A combinatorial characterization of postcritically finite Newton maps

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Thurston Maps

$F : S^2 \longrightarrow S^2$ finite degree branched cover from the 2-sphere to itself
C_F is the set of critical points

- The *postcritical set* of F is given by

$$P_F = \bigcup_{i>0} F^i(C_F).$$

- If $|P_F|$ is finite, F is said to be *postcritically finite*

Definition

A *Thurston map* is a postcritically finite orientation-preserving branched cover $F : S^2 \longrightarrow S^2$ where $\deg(F) \geq 2$.

Thurston Equivalence

Definition

Let F and G be Thurston maps. Then F is *Thurston equivalent* to G if there are orientation preserving homeomorphisms

$$h_0, h_1 : (S^2, P_F) \longrightarrow (S^2, P_G)$$

with h_0 homotopic to h_1 rel P_F, so that the following commutes:

$$
\begin{CD}
(S^2, P_F) @>{h_1}>> (S^2, P_G) \\
@V{F}VV @VV{G}V \\
(S^2, P_F) @>{h_0}>> (S^2, P_G)
\end{CD}
$$

Newton Maps

Definition

A rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 3$ is a *Newton map* if there exists a polynomial p such that for every $z \in \mathbb{C}$, $f(z) = z - p(z)/p'(z)$

Theorem (Y. Mikulich, D. Schleicher,–)

There is a bijection between the set of postcritically finite Newton maps up to affine conjugacy and the set of abstract extended Newton graphs up to Thurston equivalence.

Newton Facts

p–degree d polynomial with simple roots
$N_p(z) = z - \frac{p(z)}{p'(z)}$

- N_p is a rational function of degree d
- $N_p'(\infty) = d/(d-1)$ and is a repelling fixed point
- The roots of p correspond to finite superattracting fixed points of N_p (since $N_p' = \frac{p \cdot p''}{p' \cdot p'}$)
- A degree d rational map has $d + 1$ fixed points and $2d - 2$ critical points counting multiplicity
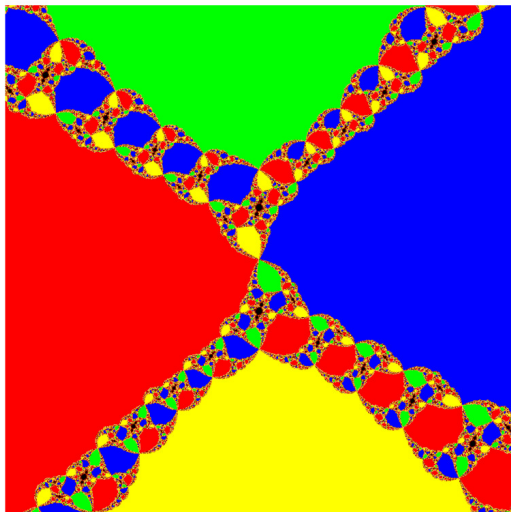
Thus the only fixed points of N_p are at ∞ and roots of p.
(Hubbard, Schleicher, Sutherland): The immediate basin of a fixed critical point of multiplicity m has exactly m accesses to infinity.
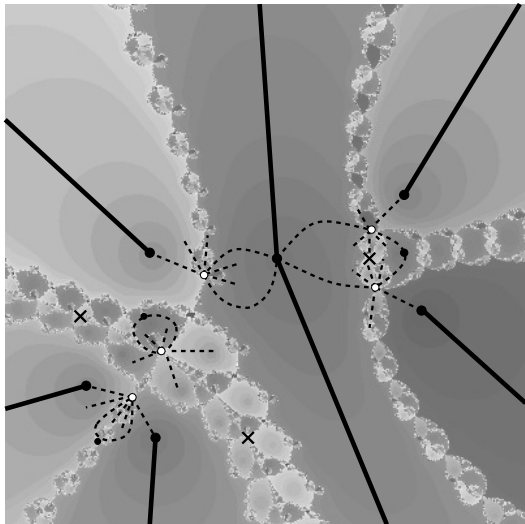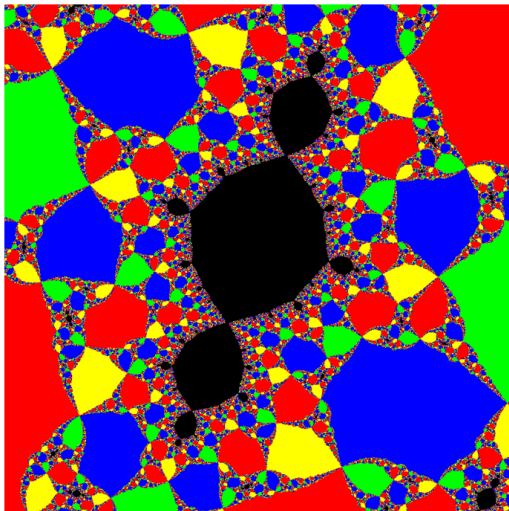
Definition (Abstract Channel Diagram)

