Reduction of Elementary Integrability of Polynomial Vector Fields

Xiang ZHANG (张 祥)

xzhang@sjtu.edu.cn

V Symposium on Planar Vector Fields

Lleida, January 12-16, 2026

January 16, 2026

- Background on elementary integrability

- New results on reduction of elementary integrability

- Sketch proofs to the new results

Background: Prelle and Singer proved in [TAMS 1983] the next results (**roughly speak**):

- If a system of ODEs defined on a differential field K has a first integral in an elementrary field extension L of K, then it must have a first integral consisting of algebraic elements over K via their constant powers and logarithms.

- Based on this result they further showed that an elementary integrable planar polynomial differential system (**PDS**) has an integrating factor which is a fractional power of a rational function.

Question:

- What about reduction of elementary integrability for higher dimensional **PDSs**?

Our answer:

- A higher dimensional PDS having k f.i. elementary F.I. has k f.i. F.I. consisting of algebraic functions.

- Any finite dimensional elementary integrable **PDS** has a rational Jacobian multiplier.

Question:

- What about reduction of elementary integrability for higher dimensional **PDSs**?

Our answer:

- A higher dimensional PDS having k f.i. elementary F.I. has k f.i. F.I. consisting of algebraic functions.

- Any finite dimensional elementary integrable **PDS** has a rational Jacobian multiplier.

Question:

- What about reduction of elementary integrability for higher dimensional **PDSs**?

Our answer:

- A higher dimensional PDS having k f.i. elementary F.I. has k f.i. F.I. consisting of algebraic functions.

- Any finite dimensional elementary integrable **PDS** has a rational Jacobian multiplier.

Definition: A differential field $(\mathscr{F}, \Delta)$ consists of

- the field $\mathscr{F}$, and
- the set Δ of commutative derivations on $\mathscr{F}$.

A diff. field (L, Δ) is

- an elementary field extension of the diff. field (K, Δ),

if this extension can be written in the **tower form**

$$K = K_0 \subset K_1 \subset \cdots \subset K_i \subset K_{i+1} \subset \cdots \subset K_r = L$$

such that

- K_{i+1} is a finite algebraic extension of K_i; or
- $K_{i+1} = K_i(t)$, where t is transcendental and satisfies that
    for each $\delta \in \Delta$, $\exists$ an $h \in K_i$ such that $\delta t / t = \delta h$; or
- $K_{i+1} = K_i(t)$, where t is transcendental and satisfies that
    for each $\delta \in \Delta$, $\exists$ an $h \in K_i$ such that $\delta t = \delta h / h$.

Figure: **Tower, 塔**
YangZhou/Jiangsu–Stone/Tang dynasty/one thousand years ago
Ying County/Shanxi–Wooden/Song dynasty/ established in 1056

Consider the autonomous rational vector field

$$\mathscr{X} := P_1(x)\frac{\partial}{\partial x_1} + \cdots + P_n(x)\frac{\partial}{\partial x_n}, \tag{1}$$

where $x = (x_1,\ldots,x_n) \in \mathbb{C}^n$, and $P_1,\ldots,P_n \in K := \mathbb{C}(x)$.

Prelle and Singer [1] in 1983 provided a reduction on elementary
first integrals of $\mathscr{X}$

---

[1]M. J. Prelle and M. F. Singer. Elementary first integrals of differential
equations. Trans. Amer. Math. Soc., 279(1):215–229, 1983

Theorem A. *Let $(L, \Delta)/(K, \Delta)$ be*

- *an elementary field extension with $C(L, \Delta) = C(K, \Delta)$.*

Assume that

- $C(L, \Delta)$ *is a proper subset of $C(L, \mathscr{X})$.*

Then $\exists$

- $w_0, w_1, \ldots, w_m \in L$, *algebraic over K;*
- $c_1, \ldots, c_m \in C(K, \Delta)$, *constants*

such that

$$\mathscr{X}(w_0) + \sum_{i=1}^{m} c_i \frac{\mathscr{X}(w_i)}{w_i} = 0,$$

and for some $\delta \in \Delta$

$$\delta(w_0) + \sum_{i=1}^{m} c_i \frac{\delta(w_i)}{w_i} \neq 0$$

Simply saying:

