Limit cycles for planar systems with invariant algebraic curves

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Objective

To study lower bounds on the number of limit cycles for planar systems with invariant algebraic curves.

Polynomial Differential Systems

Consider the polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are polynomials in x and y, and the dot denotes differentiation with respect to t.

The system (1) defines the associated vector field

$$X(x, y) = (P(x, y), Q(x, y)).$$

Invariant Algebraic Curves

Given a polynomial $f(x,y)$, the algebraic curve $f(x,y)=0$ is an *invariant algebraic curve* of system (1) if

$$\langle \nabla f(x,y), X(x,y) \rangle = f_x(x,y)\, P(x,y) + f_y(x,y)\, Q(x,y) = K(x,y)\, f(x,y),$$

for some polynomial $K(x,y)$.

The polynomial $K(x,y)$ is called the *cofactor* of the invariant algebraic curve $f(x,y)=0$.

Invariant Algebraic Curves

It is easy to see that if the polynomial vector field $X = (P, Q)$ has degree

$$n = \max\{\deg P, \deg Q\},$$

and the polynomial $f(x, y)$ has degree d, then the associated cofactor $K(x, y)$ satisfies

$$\deg K(x, y) \leq n - 1.$$

Invariant Algebraic Curves

All cofactors of a polynomial vector field of degree n belong to a vector space of polynomials of degree at most $n-1$.

The dimension of this vector space is equal to the number of monomials in a polynomial of degree at most $n-1$. Therefore, its dimension is

$$\frac{n\,(n+1)}{2}.$$

Invariant Algebraic Curves

Recall that $\nabla f(x, y)$ is normal to the curve $f(x, y) = 0$. Along this curve, the vector field $X = (P, Q)$ satisfies

$$\langle \nabla f(x, y), X(x, y) \rangle = K(x, y)\, f(x, y) = 0,$$

which implies that X is tangent to the curve at every point.

Therefore, at every point of $f(x, y) = 0$, the vector field X is tangent to the curve, so the curve is composed of trajectories of X.

This justifies the name *invariant algebraic curve*, since it remains invariant under the flow defined by X.

Invariant Algebraic Curves

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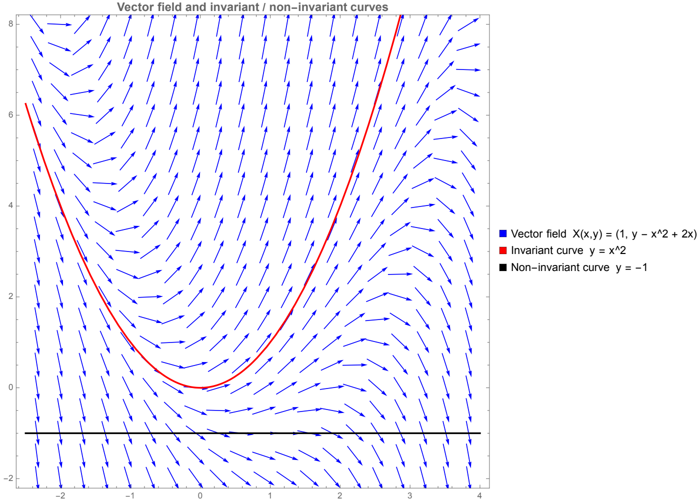

Motivation

Why study invariant algebraic curves?

Motivation

Remember that if a planar differential system is integrable on some open subset of the plane, then its phase portrait in this region is given by the level sets of its first integral, with the appropriate orientation.

Motivation

Darboux [1] presents a method for finding a first integral on the open set

$$\mathbb{R}^2 \setminus \Sigma, \quad \text{where } \Sigma = \{(x,y) \in \mathbb{R}^2 \mid (f_1 \cdot f_2 \cdots f_r)(x,y) = 0\}.$$

Here, each $f_i(x,y) = 0$ represents an irreducible invariant algebraic curve of the system, and λ_i are constants for $i = 1, 2, \ldots, r$, with $r \in \mathbb{N}$.

He proposes that a first integral can be written in the form

$$H(x,y) = \prod_{i=1}^{r} f_i(x,y)^{\lambda_i}.$$

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[1] G. Darboux, *Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré*, *Bulletin des sciences mathématiques et astronomiques*, vol. 2, no. 1, pp. 151–200, 1878.

