Constructing an entire function with Julia set of zero Lebesgue measure

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Outline

Preliminaries

Suppose f is a transcendental entire function.

- Julia set $\mathcal{J}(f)$ and escaping set $\mathcal{I}(f)$.

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- Singular values: critical values and asymptotic values.

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- Singular values: critical values and asymptotic values.
- Order of growth:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$ is the maximal modulus.

Eremenko-Lyubich Class $\mathcal{B}$

- Bounded set of singularities.
- Logarithmic change of variables.
- Expanding property.

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- An entire function f is in *Eremenko-Lyubich Class* $\mathcal{B}$ if $Sing(f^{-1})$ is bounded.
- $\mathcal{I}(f) \subset \mathcal{J}(f)$, and hence $\mathcal{J}(f) = \overline{\mathcal{I}(f)}$.

Logarithmic change of variables

Suppose $f \in \mathcal{B}$, then, by definition, there exists a $R \geq 0$ such that $Sing(f^{-1}) \subset \{z : |z| \leq e^R\}$. We use the following notations:

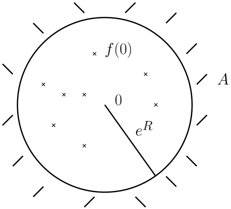
$$A = \{z \in \mathbb{C} : \ |z| > e^R\}, \quad U = f^{-1}(A),$$

$$V = \mathbb{C} \setminus U,$$

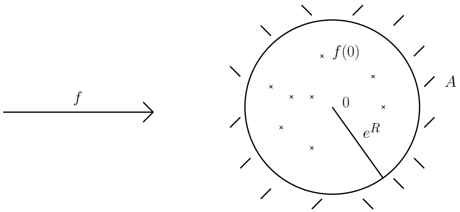
$$W = \exp^{-1}(U), \quad H = \{z \in \mathbb{C} : \mathrm{Re}\, z > R\}.$$

By logarithmic change of variables, we mean the following commutative diagram:

Logarithmic change of variables



Logarithmic change of variables



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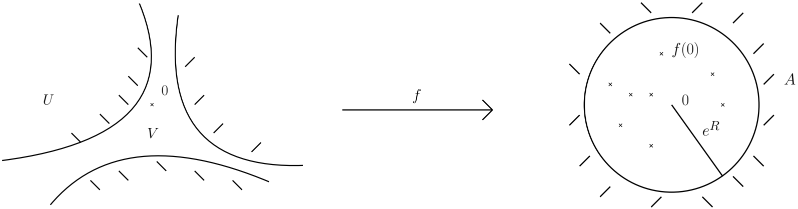

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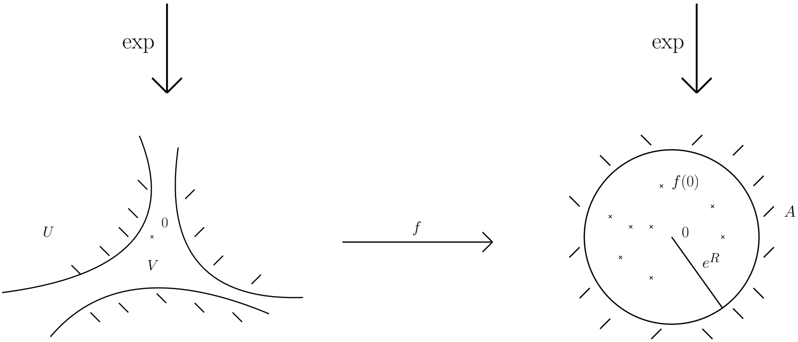