- A **rational vector field** which has an elementary first integral

  has a **first integral of the form**

$$H = w_0 + \sum_i \ln w_i^{c_i}$$

  where w_i's are algebraic elements over K, or written in Darboux-like first integral

$$F = e^H = \left(\Pi_i w_i^{c_i}\right) e^{w_0}$$

Restricted to a planar polynomial vector field $\mathscr{P}$,

$\star$) Prelle and Singer [1] further proved

Proposition A. *If a planar PVF $\mathscr{P}$ has an elementary first integral, the next hold.*

(a) *The vector field $\mathscr{P}$ has an algebraic integrating factor.*

(b) *The vector field $\mathscr{P}$ has*

    – either a nontrivial rational first integral,

    – or an elementary integrating factor R such that R^r is in K

      for some $r \in \mathbb{Z}$.

Remark:

- In 1997, Man and MacCallum [2] attempted to implement Prelle and Singer's result through symbolic integration.

- In 2007, Avellar et al [3] also provided some algorithm to find elementary first integrals of rational second order ODE.

- In 2019, Christopher et al [4] focused on when a planar elementary integrable PDS is also Darboux integrable, among others.

---

[2]Y.-K. Man and M. A. H. MacCallum. A rational approach to the Prelle-Singer algorithm. J. Symbolic Comput., 24(1):31–43, 1997

[3]J. Avellar, L. G. S. Duarte, S. E. S. Duarte, and L. A. C. P. da Mota. A semi-algorithm to find elementary first order invariants of rational second order ODEs. Appl. Math. Comput., 184(1):2–11, 2007

[4]C. Christopher, J. Llibre, C. Pantazi, and S. Walcher. On planar polynomial vector fields with elementary first integrals. J. Differential Equations, 267(8):4572–4588, 2019

Remark:

- In 1997, Man and MacCallum [2] attempted to implement Prelle and Singer's result through symbolic integration.

- In 2007, Avellar et al [3] also provided some algorithm to find elementary first integrals of rational second order ODE.

- In 2019, Christopher et al [4] focused on when a planar elementary integrable PDS is also Darboux integrable, among others.

---

[2] Y.-K. Man and M. A. H. MacCallum. A rational approach to the Prelle-Singer algorithm. J. Symbolic Comput., 24(1):31–43, 1997

[3] J. Avellar, L. G. S. Duarte, S. E. S. Duarte, and L. A. C. P. da Mota. A semi-algorithm to find elementary first order invariants of rational second order ODEs. Appl. Math. Comput., 184(1):2–11, 2007

[4] C. Christopher, J. Llibre, C. Pantazi, and S. Walcher. On planar polynomial vector fields with elementary first integrals. J. Differential Equations, 267(8):4572–4588, 2019

Remark: Reduction of elementary integrability

- was extended to the Liouvillian class

by Singer [5] in 1992 for **planar PVFs**:

- existence of an integrating factor being exponential of integration of a rational one-form

Recall that Liouvillian field extension of a differential field

- is an extension which can be written in the tower form, with t in the **third item** of elementary field extension, replaced by t satisfying $\delta t \in K_i$ for each $\delta \in \Delta$.

---

[5]M. F. Singer. Liouvillian first integrals of differential equations. Trans. Amer. Math. Soc., 17 333(2):673–688, 1992

Remark: Reduction of elementary integrability

- was extended to the Liouvillian class

by Singer [5] in 1992 for **planar PVFs**:

- existence of an integrating factor being exponential of integration of a rational one-form

Recall that Liouvillian field extension of a differential field

- is an extension which can be written in the tower form, with t in the **third item** of elementary field extension, replaced by t satisfying $\delta t \in K_i$ for each $\delta \in \Delta$.

---

[5]M. F. Singer. Liouvillian first integrals of differential equations. Trans. Amer. Math. Soc., 17 333(2):673–688, 1992

♠ Christopher [6] provided further reduction on Singer's result:

- if a planar PDS is Liouvillian integrable, it has a Darboux integrating factor.

♠ Zoladek[7] provided also an approach to Christopher's result in terms of monodromy group of the first integral.

