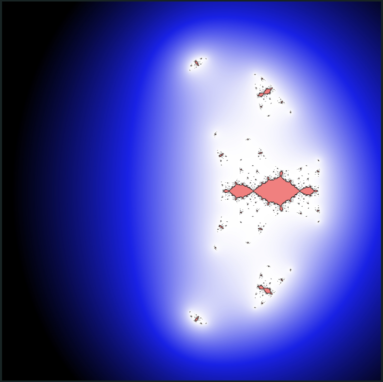
Polynomial dynamics

Iteration

Non-linear polynomial p

$$z \to p(z) \to p(p(z)) \to \cdots$$

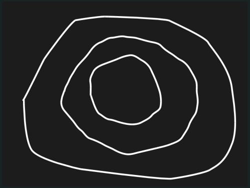


Totally invariant sets $K(p)$, $J(p)$, $F(p)$, $\Omega(p)$

Some notation

Classes of polynomials

- $\mathcal{P}_n$ formed by polynomials $p(z)$ of degree at most n.
- $\mathcal{P}_n^{>0}$ formed by polynomials $p(z)$ of degree n with leading coefficient real and postive.
- $\mathcal{P}_n^{=1}$ formed by polynomials $p(z)$ of degree n with leading coefficient equal to one.



Chebyshev polynomials

From Orthogonal Polynomials

μ gives inner product, that leads to orthogonal polynomials

$$P_0, P_1, \ldots \in \mathcal{P}_n$$

Set

$$p_n = \frac{P_n}{[z^n]P_n(z)} \in \mathcal{P}_n^{=1}.$$

Extremal property: p_n minimizes

$$||p||_{2,\mu} \text{ among } p \in \mathcal{P}_n^{=1}.$$

Chebyshev polynomials

Definition

Compact K leads to uniquely determined sequence $t_n(z;K), n = 0, 1, 2, \ldots, |K|$ of monic polynomials: $t_n(z;K)$ is the monic degree n polynomial that minimizes

$$||p_n||_{K,\infty} \ , \quad p_n \in \mathcal{P}_n^{=1}$$

Existence: Can restrict to polynomials with roots in $\mathrm{Co}(K)$.
Uniqueness: Via cardinality of extremal points.

Norm-alized

Set

$$T_n = t_n / ||t_n||_{K,\infty}.$$

Then T_n has maximal leading coefficient among

$$\{P \in \mathcal{P}_n^{>0} : ||P||_{K,\infty} \leq 1\}$$

Examples Disk and segment

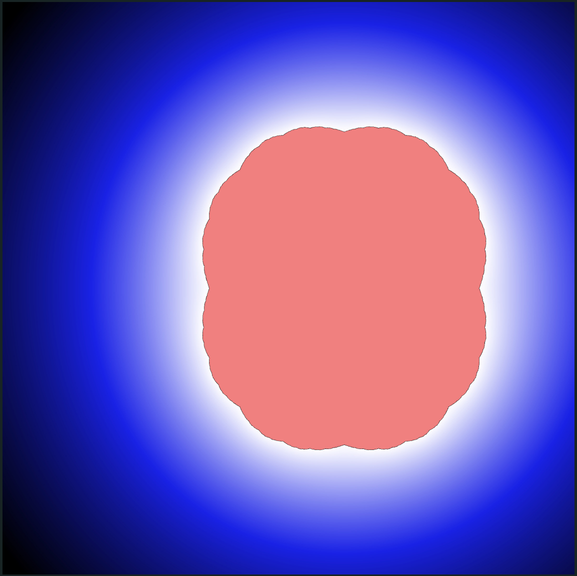
Dynamics

Prior work by Barnsley, Geronomi, Harrington for orthogonal polynomials
when μ is a particular measure.