Motivation

The derivative of this first integral along the system's orbits is given by

$$\frac{dH}{dt} = \sum_{i=1}^{r} \lambda_i \, f_1^{\lambda_1} \cdots f_i^{\lambda_i - 1} \cdot f_i^{\lambda_i + 1} \cdots f_r^{\lambda_r} \, \dot{f}_i = H \sum_{i=1}^{r} \lambda_i \frac{\dot{f}_i}{f_i}.$$

If each f_i is an invariant algebraic curve with associated cofactor K_i, i.e.,

$$\dot{f}_i = P \, f_{i,x} + Q \, f_{i,y} = K_i f_i,$$

then the derivative simplifies to

$$\frac{dH}{dt} = H \sum_{i=1}^{r} \lambda_i K_i(x, y).$$

Motivation

As we have already seen, the dimension of the cofactor space implies that if the system has

$$r \geq \frac{n(n+1)}{2} + 1$$

invariant curves, the corresponding cofactors are necessarily linearly dependent. Consequently, there exist nontrivial values of λ_i such that

$$\sum_{i=1}^{r} \lambda_i K_i(x,y) = 0,$$

which yields a first integral.

Motivation

In this context, Jouanolou [2] proved that if

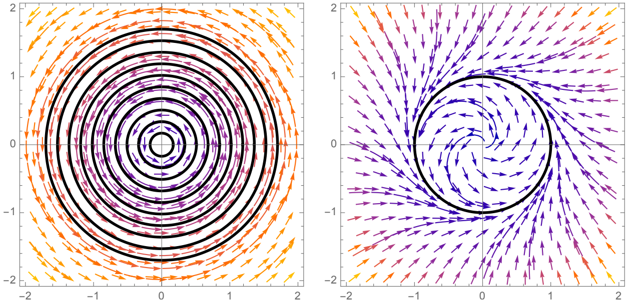
$$r \geq \frac{n(n+1)}{2} + 2,$$

then the system admits a rational first integral. Consequently, system admits an infinite family of invariant algebraic curves.

[2] J.-P. Jouanolou, *Équations de Pfaff algébriques*, vol. 708, Springer, 2006.

Limit cycles

Remember that a limit cycle is a periodic orbit of a planar differential system that is isolated from all other periodic orbits.



The investigation of the existence of limit cycles is relevant because they are the α or ω limit set of many other trajectories.

Nonexistence of Limit Cycles

It is a well-established fact that integrable planar differential systems do not admit limit.

Sufficient invariant curves $\Rightarrow$ **integrability** $\Rightarrow$ **no limit cycles**.

Invariant Regions

In this way, given an invariant algebraic curve $f = 0$:

- Any orbit starting on $f = 0$ remains on $f = 0$ for all time.
- Any orbit starting with $f \neq 0$ cannot cross the curve $f = 0$; the sign of f is preserved along the trajectory.

Therefore, the curve $f = 0$ acts as a geometric barrier, separating the phase plane into invariant regions.

And, from a geometric and intuitive viewpoint, systems admitting invariant algebraic curves are expected to possess fewer limit cycles.

Lotka–Volterra Systems

A classical example of planar differential systems with invariant algebraic curves is the Lotka–Volterra systems.

In 1925 and 1926, Alfred J. Lotka [3] and Vito Volterra [4] independently introduced a class of planar polynomial differential systems to model interactions between two populations sharing the same ecological niche.

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[3] A. J. Lotka, *Analytical Note on Certain Rhythmic Relations in Organic Systems*, *Proc. Natl. Acad. Sci. USA*, vol. 11, pp. 459–463, 1925.

[4] V. Volterra, *Variazioni e fluttuazioni del numero d'individui in specie animali conviventi*, *Mem. R. Accad. Lincei*, vol. 2, pp. 31–113, 1927.

Lotka–Volterra Systems

These systems, known as Lotka–Volterra systems, can be written as

$$\dot{x} = x\,(a_0 + a_1 x + a_2 y), \qquad \dot{y} = y\,(b_0 + b_1 x + b_2 y), \tag{2}$$

with $a_i, b_i \in \mathbb{R}$ for $i = 0, 1, 2$.

Since x and y represent population densities, the relevant dynamics occur in the positive quadrant.

Kolmogorov Systems

In 1936, Kolmogorov [5] extended the Lotka–Volterra systems to arbitrary degrees and dimensions. A planar Kolmogorov system is given by

$$\dot{x} = x\,X_m(x,y), \qquad \dot{y} = y\,Y_m(x,y), \tag{3}$$

where X_m and Y_m are polynomials of degree m.

As expected, the case $m = 1$ reduces to the classical Lotka–Volterra system.