An *abstract channel diagram* of degree $d \geq 3$ is a graph $\Delta \subset S^2$ with vertices $v_1, \ldots, v_d, v_\infty$ and edges $e_1, \ldots, e_l$ that satisfies the following properties:

(1) $l \leq 2d - 2$

(2) each edge joins v_∞ to a v_i

(3) each v_i is connected to v_∞ by at least one edge

(4) if e_i and e_j both join v_∞ to v_k, then each connected component of $S^2 \setminus \overline{e_i \cup e_j}$ contains at least one vertex of Δ

Definition (Abstract Newton Graph)

Let $\Gamma \subset S^2$ be a connected finite graph and $f : \Gamma \to \Gamma$ a graph map. The pair (Γ, f) is called an *abstract Newton graph* if it satisfies the following conditions:

(1) There exists $d_\Gamma \geq 3$ and an abstract channel diagram $\Delta \subsetneq \Gamma$ of degree d_Γ such that f fixes each vertex and each edge of Δ.

(2) The graph map f can be extended to a branched covering $f : S^2 \to S^2$ such that the following conditions (3) – (6) are satisfied.

Definition (Abstract Newton Graph cont'd)

(3) If $v_1, \ldots, v_{d_\Gamma}, v_\infty$ are the vertices of Δ, then $v_i \in \overline{\Gamma \setminus \Delta}$ if and only if $i \neq \infty$. Moreover, there are exactly $\deg_{v_i}(\overline{f}) - 1 \geq 1$ edges in Δ that connect v_i to v_∞ for $i \neq \infty$.

(4) $\sum_{x \in \Gamma'} \left(\deg_x(\overline{f}) - 1 \right) \leq 2d_\Gamma - 2$.

(5) The graph $\overline{\Gamma \setminus \Delta}$ is connected.

(6) Γ equals the component of $\overline{f}^{-N_\Gamma}(\Delta)$ that contains Δ.

Definition (Abstract Extended Newton Graph)

$\Sigma \subset S^2$ a finite connected graph, $f : \Sigma \to \Sigma$ a graph map, and Σ' the set of vertices. A pair (Σ, f) is called an *abstract extended Newton graph* if:

(1) (Edge Types) Any two different edges in Σ may only intersect at vertices of Σ. Every edge must be one of the following three types:
- An edge in the abstract Newton graph Γ
- An edge in a periodic or preperiodic abstract Hubbard tree
- A periodic or preperiodic abstract Newton ray

(2) (Abstract Newton graph) There exists a positive integer N and an abstract Newton graph Γ at level N so that $\Gamma \subseteq \Sigma$. Furthermore N is minimal so that condition (6) holds.