♠ Zhang [8] generalized the results of Singer[5] and Christopher[6] to n–dimensional Liouvillian integrable PDS and provided a reduction via Darboux Jacobian multipliers

---

[6]C. Christopher. Liouvillian first integrals of second order polynomial differential equations. Electron. J. Differential Equations, pages No. 49, 7, 1999

[7]H. Zoladek. The extended monodromy group and Liouvillian first integrals. J. Dynam. Control Systems, 4(1):1–28, 1998

[8]X. Z.. Liouvillian integrability of polynomial differential systems. Trans. Amer. Math. Soc., 368(1):607–620, 2016.

♠ Christopher [6] provided further reduction on Singer's result:

- if a planar PDS is Liouvillian integrable, it has a Darboux integrating factor.

♠ Zoladek[7] provided also an approach to Christopher's result in terms of monodromy group of the first integral.

♠ Zhang [8] generalized the results of Singer[5] and Christopher[6] to n–dimensional Liouvillian integrable PDS and provided a reduction via Darboux Jacobian multipliers

---

[6]C. Christopher. Liouvillian first integrals of second order polynomial differential equations. Electron. J. Differential Equations, pages No. 49, 7, 1999

[7]H. Zoladek. The extended monodromy group and Liouvillian first integrals. J. Dynam. Control Systems, 4(1):1–28, 1998

[8]X. Z.. Liouvillian integrability of polynomial differential systems. Trans. Amer. Math. Soc., 368(1):607–620, 2016.

Very recently,

- Aziz et al[9] extended Singer's result to **higher dimensional Lioiuvillian integrable complex rational VFs**, showing existence of a first integral obtained by two successive integrations from a one-form with coefficients in a finite algebraic extension L of the rational function field K, among others

- Christopher et al[10] further concerned **exceptional VFs**: the choice $L = K$ is not possible. They showed existence of the exceptional VFs in dimension three.

---

[9]W. Aziz, C. Christopher, H. Pantazi and S. Walcher, Liouvillian integrability of vector fields in higher dimensions, arXiv:2512.15522, 2025

[10]C. Christopher, H. Pantazi and S. Walcher, Liouvillian integrability of rational vector fields: The case of algebraic extensions, arXiv:2512.22138, 2025

Related to these kinds of reductions are algebraic integrability by Bruns [11]:

reduction of algebraic integrability to rational integrability

Lemma 3.1

Let

- $(F, \mathscr{Y})$ be a differential field,
- $(L, \mathscr{Y})$ be an algebraic extension of $(F, \mathscr{Y})$.

If $H \in L \setminus F$ satisfies $\mathscr{Y}(H) = 0$, then

- all coefficients of the minimal characteristic polynomial for H belong to $C(F, \mathscr{Y})$.

Algebraic integrable $\Leftrightarrow$ rational integrable

---

[11]H. Bruns. Uber die integrale des vielkörper-problems. Acta Math., 11:25–96, 1887

Our main results, denote by $\overline{F}$ the algebraic closure of a field F

Theorem 1.1

Let (L, Δ) be an elementary field extension of

- the differential field (K, Δ), with $C(L, \Delta) = C(K, \Delta)$.

Assume that $C(L, \Delta) \subsetneqq C(L, \mathscr{X})$, and

- $\mathscr{X}$ has $k\ (1 \le k \le n-1)$ functionally independent first integrals $H_1, \ldots, H_k$ in L.

Then there exist k functionally independent first integrals $\widetilde{H}_1, \ldots, \widetilde{H}_k$, which are of the form

$$\widetilde{H}_j = u_{0,j} + \sum_{i=1}^{n_j} c_{i,j} \ln u_{i,j},$$

where $u_{0,j}, \ldots, u_{n_j,j} \in \overline{K} \cap L$; $c_{1,j}, \ldots, c_{n_j,j} \in \mathbb{C}$, $j = 1, \ldots, k$.

Remark:

- When $k = 1$, Theorem 1.1 is the one of Prelle and Singer[1].

- Here we extend their result to any $k \in \{1, \ldots, n-1\}$ number of functionally independent elementary first integrals.

- The difficulty in proving Theorem 1.1 is to find functionally independent first integrals of the given simpler form in the elementary field extension.