Limit results for $K_{P_n} = K_n$ by [PCHP]

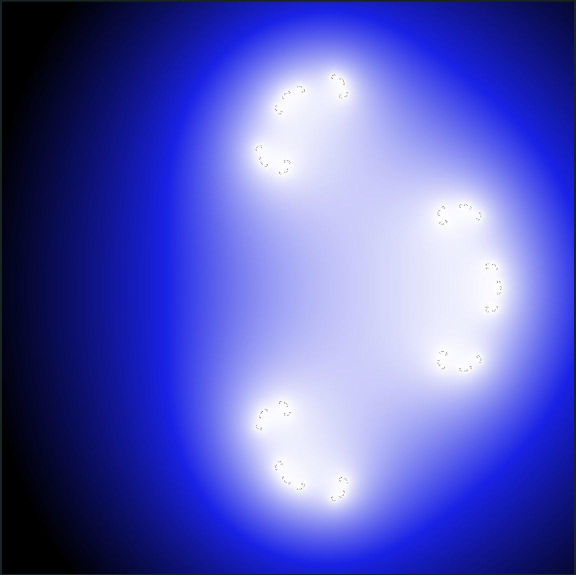
What about $K_n = K_{T_n}$?

Half circle



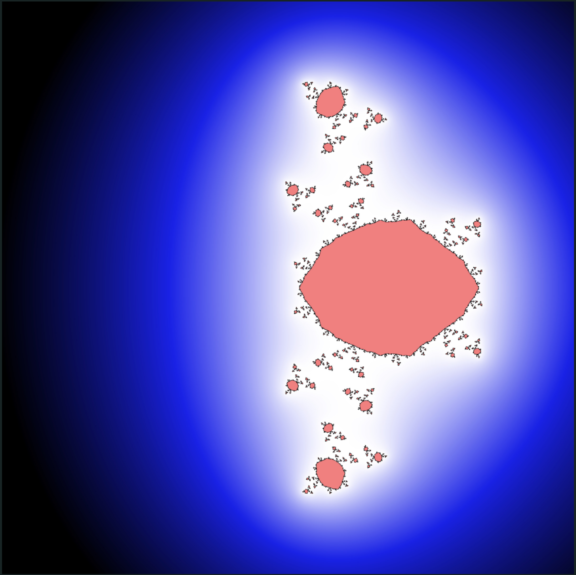
$n = 2$

Half circle



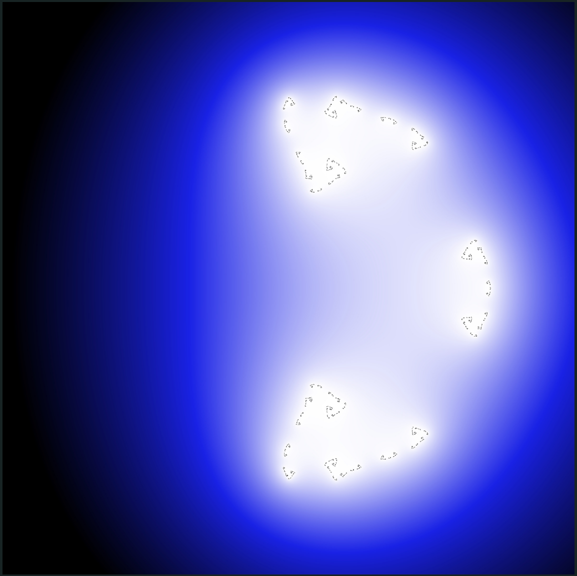
$n = 3$

Half circle



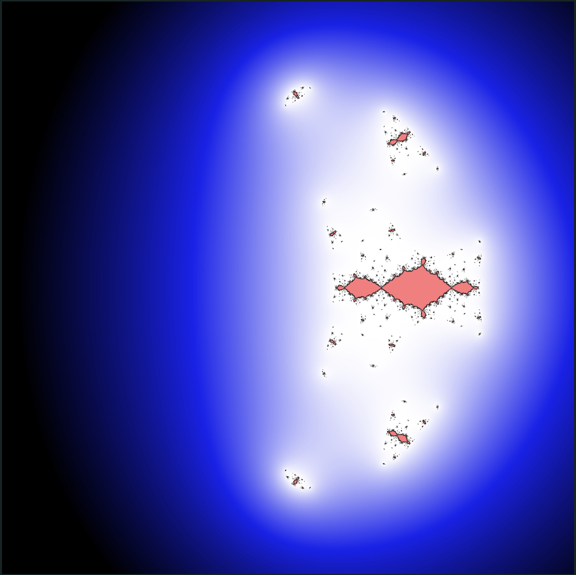
$n = 4$

Half circle



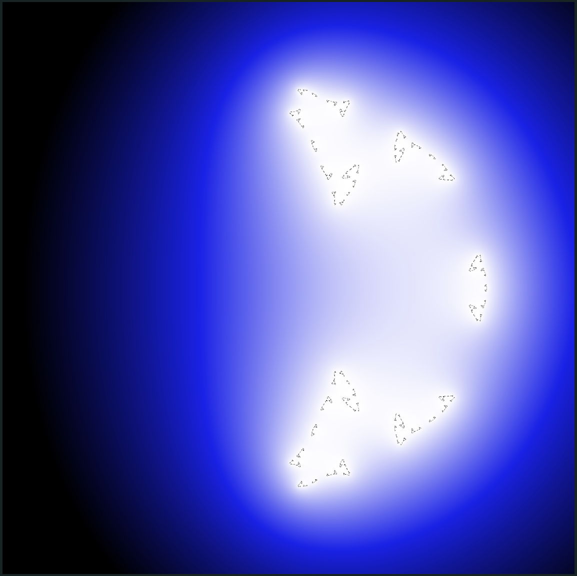
$n = 5$

Half circle



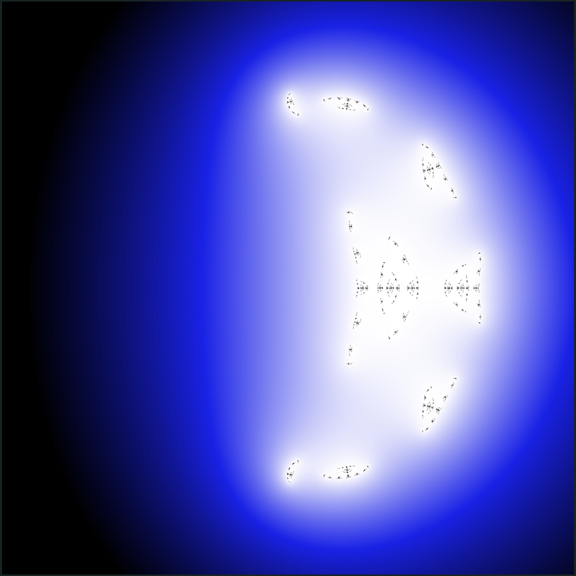
$n = 6$

Half circle



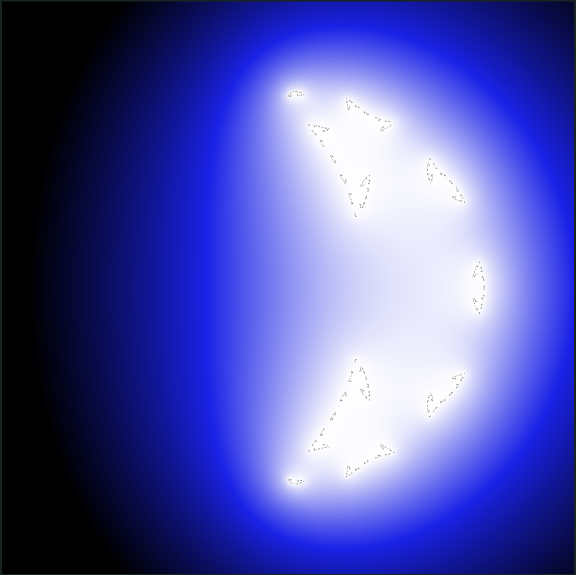
$n = 7$

Half circle



