On boundaries of multiply connected wandering domains

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Introduction

Definition (Wandering domain)

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Theorem (Sullivan 1982)

There are no wandering domains for rational functions.

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where $C > 0$ is a small constant, r_1 is large and $(r_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers that satisfies the recurrence relation

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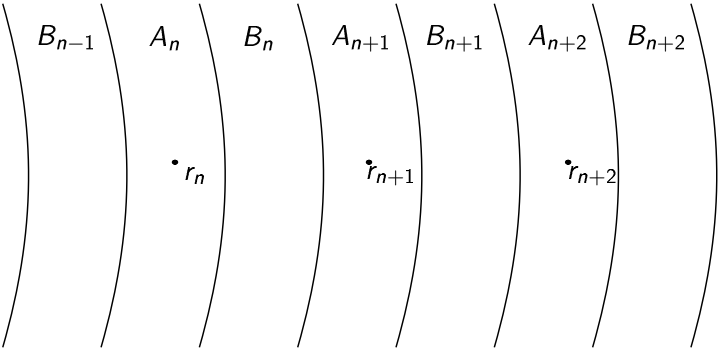
In 1976 Baker was able to show that the U_n are all different and therefore wandering domains.

$$\bullet\, r_n \xrightarrow{\quad f \quad} \bullet\, r_{n+1} \xrightarrow{\quad f \quad} \bullet\, r_{n+2}$$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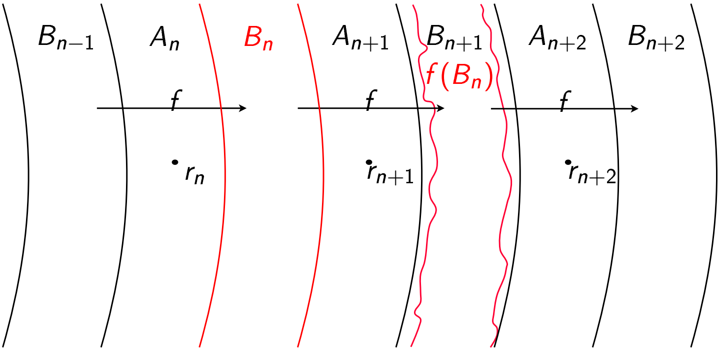

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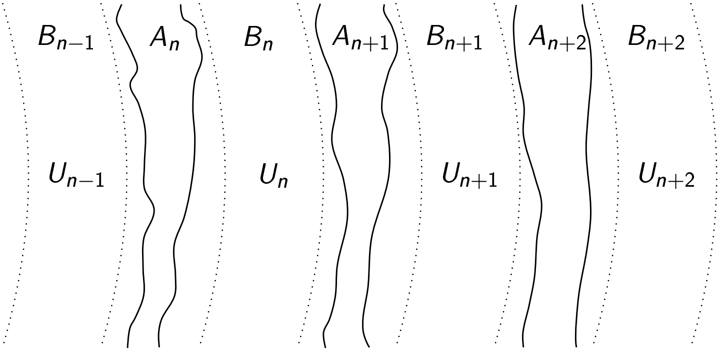

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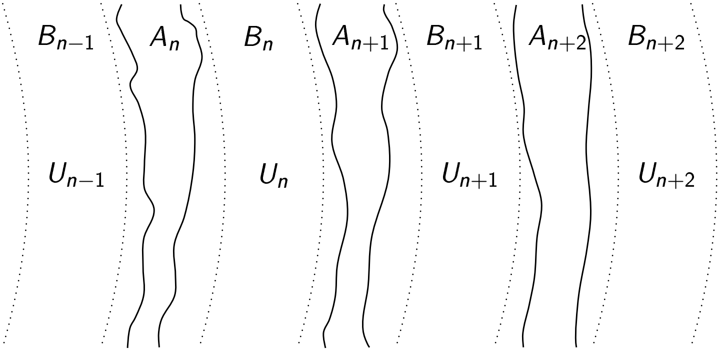

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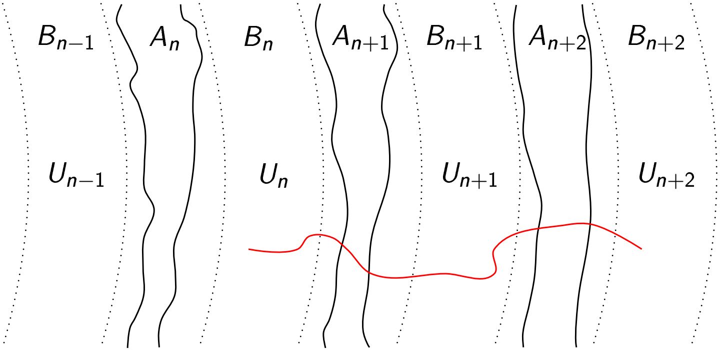

