Arnold Tongues in Degree 4 Blaschke Products

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Degree 4 Blaschke products

We want to study the degree 4 Blaschke products

$$B_a(z) = z^3 \frac{z-a}{1-\bar{a}z}$$

These products are the rational version of the double standard map:

	$\mathbb{S}^1 \to \mathbb{S}^1$	$\mathbb{C}^* \to \mathbb{C}^*$	$\hat{\mathbb{C}} \to \hat{\mathbb{C}}$
Standard map	$\theta \to \theta + \alpha + \beta \sin\theta$	$e^{i\alpha} \cdot z \cdot e^{\beta/2(z+1/z)}$	$e^{it} z^2 \frac{z-a}{1-\bar{a}z}$
Double standard map	$\theta \to 2\theta + \alpha + \beta \sin\theta$	$e^{i\alpha} \cdot z^2 \cdot e^{\beta/2(z+1/z)}$	$e^{it} z^3 \frac{z-a}{1-\bar{a}z}$

A quasi-conformal surgery construction relates these products to cubic polynomials having $z = 0$ as a superattracting fixed point.

M. Herman, *Sur la conjugaison des difféomorphismes du cercle à des rotations*, 1976

N. Fagella, *Limiting dynamics for the complex standard family*,1995

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Properties of these Blaschke products

The main properties of these Blaschke products are the following:

- They leave $\mathbb{S}^1$ invariant.
- They are symmetric with respect to $\mathbb{S}^1$, i.e., $B_a(z) = (B_a(z^*))^*$, where $z^* = 1/\overline{z}$.
- $z = 0$ and $z = \infty$ are superattracting fixed points of order 2.
- They have two "free" critical points $c_\pm = a \cdot \frac{1}{3|a|^2}\left(2 + |a|^2 \pm \sqrt{(|a|^2 - 4)(|a|^2 - 1)}\right)$.
- B_a and $B_{\xi a}$ are conjugate, where ξ is a third root of unity.
- $z_\infty = 1/\overline{a}$ and $z_0 = a$ are the only pole and zero respectively.

One can study the dynamics of these products depending on $|a|$.

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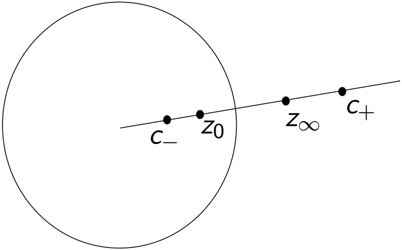
Case $|a| < 1$

$z_\infty \notin \mathbb{D} \Rightarrow B_a(\mathbb{D}) = \mathbb{D} \Rightarrow \mathcal{A}(0) = \mathbb{D},\ \mathcal{A}(\infty) = \mathbb{C} \setminus \overline{\mathbb{D}},\ \mathcal{J}(B_a) = \mathbb{S}^1$



Case $|a| = 1$

This is a degenerate case.

$z_0 = z_\infty = c_+ = c_- = a \in \mathbb{S}^1$.

$$B_a(z) = z^3 \frac{z-a}{1-z/a} = -az^3$$

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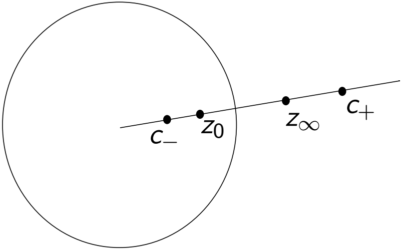

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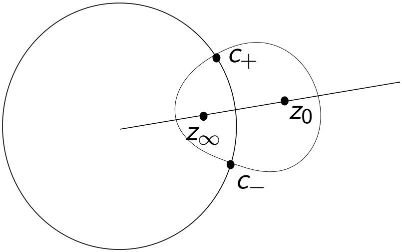
Case $1 < |a| < 2$