Based on Theorem 1.1, we have the next reduction

Theorem 1.2

If the vector field $\mathscr{X}$ is elementary integrable, i.e. it has

$n-1$ functionally independent elementary first integrals,

then $\mathscr{X}$ has

- either a rational first integral in $\mathbb{C}(x)$,
- or a rational Jacobian multiplier in $\mathbb{C}(x)$.

This result is a generalization of

- Proposition A obtained by Prelle and Singer[1]

  from two dimension to any finite dimension.

Let

- F be an extension field of a field E, denoted by F/E.

Definition: Let E be a field

- $\alpha \in F$ is an algebraic element over E if there exists a polynomial P with coefficients in E such that
  $$P(\alpha) = 0.$$

- Otherwise, α is a transcendental element over E.

- If all the elements of F are algebraic over E, we call F/E an algebraic field extension of E.

The next can be found in Sect. 3.2 of Hodge and Pedoe[12].

For the field extension F/E, and $\xi \in F$,

- (a) $E[\xi] = E$ if and only if $\xi \in E$.

- (b) If ξ is algebraic over E, then $E(\xi) = E[\xi]$.

- (c) If ξ is transcendental over E, then

     $E[\xi]$ is equivalent to an integral domain $E[X]$,

  with X an indeterminate.

- (d) In this case, $E(\xi)$ is a transcendental extension of E.

- (e) $E(\xi)$ is the smallest extension of E which contains ξ.

- (f) Similarly, we can define the field $E(\xi_1, \ldots, \xi_n)$ by
  $E(\xi_1, \ldots, \xi_{n-1})(\xi_n)$

.

---

[12]W. V. D. Hodge and D. Pedoe. Methods of algebraic geometry. Vol. I. Cambridge Univ. Press, 1994

In the above sense, for a transcendental element λ over K, let

- $P(\lambda)$ and $R(\lambda)$ be polynomials in λ with coefficients in K.

View P and R as elements in the ring $K[\lambda]$ (or the field $K(\lambda)$), we can explore

- their coprimality, division and factorization.

- their functional independence and smoothness.

For a field extension F/K,

- $y_1, \ldots, y_k \in F$ are *algebraically dependent* over E
  if there exists a nontrivial $f \in E[X_1, \ldots, X_k]$ such that
  $$f(y_1, \ldots, y_k) = 0.$$

- Otherwise, $y_1, \ldots, y_k$ are algebraically independent

Let $S = \{y_\alpha : \alpha \in I\} \subset F$, with I an index set,

- be algebraically independent over E.

If the cardinality of S is greatest among all such subsets, it is

- the transcendence degree of F over E, denote by $\text{tr}[F : E]$.

As a result,

- $\text{tr}[F : E] = 0$ means that F is algebraic over E.

Refer to Chapter VIII of Lang [13],

<div style="border:1px solid">

Lemma 2.1

Let $U \subset E \subset F$ be extension fields. Then

$$\operatorname{tr}[F : U] = \operatorname{tr}[F : E] + \operatorname{tr}[E : U].$$

</div>

Moreover, if S is maximal with respect to the inclusion ordering,

- we call S a transcendence base of F over E.

From the maximality, if S is a transcendence base of F over E, then F is algebraic over $E(S)$

---

[13] S. Lang. Algebra, GTM Vol 211, Springer-Verlag, New York, 3rd edition, 2002

The next lemma, Theorem 2.16 of Olver [14], reveals a crucial aspect of the functional independence.

Lemma 2.2

Assume that

- $M \subset \mathbb{R}^n$ is a smooth manifold,

- $g_1, \ldots, g_k$ are real smooth functions defined on M.

Then $g_1, \ldots, g_k$ are functionally dependent on M if and only if for each $x \in M$, there exists a neighborhood U of x and a smooth real function $F(z_1, \ldots, z_k)$ of k variables such that

$$F(g_1(x), \ldots, g_k(x)) \equiv 0., \qquad x \in U.$$

This result is stated in real, but it also works in complex setting.