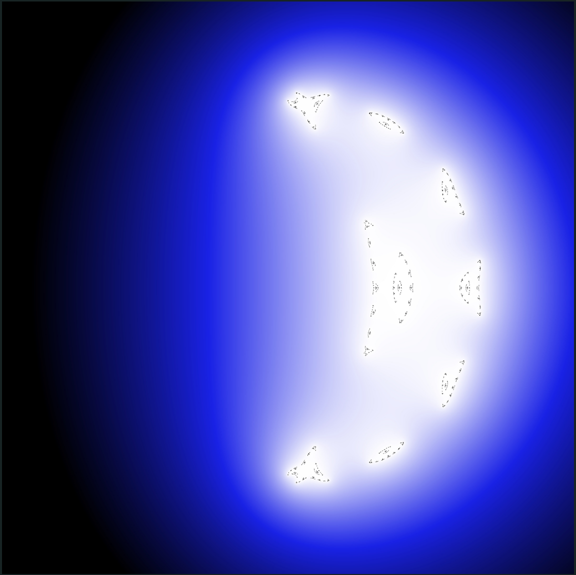
$n = 8$

Half circle



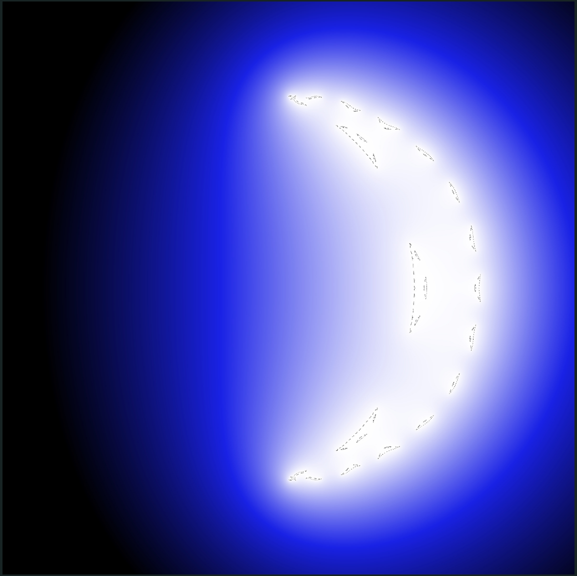
$n = 9$

Half circle



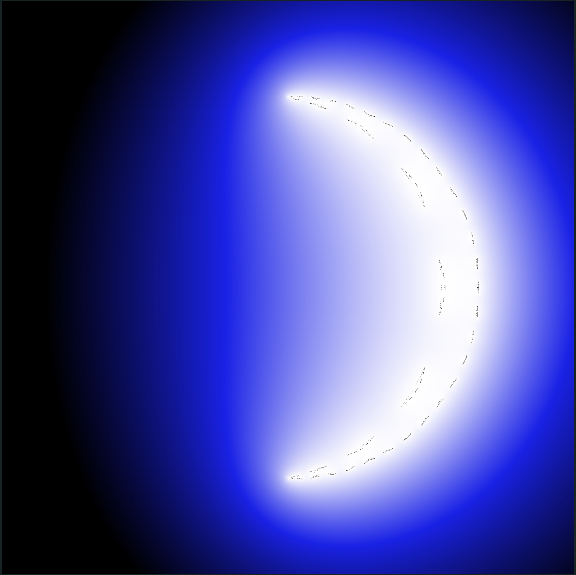
$n = 10$

Half circle



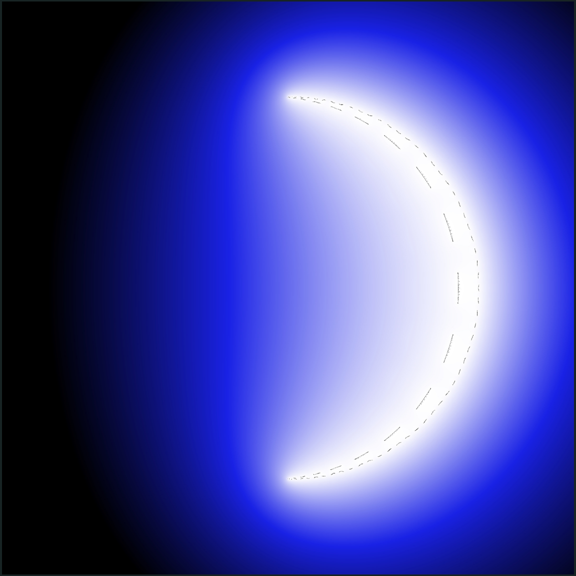
$n = 20$

Half circle



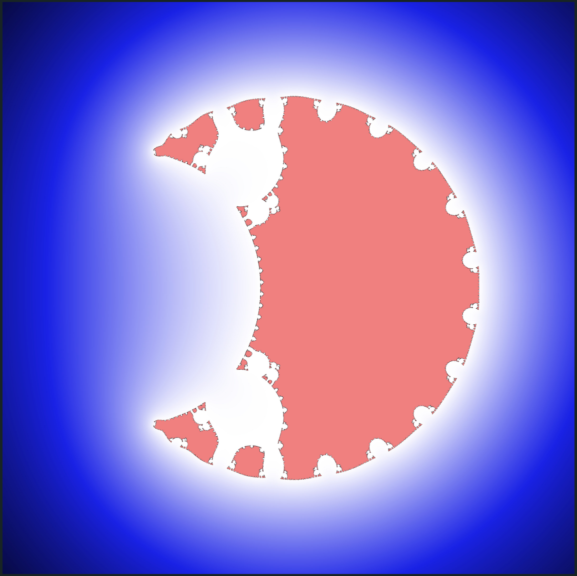
$n = 40$

Half circle



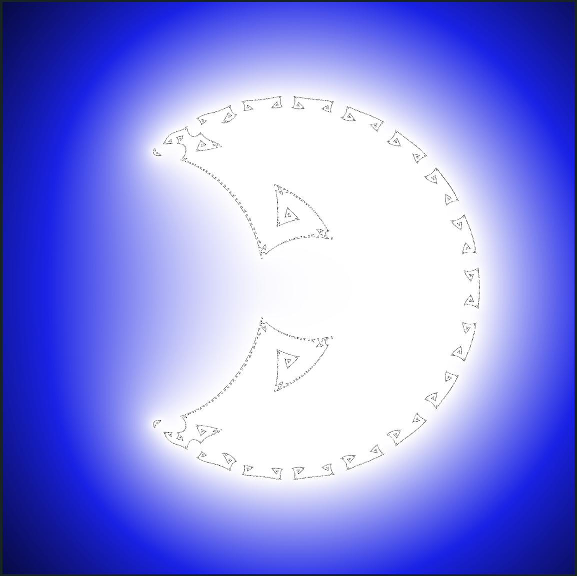
$n = 80$

Three quarter circle



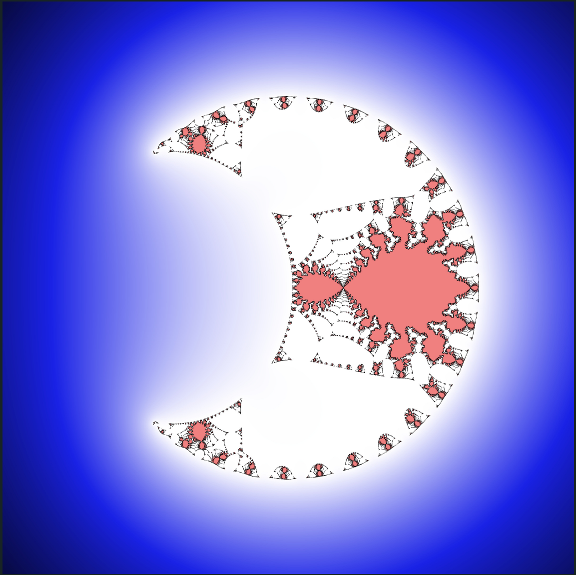
$n = 20$

Three quarter circle



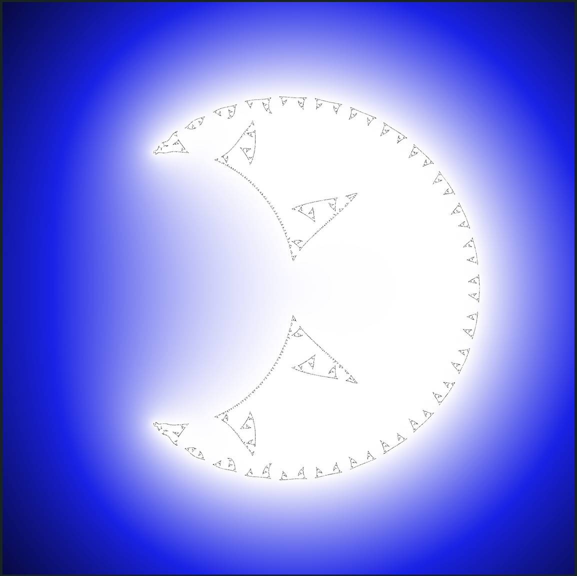