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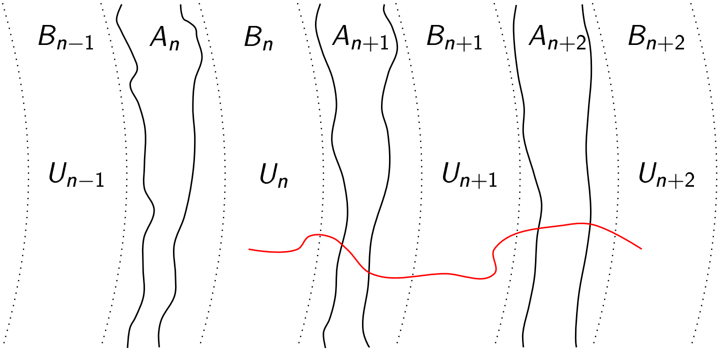

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- This implies that B_n belongs to a multiply connected Fatou component U_n.
- Assume that $U_n = U_m$ for $n \neq m$, then this implies that $U_n = U_m$ for all n, m.
- Baker showed that there are no unbounded multiply connected Fatou components.

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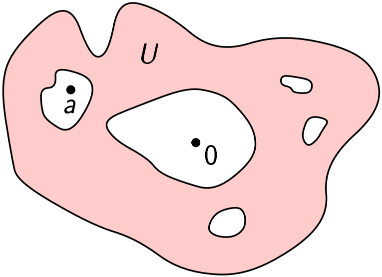
We want to show that under suitable conditions every boundary component of a multiply connected wandering domain is a curve or even a Jordan curve and therefore locally connected.

Results

Definition (Inner and outer boundary)

Let $U \subset \mathbb{C}$ be a domain and let $a \in \overline{\mathbb{C}} \setminus U$. We denote by $C(a, U)$ the component of $\overline{\mathbb{C}} \setminus U$ that contains a.



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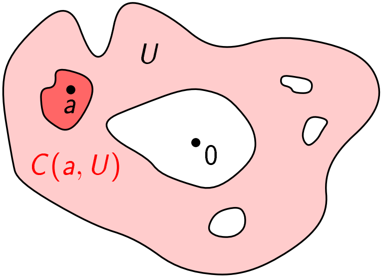

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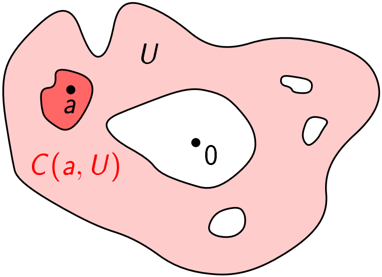

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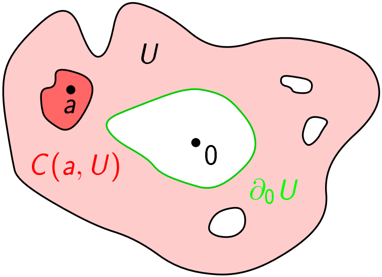

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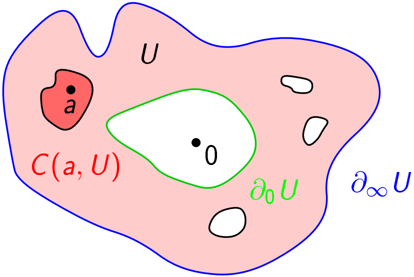

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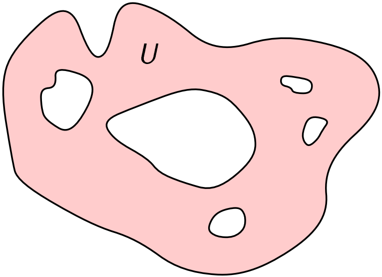

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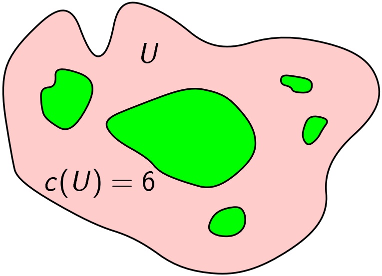

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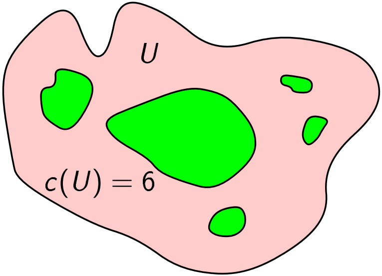

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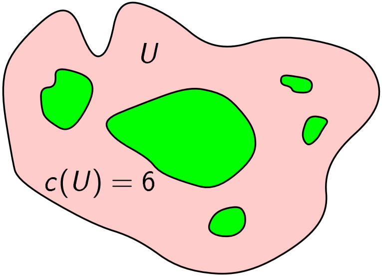

Kisaka and Shishikura showed that the eventual connectivity of a multiply connected wandering domain is either 2 or ∞.