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[5] A. Kolmogorov, *Sulla teoria di Volterra della lotta per l'esistenza*, *Giornale dell'Istituto Italiano degli Attuari*, vol. 7, pp. 74–80, 1936.

Hilbert's 16th Problem

The second part of Hilbert's 16th problem asks for the maximum number of limit cycles of planar polynomial vector fields of a given degree.

The maximum number of limit cycles of planar polynomial vector fields of degree n is usually denoted by $H(n)$.

The maximum number of limit cycles that can bifurcate from an equilibrium point is called the local cyclicity.

Hilbert's 16th Problem

Therefore, there also exists a local version of Hilbert's 16th problem.

In this context, the maximum number is denoted by $\mathcal{M}(n)$, and clearly,

$$\mathcal{M}(n) \leq H(n).$$

Hilbert's 16th Problem

We are now interested in this problem in the context of Kolmogorov systems.

Let us define $\mathcal{M}_K(n)$ as the maximal number of limit cycles that can bifurcate from a single equilibrium point of Kolmogorov system with degree

$$n = m + 1.$$

Consequently, we consider an even more restricted version of the 16th Hilbert problem, and therefore

$$\mathcal{M}_K(n) \leq \mathcal{M}(n) \leq H(n).$$

Limit Cycles in Kolmogorov Systems

In the case $n = 1$, we have a linear system then $\mathcal{M}_K(1) = 0$.

Moreover, Bautin [6] proved that $\mathcal{M}_K(2) = 0$.

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[6] N. N. Bautin, *On periodic solutions of systems of differential equations (in Russian)*, *Prikl. Mat. Mekh.*, vol. 18, p. 128, 1954.

Limit Cycles in Kolmogorov Systems

For a specific family, N. G. Lloyd et al. [7] showed that the maximum order of a weak focus is six, which implies that $\mathcal{M}_K(3) \geq 6$.

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[7] N. G. Lloyd, J. M. Pearson, E. Saéz, I. Szántó, *A cubic Kolmogorov system with six limit cycles*, *Computers & Mathematics with Applications*, vol. 44, no. 3–4, pp. 445–455, 2002.

Limit Cycles in Kolmogorov Systems

For $n = 4$, the authors in [8] provide an example showing that $\mathcal{M}_K(4) \geq 8$.

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[8] C. Du, Y. Liu, W. Huang, *Behavior of limit cycle bifurcations for a class of quartic Kolmogorov models in a symmetrical vector field*, *Applied Mathematical Modelling*, vol. 40, no. 5–6, pp. 4094–4108, 2016.

New Lower Bounds in Kolmogorov Systems

In [9], we improve the previously known lower bounds for $\mathcal{M}_K(n)$ in the quartic and quintic Kolmogorov cases.

Moreover, we present a new cubic Kolmogorov system such that

$$\mathcal{M}_K(3) \geq 6.$$

Based on the currently known results, we present the following table summarizing the values of $\mathcal{M}(n)$ and $\mathcal{M}_K(n)$.

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[9] Y. R. Carvalho, L. P. C. Da Cruz, L. F. S. Gouveia, *New lower bound for the Hilbert number in low degree Kolmogorov systems*, *Chaos, Solitons & Fractals*, vol. 175, Art. no. 113937, 2023.

New Lower Bounds in Kolmogorov Systems

Degree n	General system	Kolmogorov system
1	$\mathcal{M}(1) = 0,$	$\mathcal{M}_K(1) = 0,$
2	$\mathcal{M}(2) = 3\,[10],$	$\mathcal{M}_K(2) = 0\,[6],$
3	$\mathcal{M}(3) \geq 12\,[11],$	$\mathcal{M}_K(3) \geq 6\,[7, 9],$
4	$\mathcal{M}(4) \geq 21\,[11],$	$\mathcal{M}_K(4) \geq 13\,[9],$
5	$\mathcal{M}(5) \geq 33\,[12],$	$\mathcal{M}_K(5) \geq 22\,[9].$

[6] N. N. Bautin, *On periodic solutions of systems of differential equations (in Russian)*, *Prikl. Mat. Mekh.*, vol. 18, p. 128, 1954.

[7] N. G. Lloyd, J. M. Pearson, E. Saéz, I. Szántó, *A cubic Kolmogorov system with six limit cycles*, *Computers & Mathematics with Applications*, vol. 44, no. 3–4, pp. 445–455, 2002.