$n = 21$

Three quarter circle



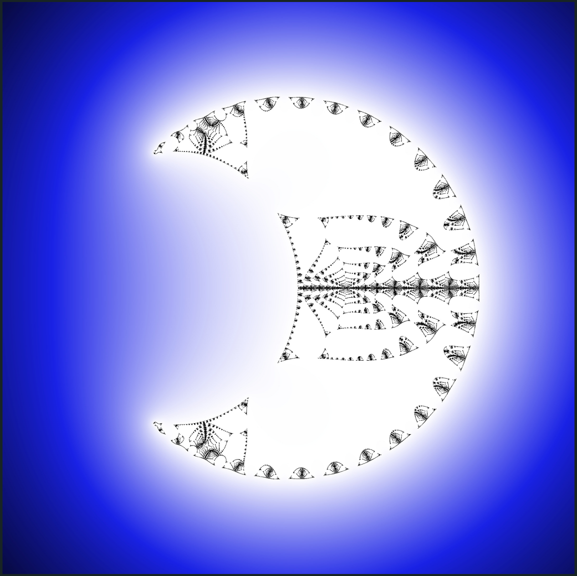
$n = 30$

Three quarter circle



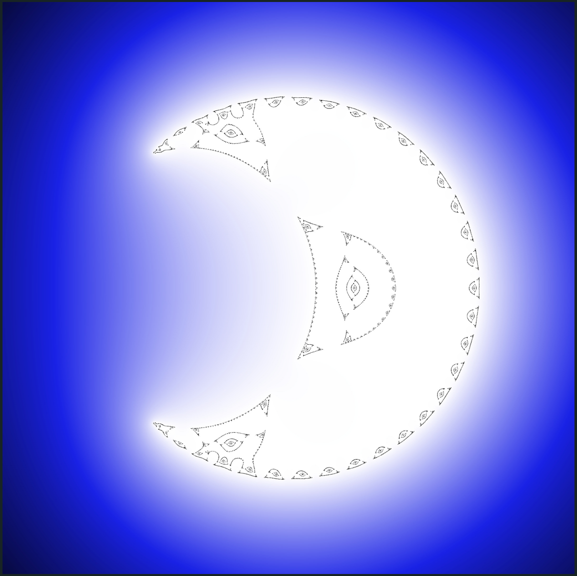
$n = 31$

Three quarter circle



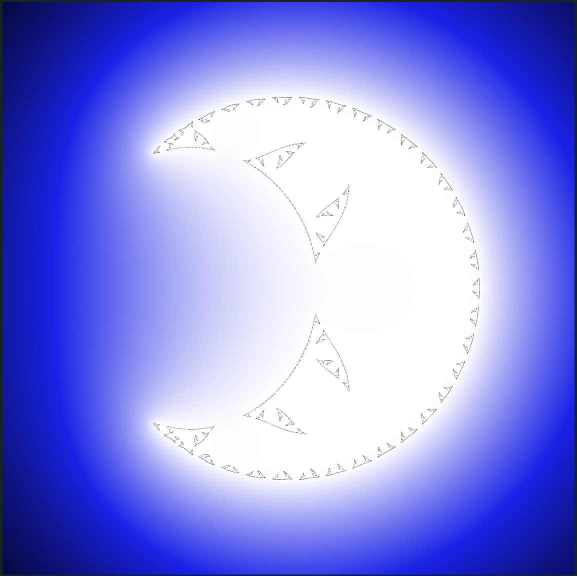
$n = 32$

Three quarter circle



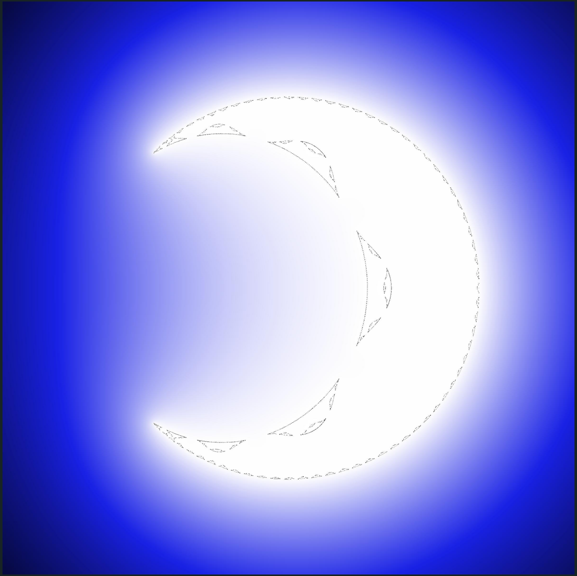
$n = 40$

Three quarter circle



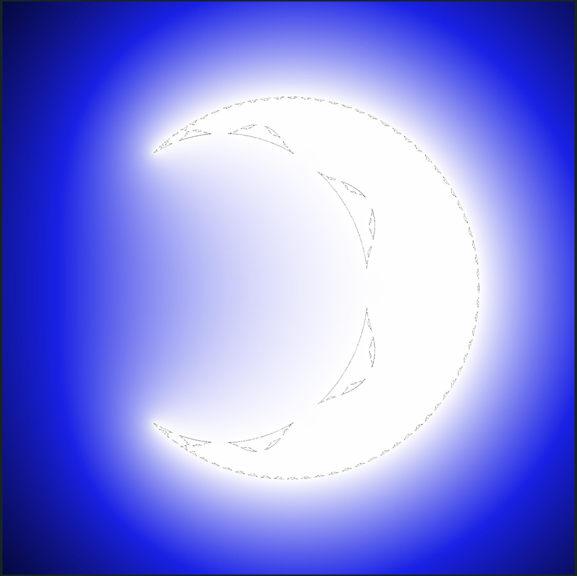
$n = 41$

Three quarter circle



$n = 80$

Three quarter circle



$n = 81$

Limits of sets

For a sequence of compact sets K_n, we set