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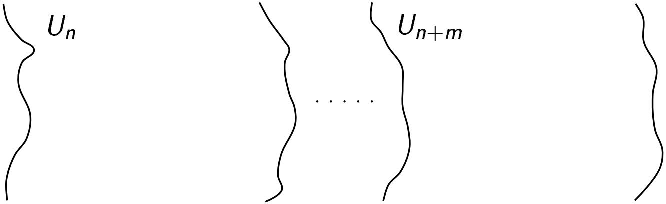

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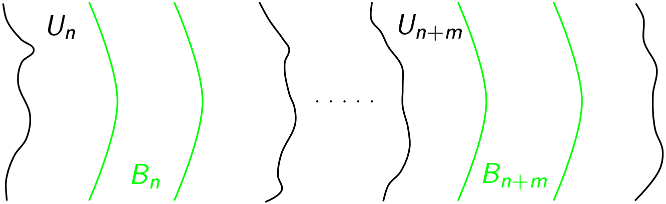

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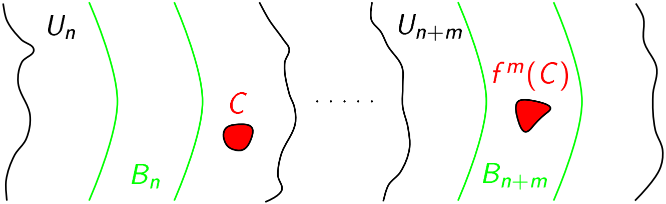

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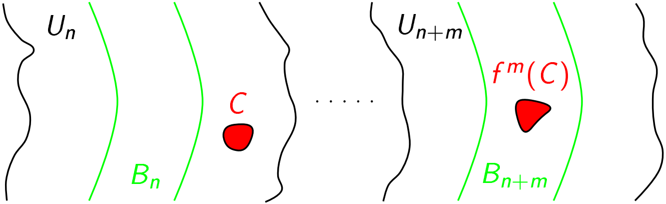

Definition (Inner connectivity)

We call $c(U_n \cap C(0, B_n))$ the *inner connectivity* of U_n and define the *eventual inner connectivity* respectively.

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Then all big boundary components are Jordan curves and $\partial_\infty U_{n-1} = \partial_0 U_n$.

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Both theorems work for Baker's first example of a wandering domain.

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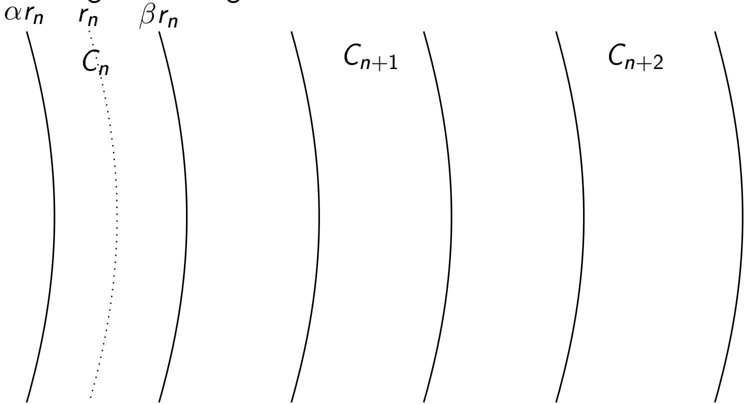

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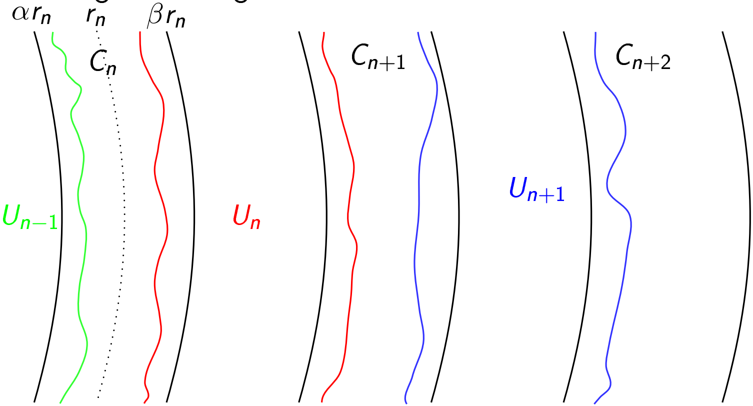