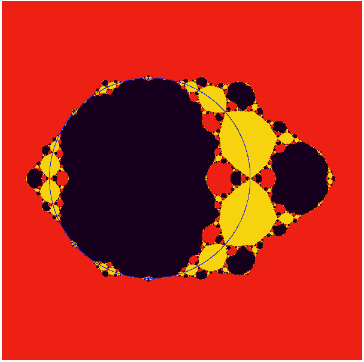
There are two different critical points:

$$c_+ = a \cdot \frac{1}{3|a|^2}\left(2 + |a|^2 + i\sqrt{(4-|a|^2)(|a|^2-1)}\right) = a \cdot k$$

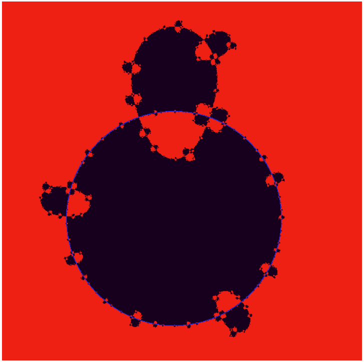
$$c_- = a \cdot \bar{k}$$

The critical points satisfy $|c_\pm| = 1$.
The critical orbits are not symmetric.





(a) Dynamical plane of $B_{3/2}$. It can be seen in yellow an attracting basin of a period 2 cycle.



(b) Dynamical plane of $B_{3/2i}$. In this case there are no other attracting basins than the ones of $z = 0$ and ∞.

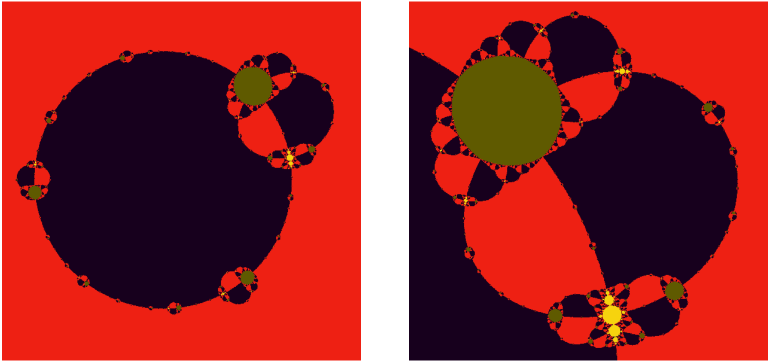
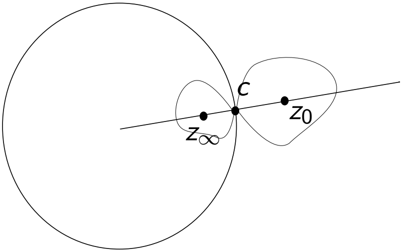


Figure: Dynamical plane of the Blaschke product $B_{1,07398+0,5579i}$.

Case $|a| = 2$

$c_+ = c_- = a/2 \in \mathbb{S}^1$
$z_\infty \in \mathbb{D}$
$z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$



$B_a|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ is a covering of degree 2.

Case $|a| > 2$

$z_\infty, c_- \in \mathbb{D}$
$z_0, c_+ \in \mathbb{C} \setminus \overline{\mathbb{D}}$
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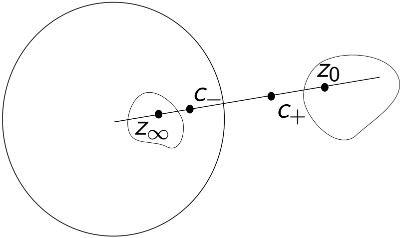
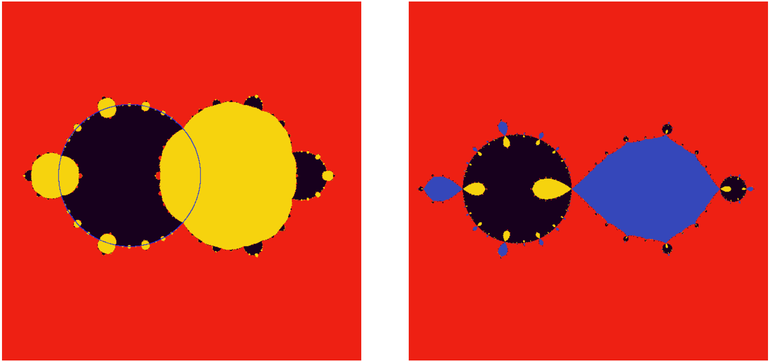


$c_- = 1/\overline{c_+} \Rightarrow$ Critical orbits are symmetric w.r.t. $\mathbb{S}^1$.