[9] Y. R. Carvalho, L. P. C. Da Cruz, L. F. S. Gouveia, *New lower bound for the Hilbert number in low degree Kolmogorov systems*, *Chaos, Solitons & Fractals*, vol. 175, Art. no. 113937, 2023.

[10] N. N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium state of the type focus or center*, *American Mathematical Society Translations*, vol. 100, no. 1, p. 19, 1954.

[11] J. Giné, L. F. S. Gouveia, J. Torregrosa, *Lower bounds for the local cyclicity for families of centers*, *Journal of Differential Equations*, vol. 275, pp. 309–331, 2021.

[12] L. F. S. Gouveia, J. Torregrosa, *Lower bounds for the local cyclicity of centers using high order developments and parallelization*, *Journal of Differential Equations*, vol. 271, pp. 447–479, 2021.

New Lower Bounds in Kolmogorov Systems

The usual method for obtaining these lower bounds is through the analysis of a degenerate Hopf bifurcation.

We now recall the technique employed to address this problem, which is based on the consideration of perturbations of specific centers.

Therefore, it is necessary to have a mechanism to identify "good centers" or, alternatively, to have a comprehensive list of families suitable for studying this problem.

Lyapunov Quantities

One approach to study a degenerate Hopf bifurcation is to compute the Lyapunov quantities by considering a planar perturbed system of the form

$$\begin{aligned} \dot{x} &= \omega x - y + \sum_{i\geq 2}^{n} P_i(x, y, \lambda), \\ \dot{y} &= x + \omega y + \sum_{i\geq 2}^{n} Q_i(x, y, \lambda), \end{aligned} \tag{4}$$

where P_i and Q_i are homogeneous polynomials of degree i in (x, y).

The perturbation parameters λ are given by the coefficients of the polynomials P_i and Q_i.

Lyapunov Quantities

One possible approach to compute the Lyapunov quantities is to search for a first integral via a Taylor series expansion.

This consists in looking for a formal solution of (4) when $\omega = 0$ of the form

$$H(x,y) = x^2 + y^2 + \sum_{i \geq 3} H_i(x,y), \tag{5}$$

where H_i are homogeneous polynomials of degree i.

Lyapunov Quantities

In general, such a first integral does not exist, since its existence requires that

$$\frac{dH}{dt} = \dot{H}(x,y) = \langle \nabla H(x,y), X(x,y) \rangle = H_x(x,y)\,\dot{x} + H_y(x,y)\,\dot{y} = 0,$$

which is a condition that is not always satisfied.

Lyapunov Quantities

The coefficients of homogeneous terms of odd degree admit a unique solution, since the determinant of the associated linear system is nonzero.

By contrast, the coefficients of even degree do not admit a unique solution, as the corresponding determinant always vanishes; in many cases, the system may have no solution at all.

Lyapunov Quantities

This lack of solvability leads to a compatibility condition, which is precisely expressed in terms of the Lyapunov quantities.

It is always possible to choose the coefficients of the formal power series (5) such that the following equation is satisfied:

$$\frac{dH}{dt} = \dot{H}(x,y) = H_x(x,y)\,\dot{x} + H_y(x,y)\,\dot{y} = \sum_{k\geq 1} L_k\,(x^2+y^2)^{2k}. \qquad (6)$$

Lyapunov Quantities

Any non-zero L_k is an obstruction for the origin to be a center.

The quantities L_k are polynomials in the perturbative parameters λ.

See [13,14] for more details.

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[13] V. G. Romanovski and D. S. Shafer, *The center and cyclicity problems: a computational algebra approach*, Birkhäuser Boston, Ltd., Boston, MA, 2009.

[14] C. Christopher, *Estimating limit cycle bifurcations from centers*, in *Differential Equations with Symbolic Computation*, Trends Math., pp. 23–35, Birkhäuser, Basel, 2005.

Family of Centers in the Kolmogorov System

To apply the techniques for obtaining limit cycles, we need centers that produce a significant number of cycles.