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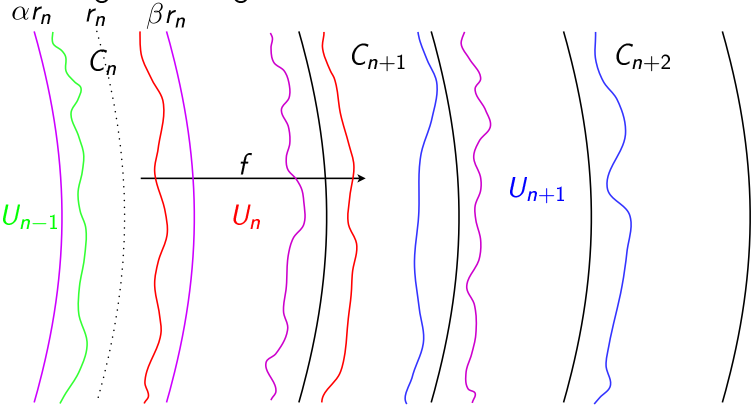

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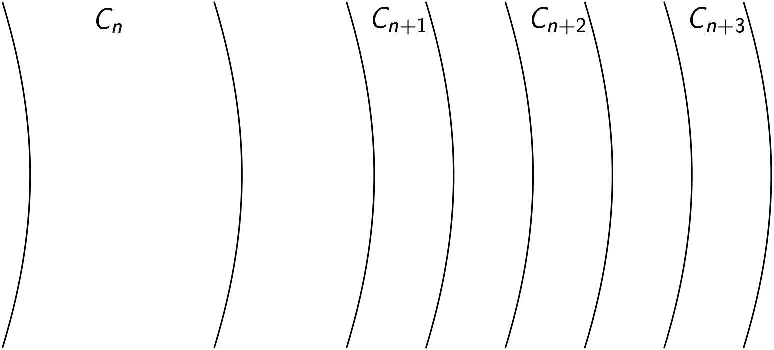

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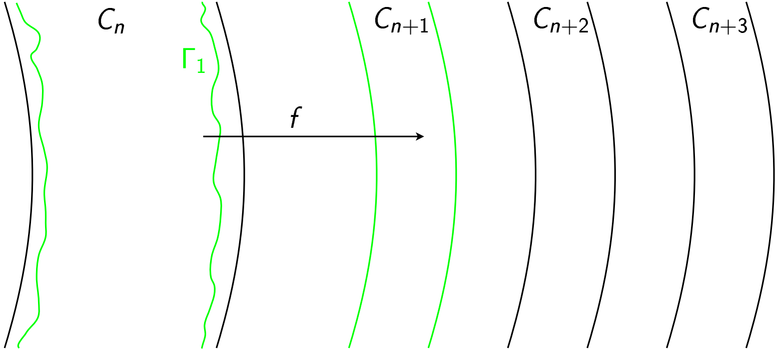

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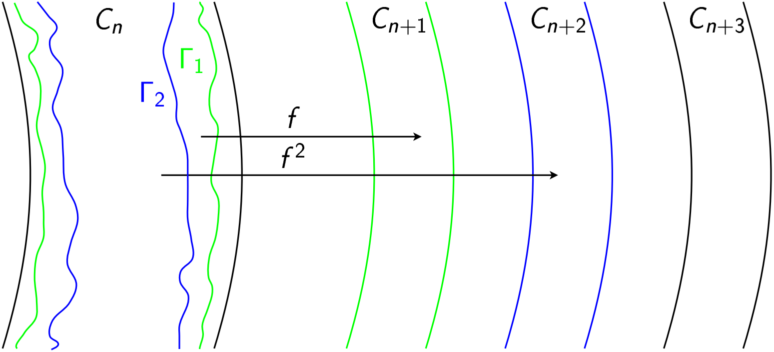

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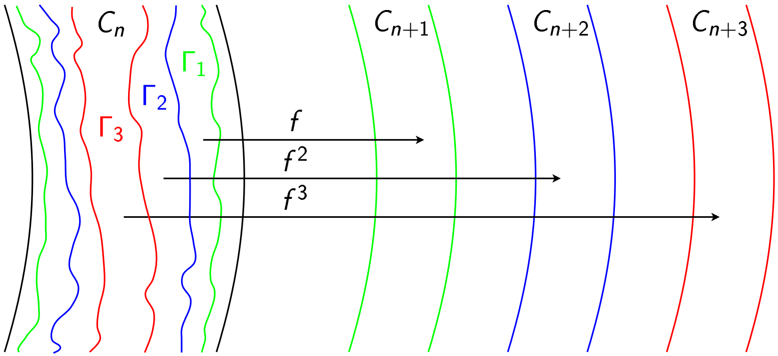