(a) Dynamical plane of $B_{5/2}$. In this case both free critical points accumulate at a fixed point on $\mathbb{S}^1$.

(b) Dynamical plane of B_4. In this case each critical orbit accumulates on a different attracting cycle.

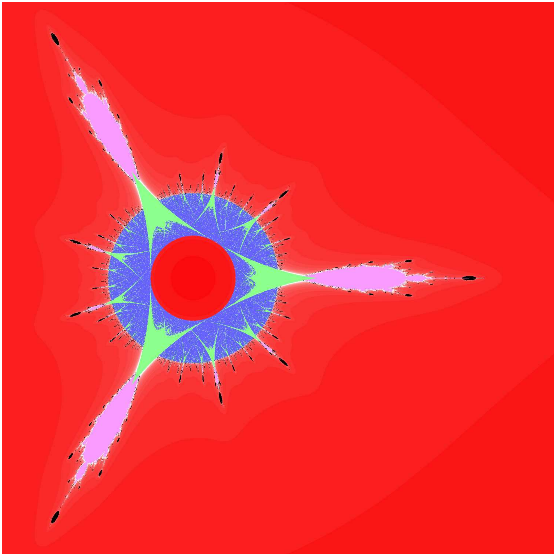
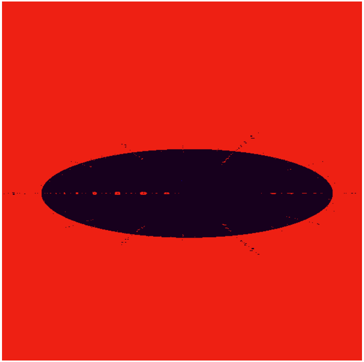
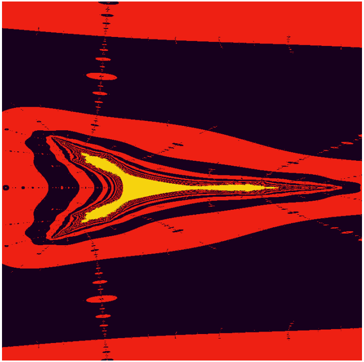


Figure: Parameter plane of B_a. It has been drawn by iterating the critical point c_+.

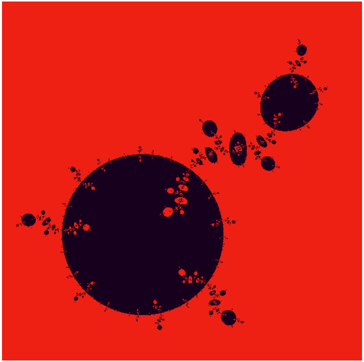
Baby Tricorns



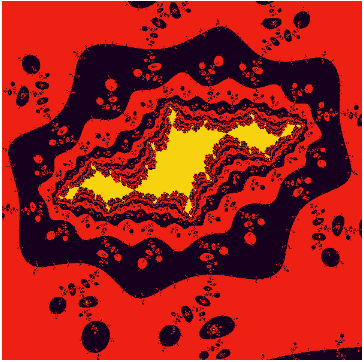
(a) Zoom in the parameter plane.



(b) Zoom in (a).



(c) Parameter plane of B_a with $a = 1.89023 + 1.70536i$.



(d) Zoom in (c).

Tongues of the Blaschke family

Let a s.t. $|a| \geq 2$. Then, $B_{a|\mathbb{S}^1}$ is a degree 2 covering of $\mathbb{S}^1$.

$B_{a|\mathbb{S}^1}$ is semiconjugate to the doubling map $\theta \to 2\theta (\text{mod } 1)$ by a unique continuous map H_a.

H_a sends periodic points to periodic points of the same period.