[14] P. J. Olver. Applications of Lie groups to differential equations, Vol 107, GTM. Springer-Verlag, New York, 2nd edition, 1993

Instead of proving Theorem 1.1 directly, we prove

Theorem 3.1

Let (K, Δ), (L, Δ) and $\mathscr{X}$ be as those in Theorem 1.1. Assume

$$\exists\, u_{0,j}, \ldots, u_{m_j,j} \in L, \text{ and } d_{1,j}, \ldots, d_{m_j,j} \in \mathbb{C}, j = 1, \ldots, k,$$

such that

- $\mathscr{X}(H_j) = 0, j = 1, \ldots, k$, where $H_j = u_{0,j} + \sum\limits_{i=1}^{m_j} d_{i,j} \ln u_{i,j}$;

- $\delta(H_j) \neq 0$ for some $\delta \in \Delta, j = 1, \ldots, k$;

- $H_1, \ldots, H_k$ are functionally independent.

Then $\exists\, w_{0,j}, \ldots, w_{\widetilde{m}_j,j} \in L, j = 1, \ldots, k$, algebraic over K,

and $c_{0,j}, \ldots, c_{\widetilde{m}_j,j} \in \mathbb{C}$, such that

- $\mathscr{X}(\widetilde{H}_j) = 0, j = 1, \ldots, k$, where $\widetilde{H}_j = w_{0,j} + \sum\limits_{i=1}^{\widetilde{m}_j} c_{i,j} \ln w_{i,j}$;

- $\delta(\widetilde{H}_j) \neq 0$ for some $\delta \in \Delta, j = 1, \ldots, k$;

- $\widetilde{H}_1, \ldots, \widetilde{H}_k$ are functionally independent.

The proof of Theorem 3.1 distinguishes two cases

- All $\delta H_j \in K$, $j \in \{1, \ldots, k\}$, $\delta \in \Delta$

- $\exists$ some $\delta H_j \notin K$

For all $\delta H_j \in K$, proof of Theorem 3.1 follows from

Proposition 3.1

Assume that

- the functionally independent first integrals $H_1, \ldots, H_k$ of $\mathscr{X}$ satisfy $\delta H_j \in K$, $j = 1, \ldots, k$, for all $\delta \in \Delta = \{\partial_{x_1}, \cdots, \partial_{x_n}\}$.

Then there exist $\widetilde{H}_j$, $j = 1, \ldots, k$, being of the form

$$\widetilde{H}_j = w_{0,j} + \sum_{i=1}^{m_j} c_{i,j} \ln w_{i,j}, \quad w_{0,j}, w_{1,j}, \ldots, w_{m_j,j} \in K, \quad c_{1,j}, \ldots, c_{m_j,j} \in \mathbb{C},$$

which are functionally independent first integrals of $\mathscr{X}$.

Proof of Proposition 3.1 needs Theorem 3 of Rosenlicht [15], i.e.

Theorem 3.2

Let

- $\widetilde{k}$ be a differential field of characteristic zero, and
- for each given derivation D of $\widetilde{k}$, let $\alpha_D \in \widetilde{k}$.

Then there exists an elementary differential field extension of $\widetilde{k}$

- having the same constants field as $\widetilde{k}$
- containing an element y

such that $Dy = \alpha_D$ if and only if there are constants $c_1, \ldots, c_\ell \in \widetilde{k}$
and elements $u_1, \ldots, u_\ell, v \in \widetilde{k}$, such that

$$\alpha_D = Dv + \sum_{i=1}^{\ell} c_i \frac{Du_i}{u_i}.$$

---

[15]M. Rosenlicht. On Liouville's theory of elementary functions. Pacific J. Math., 65(2):485–492, 1976

Proof: Applying Theorem 3.2 to H_j, $j \in \{1, \ldots, k\}$, yields that

- there are $w_{0,j}, \ldots, w_{m_j,j} \in K$ and $c_{1,j}, \ldots, c_{m_j,j} \in \mathbb{C}$,
  $m_1, \ldots, m_k \in \mathbb{N}$, such that, for all $\delta \in \Delta$,

$$\delta H_j = \delta w_{0,j} + \sum_{i=1}^{m_j} c_{i,j} \frac{\delta w_{i,j}}{w_{i,j}}, \quad j = 1, \ldots, k.$$

Set

$$\widetilde{H}_j = w_{0j} + \sum_{j=1}^{m_j} c_{ij} \ln w_{ij}, \ \ j = 1, \ldots, k.$$

Then $\delta H_j = \delta \widetilde{H}_j$, $j = 1, \ldots, k$, which means that

- $\mathscr{X}(\widetilde{H}_j) = \mathscr{X}(H_j) = 0$,
- $\widetilde{H}_1, \ldots, \widetilde{H}_k$ are functionally independent due to those of $H_1, \ldots, H_k$.