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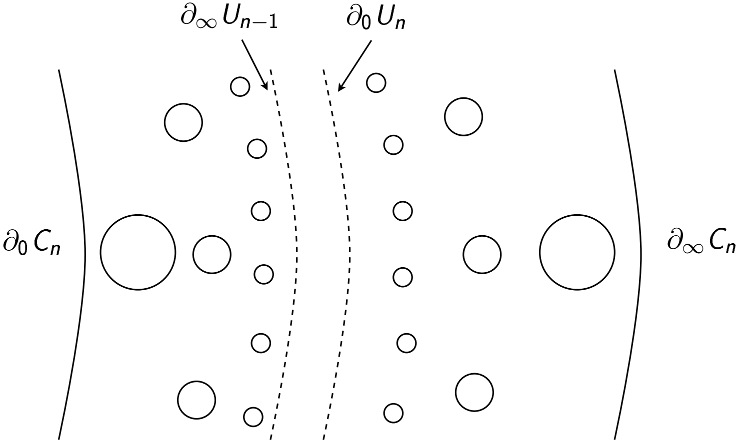
We parametrise now $\partial_0 \Gamma_k$ and $\partial_\infty \Gamma_k$ as curves by γ_k^0 and γ_k^∞ respectively. Thereby one has to check that the parametrisations are compatible with each other. Here $\text{Re}\left(\frac{z \cdot f'(z)}{f(z)}\right) > 0$ is used. It ensures that the curves are not distorted too much under iteration.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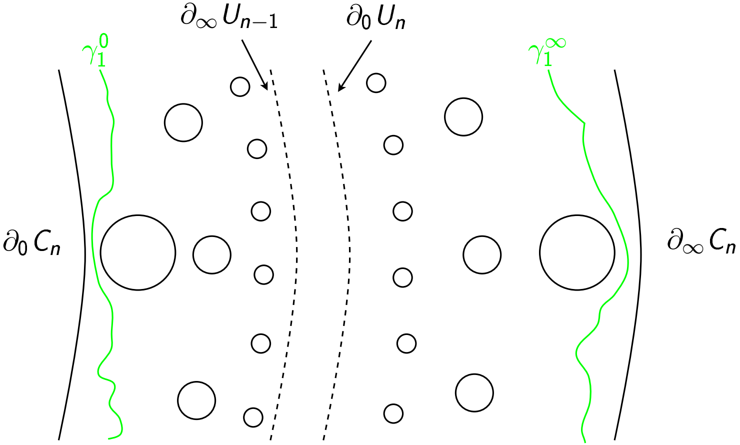

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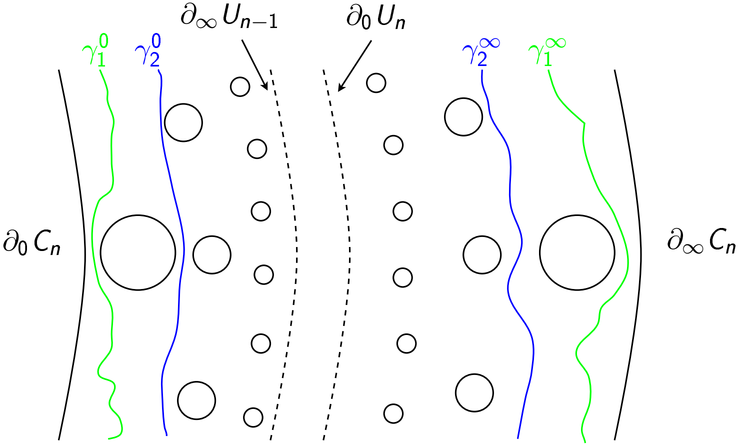