(3) (Unique extendability) f can be extended to a branched covering $\overline{f} : S^2 \to S^2$, unique up to Thurston equivalence

(4) (Topological admissibility) The total number of critical points of $\overline{f}$ counted with multiplicity is $2d_\Gamma - 2$, where d_Γ is the degree of the abstract channel diagram $\Delta \subset \Gamma$.

Definition (Abstract Extended Newton Graph (cont'd))

(5) (Periodic Hubbard trees) There is a finite collection of distinct, disjoint, possibly degenerate, abstract extended Hubbard trees $H_i \subset \Sigma$, where $H_i \cap \Gamma = \emptyset$, and for each H_i there is a minimal positive integer $m_i \geq 2$ called the period of the tree such that $\overline{f}^{m_i}(H_i) = H_i$. For all k, the forward image of $f^k(H_i)$ is always some periodic Hubbard tree.

(6) (Preperiodic Hubbard trees) There is a finite collection of distinct, disjoint, possibly degenerate abstract extended Hubbard trees $H_i' \subset \Sigma$ where $H_i' \cap \Gamma = \emptyset$, and for each i there is some minimal positive integer $\ell_i \geq 1$ so that $\overline{f}^{\ell_i}(H_i')$ is a periodic Hubbard tree. This ℓ_i is called the preperiod of H_i' and for $1 \leq k < \ell_i$, the forward image $f^k(H_i')$ is always some preperiodic Hubbard tree.

(7) (Trees separated) Any two different abstract extended Hubbard trees lie in different complementary components of Γ.

Definition (Abstract Extended Newton Graph (cont'd))

(8) **(Periodic Newton Rays)** For every periodic abstract extended Hubbard tree H_i of period m_i and every fixed point ω_i of H_i there exists a periodic abstract Newton ray $\mathcal{R}_i$ that lands at ω_i (note that it is not required that $\mathcal{R}_i \subset \Sigma$). For every β-fixed point ω_i of H_i, the graph Σ contains exactly one periodic abstract Newton ray $\mathcal{R}_i$ of period m_i that lands at ω_i, namely the rightmost such Newton ray.

(9) **(Preperiodic Newton Rays)** For every preperiodic tree H_i' of preperiod ℓ_i where $\overline{f}^{\ell_i}(H_i') = H_j$, the following holds: for every $\omega_i' \in H_i'$ so that $\overline{f}^{\ell_i}(\omega_i')$ is a β-fixed point of H_j, there exists exactly one preperiodic abstract Newton ray in Σ that lands at ω_i', and whose image under $\overline{f}^{\ell_i}$ is a ray of period m_j landing on H_j.

Background for Thurston's Theorem

A simple closed curve γ in $S^2 \setminus X$ is *essential* if each component of $S^2 \setminus \gamma$ intersects X in at least two points. A *multicurve* is a collection $\Gamma = \{\gamma_1, ... \gamma_k\}$ of disjoint essential curves where the elements of the collection are pairwise non-homotopic. Use $\mathscr{C}_F$ to denote the set of homotopy classes of essential simple closed curves in $S^2 \setminus X$. Denote by $\mathbb{R}[\mathscr{C}_F]$ the free $\mathbb{R}$-module over $\mathscr{C}_F$.

The *Thurston linear map* $\lambda_F : \mathbb{R}[\mathscr{C}_F] \longrightarrow \mathbb{R}[\mathscr{C}_F]$ is defined by

$$\lambda_F(\gamma) = \sum_{\gamma'} \sum_{\gamma' \simeq \delta \subset F^{-1}(\gamma)} \frac{1}{\deg(F : \delta \rightarrow \gamma)} \cdot \gamma'$$

where γ and γ' are essential curves and the outer sum is over all γ' homotopic to preimages of γ. A *Thurston obstruction* is a nonempty multicurve Γ so that $\mathbb{R}[\Gamma]$ is invariant under λ_F, and the spectral radius of λ_F is greater than or equal to 1.

Theorem (Thurston)

Let F be a Thurston map not equivalent to a Lattès map. Then F is Thurston equivalent to a rational function if and only if there are no obstructions. If this rational function exists, it is unique up to Möbius conjugation.