$$\limsup K_n = \cap_n \overline{\cup_{k>n} K_n}.$$

For a sequence of connected open sets Ω_n containing ∞, we set

$$\limsup \Omega_n = \cup\{V : V \text{ admissible }\},$$

where admissible means that V is

- V is an open connected set containing ∞
- $V \subset \Omega_n$ for infinitely many values of n.

Theorem [PCHP] in preparation

Suppose

- K non-polar and compact.
- T_n normalized Chebyshev polynomials
- K_n filled Juliaset, Ω_n complement

Then

- $\limsup K_n \subset \mathrm{Co}(K)$
- $\limsup \Omega_n \cap K = \emptyset$

Norm bounds

Easy to get upper bound on $||t_n||_{K,\infty}$.
Recall $T_n(z;K)$ is the polynomial with maximal leading coefficient among

$$\{P \in \mathcal{P}_n^{>0} : ||P||_{K,\infty} \leq 1\}$$

If $K' \subset K$ then $[z^n]T_n(z;K') \geq [z^n]T_n(z;K)$.
When $K' = \{\zeta_0, \ldots, \zeta_n\}$ of card. $n+1$, then

$$T_n(z;K') = \sum_{k=0}^{n} \prod_{j \neq k} \frac{z - \zeta_j}{|\ \zeta_k - \zeta_j\ |}$$

Leading coefficient is