This allows us to state the following result:

Family of Centers in the Kolmogorov System

Proposition 1.1 ([9]). For the cubic Kolmogorov system

$$\dot{x} = x(y-1)(a_2b_1x + a_1a_3y + a_1b_1 - a_1a_3 - a_2b_1)/(-b_1^2),$$
$$\dot{y} = y(x-1)(b_1b_2x + a_1b_3y + a_1b_1 - a_1b_3 - b_1b_2)/(a_1^2),$$

the equilibrium point $(1,1)$ is a center if, and only if, one of the following conditions holds:

($\mathcal{C}_1$) $a_2 = b_2 = 0$;
($\mathcal{C}_2$) $a_2 = b_3 = 0$;
($\mathcal{C}_3$) $a_3 = b_2 = 0$;
($\mathcal{C}_4$) $a_3 = b_3 = 0$;
($\mathcal{C}_5$) $a_2 - b_2 = b_3 - a_3 = 0$;
($\mathcal{C}_6$) $a_3 - b_2 = b_1 - a_1 = b_3 - a_2 = 0$;
($\mathcal{C}_7$) $a_3 + b_2 = b_1 + a_1 = a_2 + b_3 = 0$;
($\mathcal{C}_8$) $\mathbf{a_1 + (a_3^2 - b_2^2)/b_2 = a_2 + b_2 = b_1 - (a_3^2 - b_2^2)/a_3 = b_3 + a_3 = 0}$.

[9] Y. R. Carvalho, L. P. C. Da Cruz, L. F. S. Gouveia, *New lower bound for the Hilbert number in low degree Kolmogorov systems*, *Chaos, Solitons & Fractals*, vol. 175, Art. no. 113937, 2023.

Family of Centers in the Kolmogorov System

Outline of the proof. The solution of the Center–Focus Problem for this specific system is divided into steps:

(i) Compute a finite number of Lyapunov quantities.

(ii) Determine the conditions on the system parameters that annul these Lyapunov quantities.

(iii) Verify whether the conditions found are sufficient to guarantee that the origin is a center.

Small-Amplitude Limit Cycles

In some of our results, we consider centers in polynomial families of planar vector fields. More specifically, we consider the following perturbed system

$$\begin{aligned} x' &= P_c(x,y,\mu) + \omega x + \sum_{i\geq 2}^{n} P_i(x,y,\lambda), \\ y' &= Q_c(x,y,\mu) + \omega y + \sum_{i\geq 2}^{n} Q_i(x,y,\lambda), \end{aligned} \tag{7}$$

where P_i and Q_i are homogeneous polynomials of degree i in (x, y) such that

$$P_i(x,y,0) = Q_i(x,y,0) = 0, \quad \text{for all } i \geq 2.$$

Small-Amplitude Limit Cycles

The number of limit cycles may vary with the parameter μ.

We can use the previously described algorithm to compute the Lyapunov quantities $L_k(\mu, \lambda)$ for the perturbed system (11) by taking $\omega = 0$.

These quantities are polynomials in λ and satisfy

$$L_k(\mu, 0) = 0.$$

$$\begin{aligned} x' &= P_c(x, y, \mu) + \omega x + \sum_{i \geq 2}^{n} P_i(x, y, \lambda), \\ y' &= Q_c(x, y, \mu) + \omega y + \sum_{i \geq 2}^{n} Q_i(x, y, \lambda), \end{aligned} \tag{11}$$

Small-Amplitude Limit Cycles

Then, we can perform a Taylor expansion at $\lambda = 0$ to obtain

$$L_k(\mu,\lambda) = \underbrace{L_k(\mu,0)}_{0} + \underbrace{\sum_i \frac{\partial L_k}{\partial \lambda_i}(\mu,0)\,\lambda_i}_{\text{linear term in } \lambda} + \text{higher-order terms.}$$

Denoting the linear terms in λ by $L_k^{[1]}(\mu,\lambda)$ we can use the Implicit Function Theorem to provide a lower bound for the number of limit cycles.

Small-Amplitude Limit Cycles

This approach appears in the works of Chicone and Jacobs [15] and Han [16].

It was further elaborated by Christopher [14].

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[14] C. Christopher, *Estimating limit cycle bifurcations from centers*, in *Differential Equations with Symbolic Computation*, Trends Math., pp. 23–35, Birkhäuser, Basel, 2005.

[15] C. Chicone and M. Jacobs, *Bifurcation of critical periods for plane vector fields*, Trans. Amer. Math. Soc., **312**(2), pp. 433–486, 1989.

[16] M. Han, *Liapunov constants and Hopf cyclicity of Liénard systems*, Ann. Differential Equations, 15(2):113–126, 1999.

Small-Amplitude Limit Cycles

In essence, this method asserts that, by fixing a generic value μ in the center variety, if the ordered set

$$\{L_1^{[1]}(\mu,\lambda), L_2^{[1]}(\mu,\lambda), \ldots, L_k^{[1]}(\mu,\lambda)\}$$

has rank k, then, after introducing the trace parameter ω, one can obtain k small-amplitude limit cycles bifurcating from the center.