Definition

We say that a, $|a| \geq 2$, is of type τ if $B_a|_{\mathbb{S}^1}$ has an attracting cycle and $H_a(x_0) = \tau$, where x_0 is the marked point of the attracting cycle.

Definition

We define the tongue $T_\tau = \{a| \quad 2 \leq |a|, a \text{ has type } \tau\}$.

Remark

Since H_a sends periodic points to periodic points, any realisable type $\tau \in \mathbb{S}^1$ is a periodic point of the doubling map $\theta \to 2\theta (\text{mod } 1)$.

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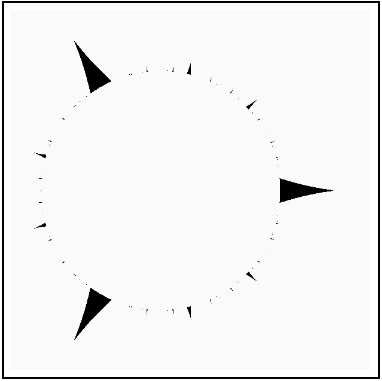
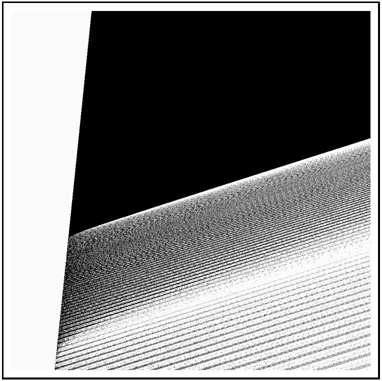


(e) Tongues



(f) Zoom in the tongues

Figure: In (a) we show the tongues of our Blaschke family. In (b) we zoom near the boundary of the big tongue intersecting the real line.

Theorem

Given any periodic point τ of the doubling map, T_τ is not empty and consists of three connected and simply connected components, each containing a unique parameter a_0 such that B_{a_0} has a superattracting cycle in $\mathbb{S}^1$. Moreover $|a_0| = 2$.

Idea of the proof

1. Perform a qc surgery connecting any $a \in T_\tau$ with a parameter $a_0 \in T_\tau$ having a superattracting cycle.
2. See that there exist only 3 parameters in T_τ having superattracting cycles.

Theorem

Given a tongue T_τ, it's boundary is the union of two curves which depend injectively on $|a|$. The parameter on the intersection of these curves is called the tip of the tongue.

A. Dezotti, *Connectedness of the Arnold tongues for double standard maps*, 2010

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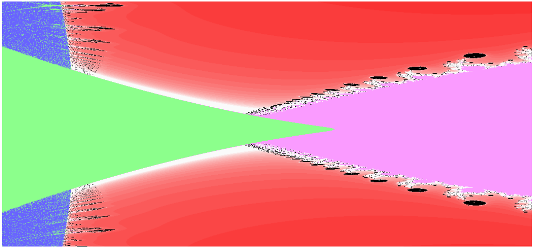


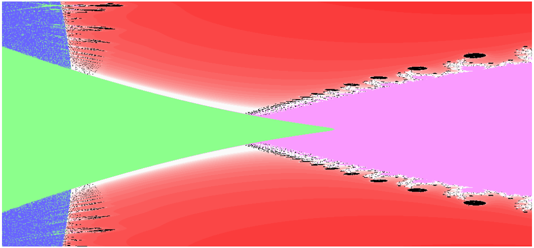
Theorem

Given any tongue T_τ, there exists a neighbourhood U of the tip of the tongue in which only one of the following can occur:

- $a \in T_\tau \Rightarrow B_{a|\mathbb{S}^1}$ *has an attracting periodic cycle.*
- $a \in \partial T_\tau$ *and* B_a *has a parabolic periodic cycle in* $\mathbb{S}^1$.
- $a \notin \overline{T_\tau}$ *and* B_a *has two different attracting periodic cycles outside* $\mathbb{S}^1$.

The result is obtained by using the index of the fixed points of B_a^n.