The results that we want to prove. □

$\exists$ some $\delta H_j \notin K$. We prove Theorem 3.1 by induction

- on transcendental degree $\text{tr}[L:K]$ replacing the order of the tower in the definition of the field extension

$$\spadesuit \ \text{tr}[L:K] = 1 \quad \left\{ \begin{array}{l} \text{k=2 f.i.F.I.} \\ \mathscr{X}(t) = 0 \\ \mathscr{X}(t) \neq 0 \end{array} \right. \quad \left\{ \begin{array}{l} \text{k>2 f.i.F.I.} \\ \mathscr{X}(t) = 0 \\ \mathscr{X}(t) \neq 0 \end{array} \right.$$

with t a transcendental element

$\spadesuit \ \text{tr}[L:K] = m-1$

$\spadesuit \ \text{tr}[L:K] = m$

$\exists$ some $\delta H_j \notin K$. We prove Theorem 3.1 by induction

- on transcendental degree tr$[L:K]$ replacing the order of the tower in the definition of the field extension

$\spadesuit$ tr$[L:K] = 1$ $\left\{ \begin{array}{l} \text{k=2 f.i.F.I.} \\ \mathscr{X}(t) = 0 \\ \mathscr{X}(t) \neq 0 \end{array} \right.$ $\left\{ \begin{array}{l} \text{k>2 f.i.F.I.} \\ \mathscr{X}(t) = 0 \\ \mathscr{X}(t) \neq 0 \end{array} \right.$

with t a transcendental element

$\spadesuit$ tr$[L:K] = m-1$

$\spadesuit$ tr$[L:K] = m$

♠ $\mathrm{tr}[L:K] = 1$

Consider the tower $K \subset K(t) \subset L$, with

- L an algebraic extension of $K(t)$,

- t an transcendental element over K satisfying

  –) either $\delta t/t = \delta v$ for some $v \neq 0$ in K and all δ in Δ,

  –) or $\delta t = \delta v/v$ for some $v \neq 0$ in K and all δ in Δ

We distinguish

- the number k of the functionally independent first integrals of the vector field $\mathscr{X}$.

Case $1: k = 1$. The result is Theorem of Prelle-Singer[1], i.e. Theorem A stated above.

Case $2: k = 2$.

We distinguish

- the number k of the functionally independent first integrals of the vector field $\mathscr{X}$.

Case $1 : k = 1$. The result is Theorem of Prelle-Singer[1], i.e. Theorem A stated above.

Case $2 : k = 2$.

Figure: **Fanjing Mountain/Guizhou**:

Proposition 3.2

Consider the tower of the differential field extension

- $K \subset K(t) \subset L$ with t an transcendental element over K,
- L an algebraic extension of $K(t)$.

Let

- H_1 and H_2 be two functionally independent first integrals of $\mathscr{X}$ in L.

If $\mathscr{X}(t) = 0$, then Theorem 3.1 holds.

Proof: By the assumption of the proposition, let

$$H_1^\ell + a_{\ell-1}H_1^{\ell-1} + \cdots + a_1 H_1 + a_0 = 0,$$
$$H_2^m + b_{m-1}H_2^{m-1} + \cdots + b_1 H_2 + b_0 = 0 \tag{2}$$

be minimal algebraic Eq over $K(t)$ that H_1 and H_2 satisfy.

By Lemma 3.1, we know that

- $S_1 := \{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}\} \subset C(K(t), \mathscr{X})$.

Moreover,

- S_1 has at least two funct. indep. elements, saying a and b

Consequently,

- $\mathscr{X}$ has two funct. indep. first integrals in $K(t)$

Since t is transcendental over K,

- the ring $K[t]$ is factorial

Write a and b in the form

$$a = h_a \prod_{j=1}^{r} Q_j^{r_j}, \quad b = h_b \prod_{i=1}^{l} R_i^{l_i}, \tag{3}$$

with $h_a, h_b \in K$,

- $Q_1, \ldots, Q_r \in K[t]$ (resp. $R_1, \ldots, R_l \in K[t]$) pairwise distinct monic irreducible polynomials,

- $r_j, l_i \in \mathbb{Z} \setminus \{0\}, j = 1, \ldots, r, i = 1, \ldots, l$.