Small-Amplitude Limit Cycles

For certain special families, non-generic critical values of μ exist for which the cyclicity may increase.

The following result addresses this type of degenerate bifurcation.

Small-Amplitude Limit Cycles

Proposition 2.5 ([12]). We assume that, after an appropriate change of coordinates in the parameter space, the Lyapunov quantities admit the representation

$$W_j = \begin{cases} \lambda_j + O_2(\lambda), & j = 1, \dots, k-1, \\ \sum_{l=1}^{k-1} g_{j,l}(\mu)\lambda_l + f_{j-k}(\mu)\lambda_k + O_2(\lambda), & j = k, \dots, k+\ell, \end{cases}$$

where $O_2(\lambda)$ denotes terms of degree at least 2 in λ, whose coefficients depend analytically on μ.

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[12] L. F. S. Gouveia, J. Torregrosa, *Lower bounds for the local cyclicity of centers using high order developments and parallelization*, *Journal of Differential Equations*, vol. 271, pp. 447–479, 2021.

Small-Amplitude Limit Cycles

Proposition 2.5 ([12]). If there exists a point μ^* such that

$$f_0(\mu^*) = \cdots = f_{\ell-1}(\mu^*) = 0, \qquad f_\ell(\mu^*) \neq 0,$$

and the Jacobian matrix of $(f_0, \ldots, f_{\ell-1})$ with respect to μ has rank ℓ at μ^*, then the system (11) has $k + \ell$ small-amplitude hyperbolic limit cycles bifurcating from the origin.

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[12] L. F. S. Gouveia, J. Torregrosa, *Lower bounds for the local cyclicity of centers using high order developments and parallelization*, *Journal of Differential Equations*, vol. 271, pp. 447–479, 2021.

$$\begin{aligned} x' &= P_c(x, y, \mu) + \omega x + \sum_{i \geq 2}^{n} P_i(x, y, \lambda), \\ y' &= Q_c(x, y, \mu) + \omega y + \sum_{i \geq 2}^{n} Q_i(x, y, \lambda), \end{aligned} \tag{11}$$

Small-Amplitude Limit Cycles

Proposition 4.1 ([9]). There exist parameter perturbations such that the center family $\mathcal{C}_8$, given by

$$\dot{x} = \frac{x(y-1)(b_2^2 x + a_3^2 y - 2b_2^2)a_3}{(a_3 - b_2)(a_3 + b_2)b_2}, \qquad \dot{y} = \frac{y(x-1)(b_2^2 x + a_3^2 y - 2a_3^2)b_2}{(a_3 - b_2)(a_3 + b_2)a_3},$$

can produce at least six small-amplitude limit cycles bifurcating from $(1,1)$.

[9] Y. R. Carvalho, L. P. C. Da Cruz, L. F. S. Gouveia, *New lower bound for the Hilbert number in low degree Kolmogorov systems*, *Chaos, Solitons & Fractals*, vol. 175, Art. no. 113937, 2023.

Small-Amplitude Limit Cycles

Five arise from the linear part of the Lyapunov quantities, and one arises from the system parameters.

Therefore, to highlight these facts, we may write

$$\mathcal{M}_K(3) \geq 6 = 5 + 1.$$

Small-Amplitude Limit Cycles

Recall that it is essential to identify centers capable of producing a significant number of limit cycles.

A natural way to achieve this for higher-degree systems is to build upon lower-degree ones by multiplying them by curves of equilibria.

Consequently, for a suitable choice of parameters in the family $(\mathcal{C}_8)$, combined with an appropriate line of equilibria, we obtain the following result.

Small-Amplitude Limit Cycles

Proposition 4.3 ([9]) .There exist parameter perturbations such that the following family of centers,

$$\dot{x} = \frac{x(y-1)(4x-8+y)\,(2bx-ay+a-2b-3)}{18},$$
$$\dot{y} = \frac{2\,y(x-1)(4x-2+y)\,(2bx-ay+a-2b-3)}{9}, \tag{12}$$

can produce at least thirteen small-amplitude limit cycles bifurcating from the center at $(1,1)$.

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[9] Y. R. Carvalho, L. P. C. Da Cruz, L. F. S. Gouveia, *New lower bound for the Hilbert number in low degree Kolmogorov systems*, *Chaos, Solitons & Fractals*, vol. 175, Art. no. 113937, 2023.

Small-Amplitude Limit Cycles

For an appropriate choice of the perturbation parameters, eleven limit cycles originate from the linear part of the Lyapunov quantities, whereas two arise from the system parameters.