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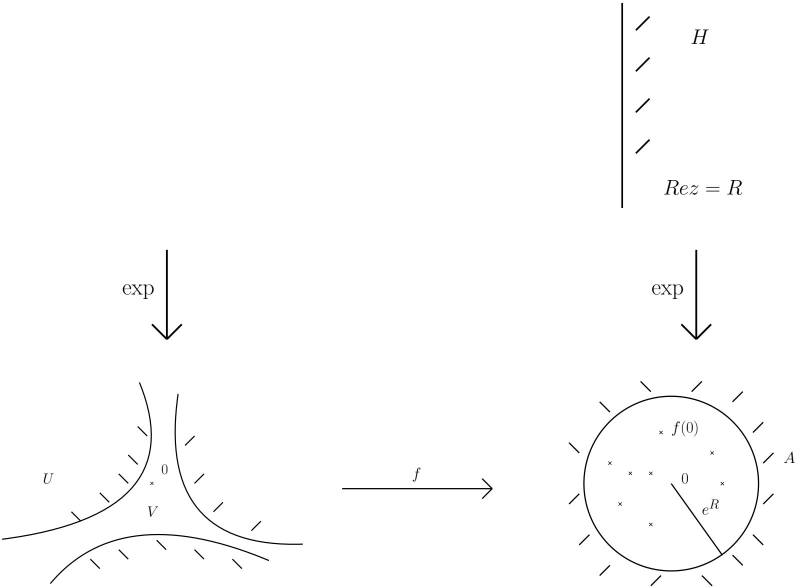

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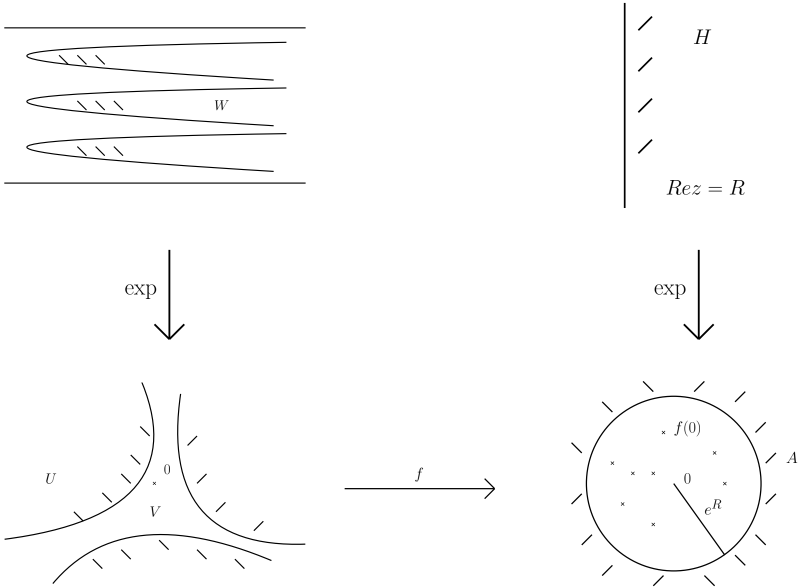

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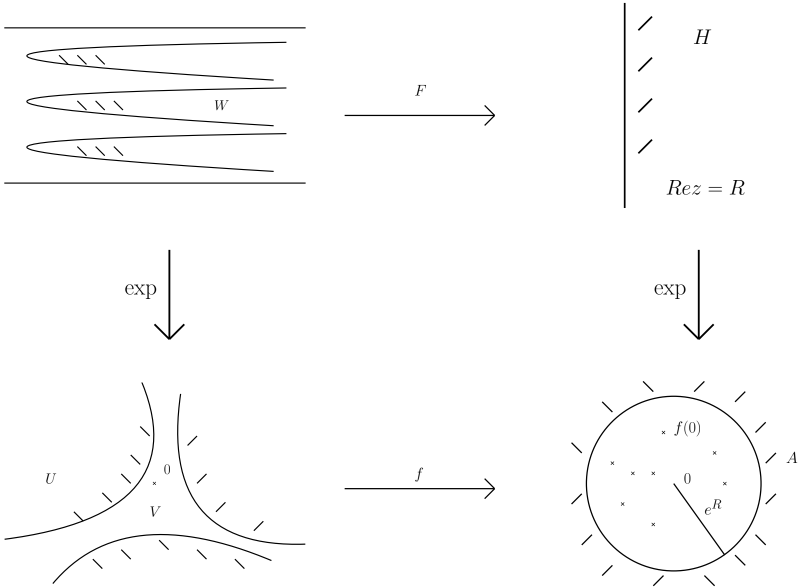