Definition (Intersection Number)

Let α and β each be an arc or a simple closed curve in (S^2, X). Their *intersection number* is

$$\alpha \cdot \beta := \min_{\alpha' \simeq \alpha, \, \beta' \simeq \beta} \#\{(\alpha' \cap \beta') \setminus X\} \, .$$

The intersection number extends bilinearly to arc systems and multicurves.

We will now prove that every abstract extended Newton graph gives rise to a postcritically finite Newton map.

Proof: Let Π be an irreducible Thurston obstruction, i.e. for $\gamma, \gamma' \in \Pi$, there exists $n > 0$ so that some component of $\overline{f}^{-n}(\gamma)$ is homotopic to γ' rel the vertices of Σ. We will show this results in a contradiction.

- Case 1: $\Pi \cdot \Delta \neq 0$
- Case 2: $\Pi \cdot \Delta = 0$

A set of pairwise non-isotopic arcs in (S^2, X) is called an *arc system*. Two arc systems Λ, Λ' are *isotopic* if each curve in Λ is isotopic relative to X to a unique element of Λ' and vice versa.

Denote by $\tilde{\Lambda}(f^{\circ n})$ the union of those components of $f^{-n}(\Lambda)$ that are isotopic to elements of Λ relative X, and define $\tilde{\Pi}(f^{\circ n})$ similarly.

### Theorem (Kevin Pilgrim, Tan Lei)

Let (f, X) be a marked branched covering, Π an irreducible Thurston obstruction and Λ an irreducible arc system. Suppose furthermore that $\#(\Pi \cap \Lambda) = \Pi \cdot \Lambda$. Then, exactly one of the following is true:

1. *$\Pi \cdot \Lambda = 0$ and $\Pi \cdot f^{-n}(\Lambda) = 0$ for all $n \geq 1$.*

2. *$\Pi \cdot \Lambda \neq 0$ and for $n \geq 1$, each component of Π is isotopic to a unique component of $\tilde{\Pi}(f^{\circ n})$. The mapping $f^{\circ n} : \tilde{\Pi}(f^{\circ n}) \to \Pi$ is a homeomorphism and $\tilde{\Pi}(f^{\circ n}) \cap (f^{-n}(\Lambda) \setminus \tilde{\Lambda}(f^{\circ n})) = \emptyset$.*

This theorem implies that $\Pi \cdot (\Gamma \setminus \Delta) = \emptyset$, and that Π may not intersect the Newton rays.

Proof of Case 1

Π is an irreducible Thurston obstruction, $\gamma_1 \in \Pi$ so that $\gamma_1 \cdot \Delta \neq 0$
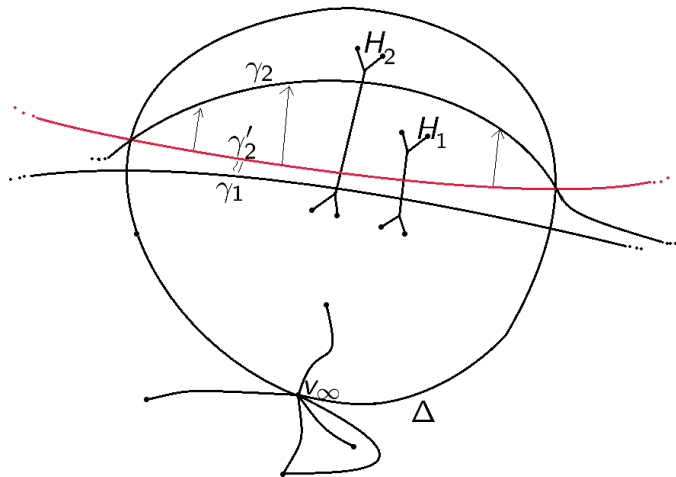
Proof of Case 2

Thank you for your attention!