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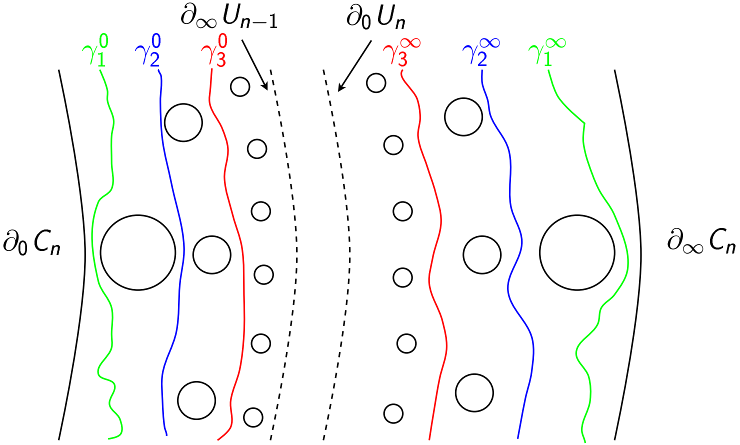
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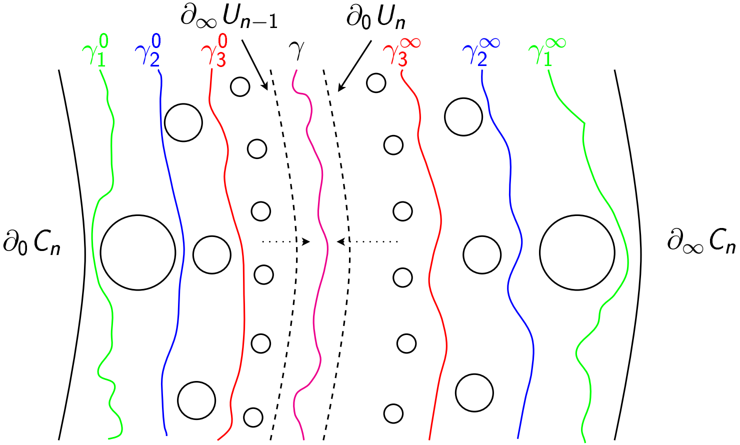

Then we use that f^{-k} is contracting to show that the curves γ_k^0 and γ_k^∞ converge uniformly to the same curve γ with

$$\mathrm{trace}(\gamma) = \bigcap_{k\in\mathbb{N}} \Gamma_k.$$

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We have proven theorem 1, so it remains to prove theorem 2.

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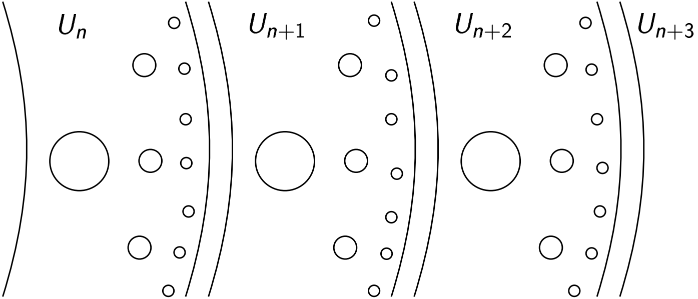

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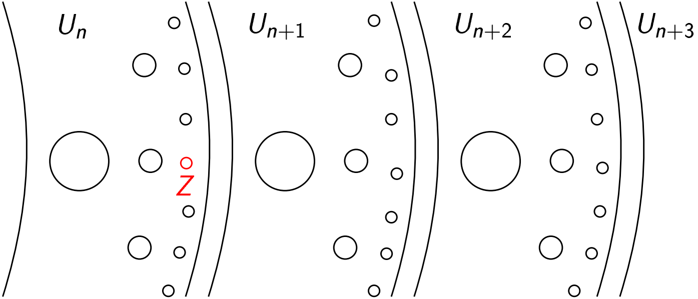

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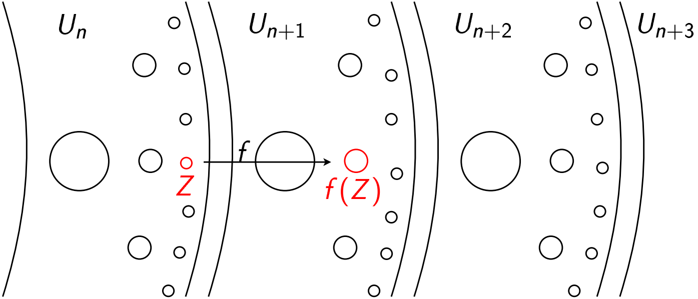

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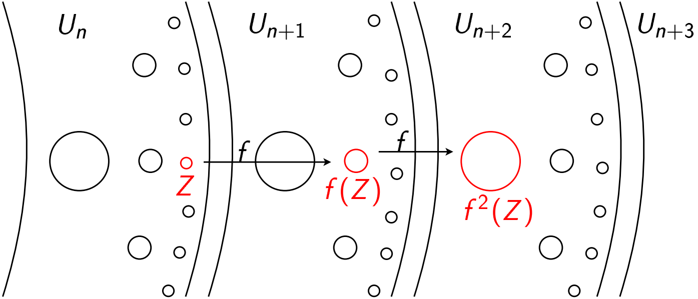