$$E(\zeta_0, \ldots, \zeta_n) = \sum_{k=0}^{n} \prod_{j \neq k} |\ \zeta_k - \zeta_j\ |^{-1}$$

Norm bounds

So when $K' = \{\zeta_0, \ldots, \zeta_n\} \subset K$ then

$$E(\zeta_0, \ldots, \zeta_n) \geq [z^n] T_n(z; K)$$

or equivalently

$$E(\zeta_0, \ldots, \zeta_n)^{-1} \leq ||t_n||_{K,\infty}$$

Conjecture (wimpy version)

When K is an arc of circle, and $E(\zeta_0, \ldots, \zeta_n)$ is minimal then

$$T_n(z; K) = T_n(z; \{\zeta_0, \ldots, \zeta_n\})$$

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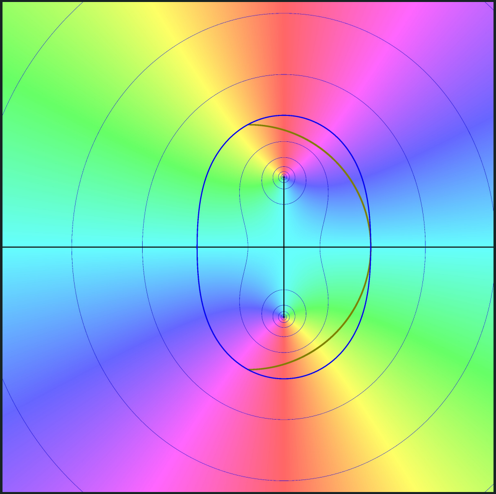