J. H. Hubbard, D. Schleicher, *Multicorns are not path connected*, 2012.



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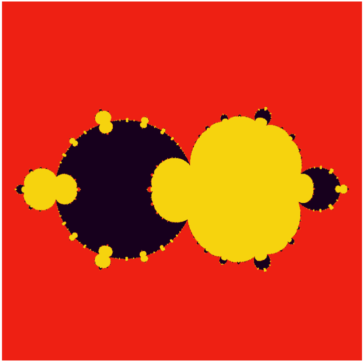
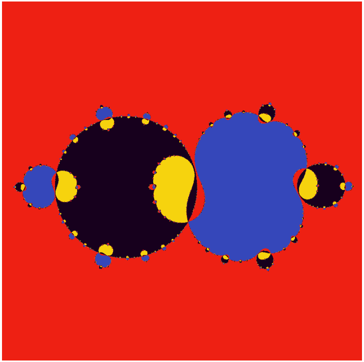
- $a \in \partial T_\tau$ *and* B_a *has a parabolic periodic cycle in* $\mathbb{S}^1$.
- $a \notin \overline{T_\tau}$ *and* B_a *has two different attracting periodic cycles outside* $\mathbb{S}^1$.

The result is obtained by using the index of the fixed points of B_a^n.

J. H. Hubbard, D. Schleicher, *Multicorns are not path connected*, 2012



(a) Dynamical plane of B_a, $a \in T_0$, $a = 2.83638 + i0.007233$.



(b) Dynamical plane of B_a, $a \notin \overline{T_0}$, $a = 2.81958 + 0.019833$.

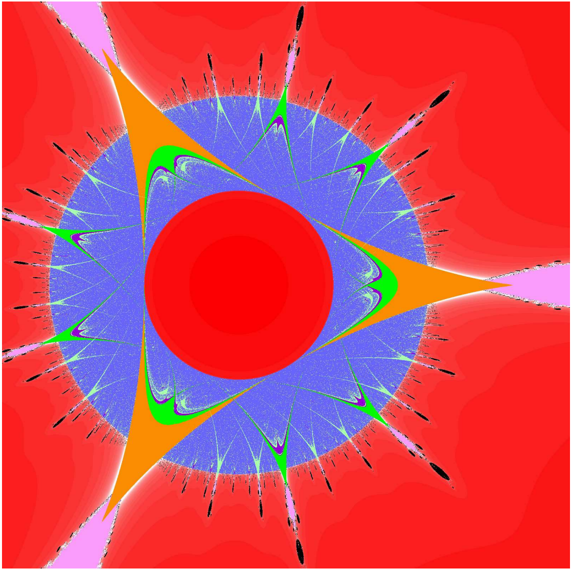


Figure: Parameter plane of the Blaschke family. The parameters correspond to $-2.5 < \mathrm{Re}(a) < 3.5$ and $-3 < \mathrm{Im}(a) < 3$.

Definition

An *extended tongue* ET_τ is defined to be the set of parameters for which the attracting cycle of T_τ can be continued.

Notice that, since there are two different critical points moving independently for $1 < |a| < 2$, two different tongues may intersect each other.

Theorem

Given an extended tongue ET_τ, its exterior boundary is a local graph with respect to $|a|$ which extends until reaching $|a| = 1$.

Corollary

All extended tongues extend until reaching the unit circle.

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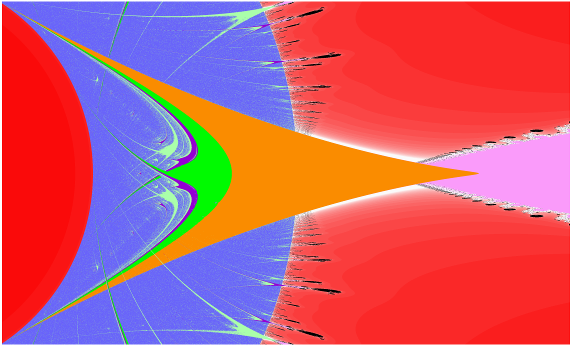

Theorem

The boundary of the extended fixed tongue ET_0 consists of an exterior component of parameters with multiplier 1 and an interior component of parameters with multiplier -1. A period doubling bifurcation takes place along the inner boundary.

Thank you for your attention!