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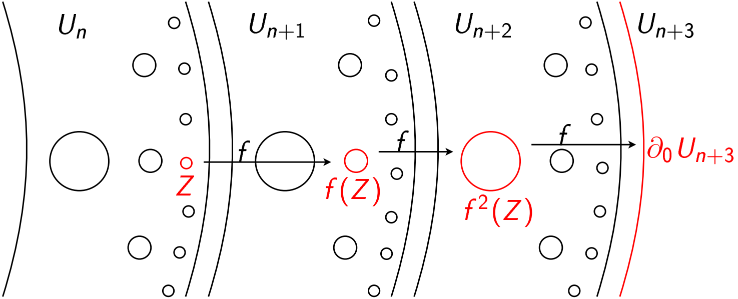

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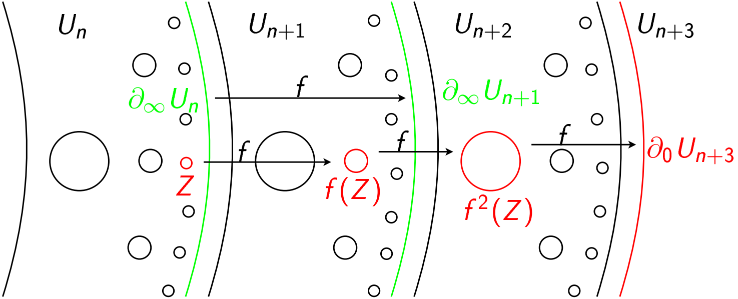

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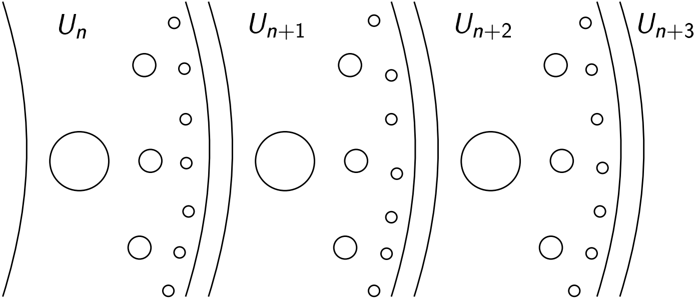

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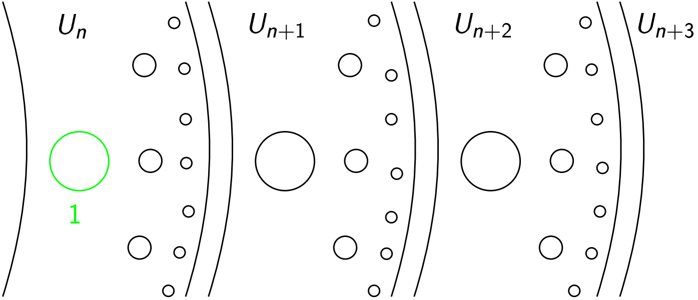

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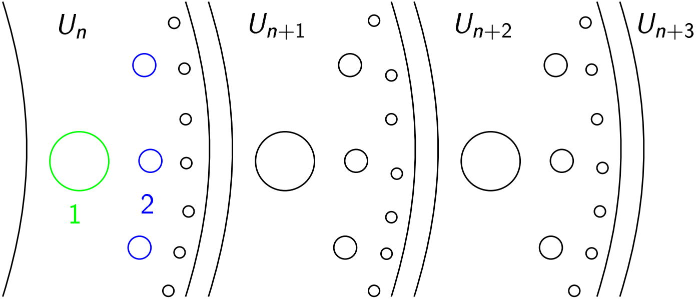

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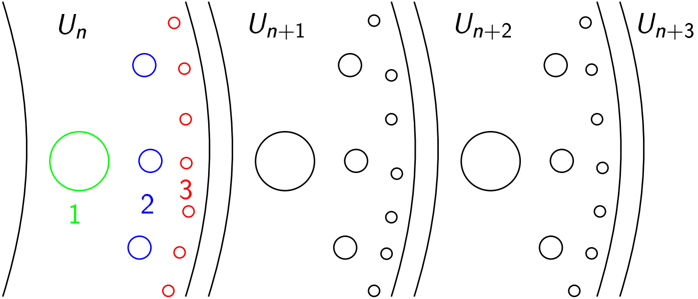

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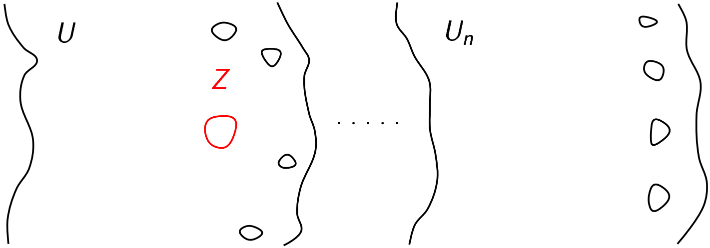