$n = 2$

Conjecture (wimpy version)

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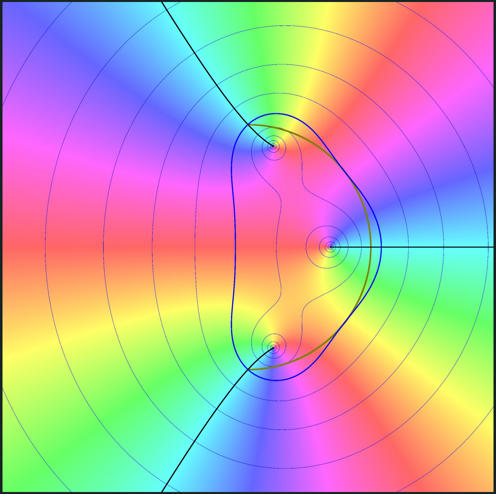

$n = 3$

Conjecture (wimpy version)

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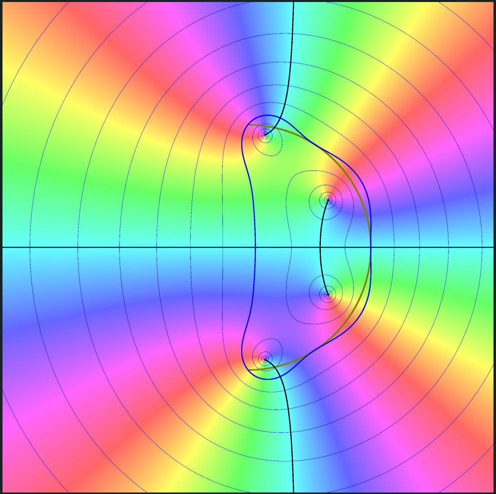
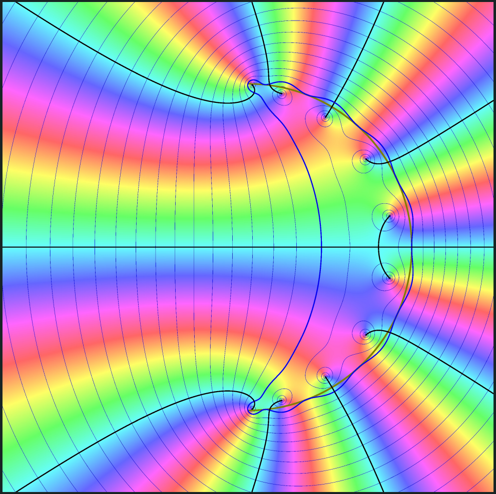


$n = 4$

Conjecture (wimpy version)

When K is an arc of circle, and $E(\zeta_0, \ldots, \zeta_n)$ is minimal then

$$T_n(z; K) = T_n(z; \{\zeta_0, \ldots, \zeta_n\})$$



$n = 10$

Conjecture (wimpy version)

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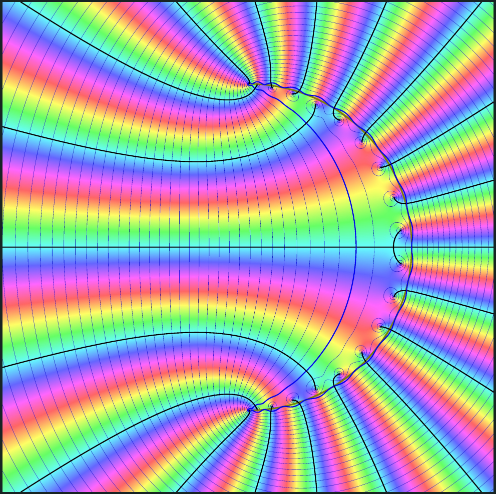

$n = 20$

Final remarks

- Work in progress
- Conference
  Holomorphic Days
  June 2. – 3.
  Copenhagen, Denmark