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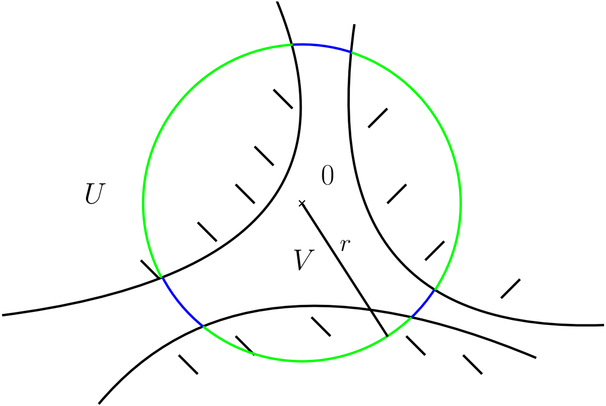

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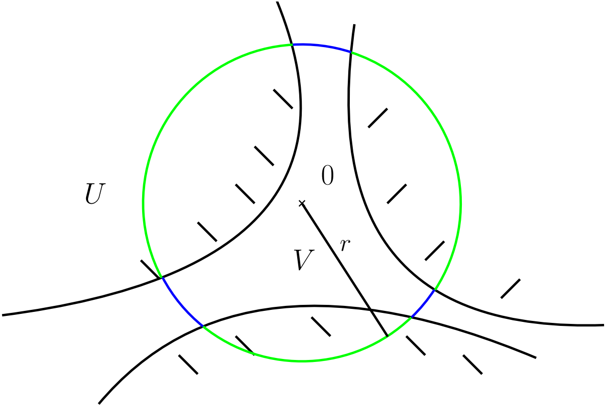


- For a given $r > 0$ sufficiently large, define

$$\theta(r) = \text{meas}\{\, t \in [0, 2\pi] : re^{it} \in V \}.$$

Expanding property

Theorem (Eremenko-Lyubich, 1992)

Suppose F is obtained from f by using logarithmic change of variables, then

$$|F'(z)| \geq \frac{\operatorname{Re} F(z) - R}{4\pi}.$$

Lebesgue measure of Julia sets and Escaping sets

- McMullen's result.
- Eremenko-Lyubich condition.
- An example.
- Aspenberg-Bergweiler condition.

McMullen's result

Theorem (McMullen, 1987)

$$\text{area}\, \mathcal{J}(\sin(\alpha z + \beta)) > 0, \textit{ for any } \alpha \neq 0, \beta \in \mathbb{C}.$$

Eremenko-Lyubich condition

Theorem (Eremenko-Lyubich, 1992)

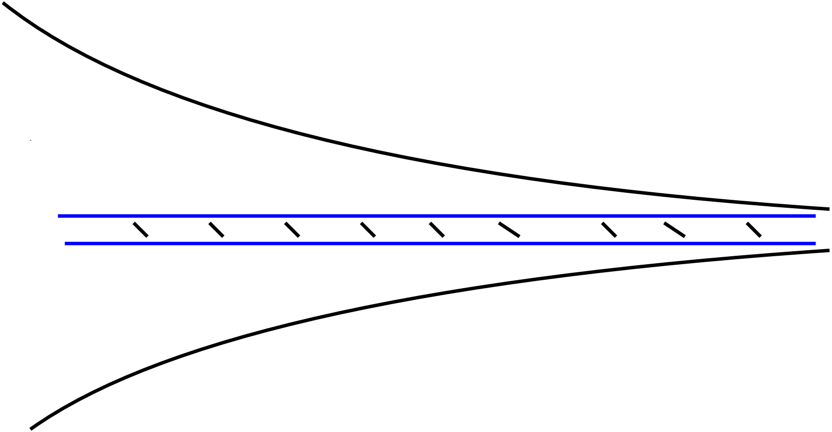
Suppose $f \in \mathcal{B}$ *is a transcendental entire function satisfying*

$$\liminf_{r\to\infty} \frac{1}{\log r} \int_1^r \theta(t) \frac{dt}{t} > 0, \qquad (1)$$