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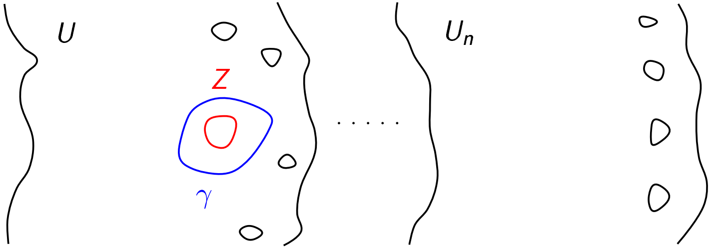

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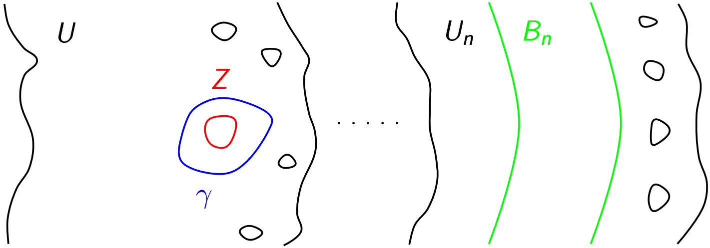

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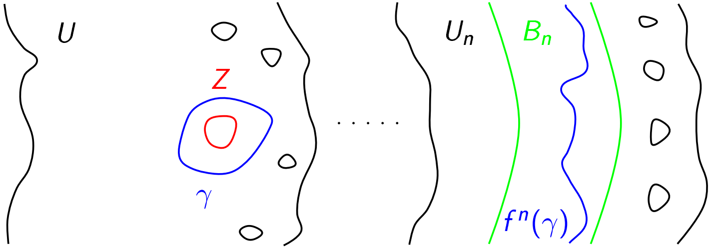

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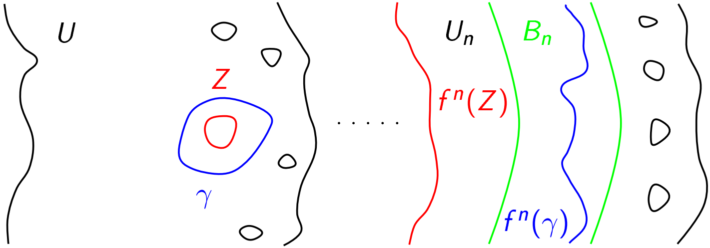

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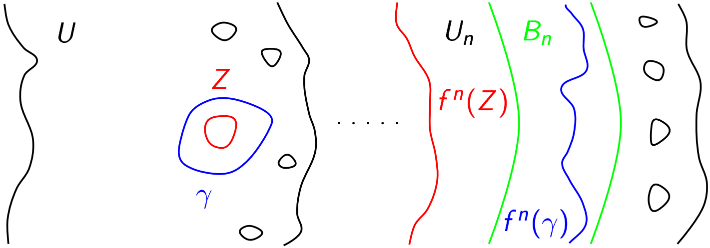

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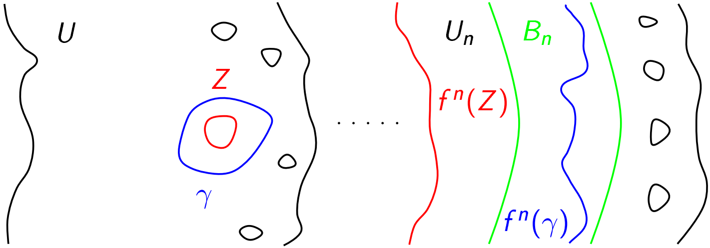

This implies that every boundary component of U will be eventually mapped onto a big boundary component.
Under the conditions of theorem 1 those big boundary components are Jordan curves, so every boundary component of U is either a curve or even a Jordan curve if there are no critical points in its forward orbit.

Examples

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Bergweiler's and Zheng's example

$$f(z) = C \cdot z^k \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right),$$

where $C > 0$, $k \in \mathbb{N}$ and $(a_j)_{j \in \mathbb{N}}$ is a complex sequence with $|a_j| = r_j$ and $(r_j)_{j \in \mathbb{N}}$ is a fast growing sequence of positive real numbers.

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This example includes the first example of Baker.

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Bergweiler and Zheng showed that Baker's first example of a wandering domain has also infinite connectivity.

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Our proofs use some of Bishop's ideas. But the arguments to show that the boundaries are $C^1 - curves$ do not work for the other examples.

Thank you for your attention.