Thus, to make this decomposition explicit, we write

$$\mathcal{M}_K(4) \geq 13 = 11 + 2.$$

Small-Amplitude Limit Cycles

It is possible that the linear parts of the subsequent Lyapunov quantities are linear combinations of the first k ones.

Small-Amplitude Limit Cycles

Addressing this type of problem, Christopher [14] extended his results on the generation of limit cycles from the linear part of the Lyapunov quantities.

In particular, he showed that, in some cases, it is possible to obtain additional limit cycles by using higher-order Taylor expansions of the Lyapunov constants.

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[14] C. Christopher, *Estimating limit cycle bifurcations from centers*, in *Differential Equations with Symbolic Computation*, Trends Math., pp. 23–35, Birkhäuser, Basel, 2005.

Small-Amplitude Limit Cycles

With this in mind, we were able to establish and prove the following result.

Small-Amplitude Limit Cycles

Proposition 4.6 ([9]). For the center of the Kolmogorov quintic system

$$\begin{aligned}\dot{x} &= \quad \frac{1}{A}\,(8x+4y-21)\,P(x,y)\,x\,(y-1),\\ \dot{y} &= -\frac{1}{A}\,(8x+4y-21)\,P(x,y)\,y\,(x-1),\end{aligned} \tag{8}$$

where

$$P(x,y) = 29x^2 - 40xy + 40y^2 + 162x + 140y + 479.$$

there exist quintic polynomial perturbations such that at least 22 limit cycles bifurcate from the point $(1,1)$, where $A = 7290$.

[9] Y. R. Carvalho, L. P. C. Da Cruz, L. F. S. Gouveia, *New lower bound for the Hilbert number in low degree Kolmogorov systems*, *Chaos, Solitons & Fractals*, vol. 175, Art. no. 113937, 2023.

Piecewise-Smooth Kolmogorov Systems

Since it is clear that $\mathcal{M}_K(1) = 0$ and Bautin [6] showed that $\mathcal{M}_K(2) = 0$, there has also been interest in the study of Kolmogorov systems in a piecewise-smooth point of view.

For example, we highlight the following works.

[17] Y. R. Carvalho, L. F. S. Gouveia, and O. Makarenkov, *Crossing limit cycles in piecewise smooth Kolmogorov systems: An application to Palomba's model*, *Communications in Nonlinear Science and Numerical Simulation*, vol. 143, Art. no. 108646, 2025.

[18] L. P. C. Da Cruz, R. Oliveira, and J. Torregrosa, *Limit cycles in piecewise quadratic Kolmogorov systems*, *Communications in Nonlinear Science and Numerical Simulation*, Art. no. 109285, 2025.

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[6] N. N. Bautin, *On periodic solutions of systems of differential equations (in Russian)*, *Prikl. Mat. Mekh.*, vol. 18, p. 128, 1954.

Work in Progress

Consider the planar Kolmogorov system

$$\dot{x} = x\,X_m(x,y), \qquad \dot{y} = y\,Y_m(x,y), \tag{9}$$

where X_m and Y_m are polynomials of degree m.

Work in Progress

Observe that the degree-two curve $f(x, y) = xy = 0$ is invariant, since the associated vector field $X(x, y) = (x\, X_m(x, y),\, y\, Y_m(x, y))$ satisfies

$$\begin{aligned}\langle \nabla f(x,y), X(x,y)\rangle &= \langle (y,x), (x\, X_m(x,y),\, y\, Y_m(x,y))\rangle \\ &= yx\, X_m(x,y) + xy\, Y_m(x,y) \\ &= xy\, \underbrace{\big(X_m(x,y) + Y_m(x,y)\big)}_{\text{cofactor}}.\end{aligned}$$

Work in Progress

When perturbing centers within this class of systems, suitable restrictions on the perturbation parameters are required to obtain meaningful results.

Indeed, after the perturbation, it is required that the curve $xy = 0$ remains invariant.

Work in Progress

These restrictions on the perturbation parameters, together with the techniques we have discussed, provide further evidence of why the number of limit cycles tends to be smaller than in the case of arbitrary perturbations.

We have also seen geometrically that the existence of invariant curves influences the regions in which limit cycles can appear.

In general, these facts are valid for systems with invariant curves.

Work in Progress

For planar quadratic polynomial systems with an invariant straight line, there can be at most one limit cycle; see, for instance:

[19] W. A. Coppel, *Some quadratic systems with at most one limit cycle*, *Dynam. Report. Expositions Dynam. Systems (N.S.)*, vol. 2, pp. 61–88, 1989.