then $\operatorname{area} \mathcal{I}(f) = 0$*.*

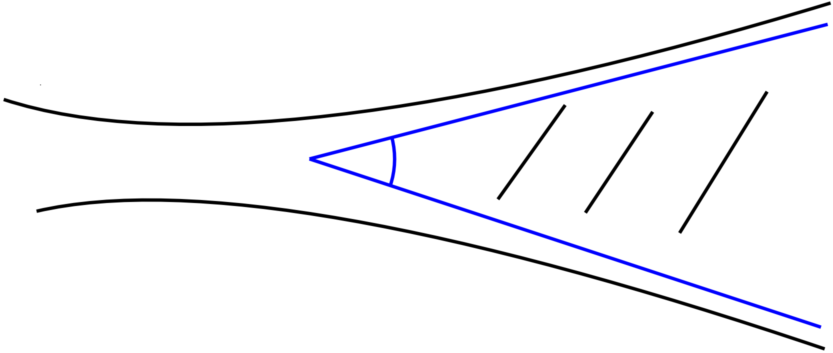
We call condition (1) the *Eremenko-Lyubich condition*.

Eremenko-Lyubich condition



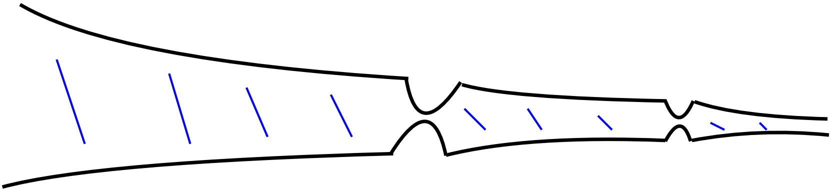
Strip with certain width

Eremenko-Lyubich condition



Sector with certain opening

Eremenko-Lyubich condition



Wierd case of $EL-condition$

An explicit example

Mittag-Leffler's function:

$$ML_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \ \alpha \in (0, 2).$$

- $\rho(ML_\alpha) = \frac{1}{\alpha}$.

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- $ML_\alpha \in \mathcal{B}$ (Aspenberg, Bergweiler).
- $\text{area}\, \mathcal{I}(ML_\alpha) = 0$ (Eremenko-Lyubich condition).

Aspenberg-Bergweiler condition

Theorem (Aspenberg-Bergweiler, 2012)
Let $f \in \mathcal{B}$ and suppose that f has N logarithmic tracts. If there exists $m \in \mathbb{N}$ such that

$$\log \log M(r, f) \leq \left(\frac{N}{2} + \frac{1}{\log^m r} \right) \log r$$

for large r, then $\operatorname{area} \mathcal{I}(f) > 0$.

Note that *Denjoy-Carleman-Ahlfors Theorem* implies

$$N \text{ asymptotic spots } \implies \rho(f) \geq \frac{N}{2}.$$

Aspenberg-Bergweiler condition

Is the Aspenberg-Bergweiler condition best possible?

Main results and ideas of proof

- Main results.
- Ideas of proof.

Main results

Theorem

Let $f \in \mathcal{B}$ *with* $\rho(f) < 1$ *and have a logarithmic tract* U*. Suppose* $\theta(r) \geq \theta_0(r)$ *for large* $r > 0$*, where* $\theta_0(r)$ *is continuous, decreasing and satisfies*

$$\sum_{k=1}^{\infty} \theta_0\left(E^k(0)\right) = \infty,$$

where $E(z) = \exp(z)$*. Then area* $\mathcal{I}(f) = 0$*.*

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- $\text{area}\, \mathcal{I}(f) = 0$.

Main results

Theorem (Optimality of Aspenberg-Bergweiler condition)

There exists an entire function in class $\mathcal{B}$ with $\rho(f) = \frac{1}{2}$ for which the escaping set has measure zero.

Idea of proof

Consider a sequence $\{a_n\}$ satisfying

$$1 \leq a_0 \leq a_1 \leq \cdots \leq a_n \leq \ldots,$$

and $\varepsilon(r)$ which is decreasing and tends to zero slower than any of $1/\log^m$ for $m \in \mathbb{N}$. Define

$$f(z) = \prod_{j=0}^{\infty}(1 - \frac{z}{a_j})$$

such that

$$n(r, f) = r^{\rho(r)} + O(1),$$

where $\rho(r) = \frac{1}{2} + \varepsilon(r)$.

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- Applying the first theorem, area $\mathcal{I}(f) = 0$.

Thank you !
Gracias !