[20] L. A. Cherkas and L. I. Zhilevich, *The limit cycles of certain differential equations*, *Differentsial'nye Uravneniya*, vol. 8, no. 7, pp. 1207–1213, 1972.

Work in Progress

Planar quadratic polynomial systems with an invariant hyperbola do not admit limit cycles.

By contrast, quadratic systems with an invariant ellipse or parabola have at most one limit cycle; see the following work:

[21] A. Gasull and H. Giacomini, *Number of limit cycles for planar systems with invariant algebraic curves*, *Qualitative Theory of Dynamical Systems*, vol. 22, no. 2, p. 44, 2023.

Work in Progress

Planar quadratic polynomial systems with an invariant algebraic curve of degree three have at least one limit cycle; for instance, some examples are highlighted in the following work.

[22] J. Chavarriga and I. García, *Existence of limit cycles for real quadratic differential systems with an invariant cubic*, *Pacific J. Math.*, vol. 223, no. 2, pp. 201–218, 2006.

In this work, it is also shown that the limit cycle is never contained within the invariant algebraic curve.

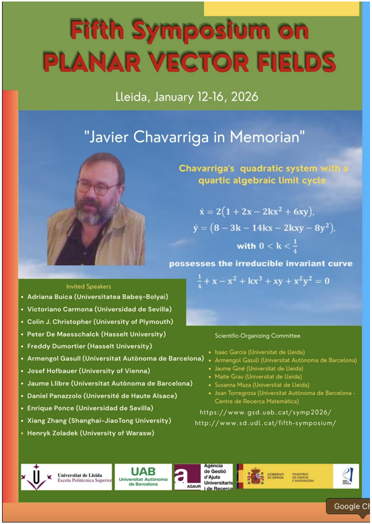
Work in Progress

And what about planar quadratic polynomial systems with an invariant algebraic curve of degree four?

Work in Progress

Did you see the poster for the event?

Work in Progress



Work in Progress

The quadratic polynomial system shown on the event poster is an example of a quadratic system with an invariant curve of degree four that has at least one limit cycle.

More details and other examples can be seen in the following references.

[23] A. I. Yablonskii, *On limit cycles of a certain differential equation*, *Differentsial'nye Uravneniya*, vol. 2, no. 3, pp. 335–344, 1966.

[24] V. F. Filiptsov, *Algebraic limit cycles*, *Differentsial'nye Uravneniya*, vol. 9, pp. 1281–1288, 1973 (in Russian; translated in *Differential Equations*, vol. 9, pp. 983–988, 1973).

[25] J. Chavarriga, J. Llibre, and J. Sorolla, *Algebraic limit cycles of degree 4 for quadratic systems*, *Journal of Differential Equations*, vol. 200, no. 2, pp. 206–244, 2004.

Work in Progress

Planar quadratic polynomial systems with an invariant algebraic curve of degree five or six also have at least one limit cycle, as can be seen in the following works.

[26] M. Alberich-Carramiñana, A. Ferragut, and J. Llibre, *Quadratic planar differential systems with algebraic limit cycles via quadratic plane Cremona maps*, *Advances in Mathematics*, vol. 389, 107924, 2021.

[27] C. Christopher, J. Llibre, and G. Świrszcz, *Invariant algebraic curves of large degree for quadratic systems*, *Journal of Mathematical Analysis and Applications*, vol. 303, no. 2, pp. 450–461, 2005.

Work in Progress

In the world of planar polynomial systems with invariant algebraic curves, several directions can be explored.

One of them is the number of limit cycles in families with invariant algebraic curves of degree d.

Work in Progress

The cubic family still presents challenges related to the center-focus problem and local cyclicity.

Thus, in a joint work in progress with Armengol Gasull and Joan Torregrosa, we are interested in this type of problem for cubic systems.

Work in Progress

In summary, we are first interested in the number of limit cycles in cubic polynomial systems with an algebraic invariant curve of degree d.

Work in Progress

Preliminary result: Denoting by $H_d(3)$ the number of limit cycles for cubic polynomial systems with an invariant algebraic curve of degree d, we have:

$$H_1(3) \geq 9, \quad H_2(3) \geq 7, \quad H_3(3) \geq 5, \quad H_4(3) \geq 3, \quad \text{and} \quad H_5(3) \geq 1.$$

Work in Progress

Thank you for your attention!

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