Set $S_2 := \{h_a, Q_1, \ldots, Q_r; h_b, R_1, \ldots, R_l\}$. Then

- there exist $Q, R \in S_2$ satisfying $\mathrm{d}Q \wedge \mathrm{d}R \neq 0$.

Substituting the expression of a into $\mathscr{X}(a) = 0$ gives

$$\frac{\mathscr{X}(h_a)}{h_a} \prod_{j=1}^{r} Q_j + \sum_{j=1}^{r} r_j Q_1 \cdots \mathscr{X}(Q_j) \cdots Q_r = 0. \tag{4}$$

This induces

- $Q_j | \mathscr{X}(Q_j), j = 1, \ldots, r$. Similarly $R_i | \mathscr{X}(R_i)$, $i = 1, \ldots, l$.

Set

- $q_j := \deg(Q_j(t))$ then $q_j \geq 1$.

Since $Q_j(t)$ is monic and $\mathscr{X}(t) = 0$, it follows that

- $\deg(\mathscr{X}(Q_j(t))) \leq q_j - 1$.

This together with $Q_j | \mathscr{X}(Q_j)$ shows that

- $\mathscr{X}(Q_j(t)) = 0 \implies \mathscr{X}(h_a) = 0$

Similarly, we have

- $\mathscr{X}(R_i) = 0$ and $\mathscr{X}(h_b) = 0$.

Consequently,

- Q and R satisfy that $\mathscr{X}(Q) = \mathscr{X}(R) = 0$.

If $\{Q, R\} = \{h_a, h_b\}$, then

- h_a and h_b are funct. indep. rational first integrals of $\mathscr{X}$.

If $\{Q, R\} \neq \{h_a, h_b\}$, assume $\deg Q \geq 1$ and set

$$\begin{aligned} Q &= t^q + A_{q-1}t^{q-1} + \cdots + A_1 t + A_0 \\ R &= t^p + B_{p-1}t^{p-1} + \cdots + B_1 t + B_0 \end{aligned} \tag{5}$$

where $A_j, B_i \in K$. Taking the derivation $\mathscr{X}$ on Q yields

$$0 = \mathscr{X}(A_{q-1})t^{q-1} + \cdots + \mathscr{X}(A_1)t + \mathscr{X}(A_0). \tag{6}$$

By this, one gets

- $\mathscr{X}(A_j) = 0$.

Similarly, one has

- $\mathscr{X}(B_i) = 0$.

Let

$$S_3 := \{A_0, \ldots, A_{q-1}; B_0, \ldots, B_{p-1}\}.$$

From independence of Q and R, there exists some $A \in S_3$

- such that A and t are functionally independent.

Consequently,

- if $t = \mathrm{In}v$, then A and $\mathrm{In}v$ are the desired first integrals
- if $t = e^v$, then A and v are the desired first integrals.

The proposition follows. □

Similarly, one has

- $\mathscr{X}(B_i) = 0$.

Let

$$S_3 := \{A_0, \ldots, A_{q-1}; B_0, \ldots, B_{p-1}\}.$$

From independence of Q and R, there exists some $A \in S_3$

- such that A and t are functionally independent.

Consequently,

- if $t = \ln v$, then A and $\ln v$ are the desired first integrals
- if $t = e^v$, then A and v are the desired first integrals.

The proposition follows.

$\square$

Similarly, one has

- $\mathscr{X}(B_i) = 0$.

Let

$$S_3 := \{A_0, \ldots, A_{q-1}; B_0, \ldots, B_{p-1}\}.$$

From independence of Q and R, there exists some $A \in S_3$

- such that A and t are functionally independent.

Consequently,

- if $t = \ln v$, then A and $\ln v$ are the desired first integrals
- if $t = e^v$, then A and v are the desired first integrals.

The proposition follows. □

Summarizing the proof of Proposition 3.2 gives the next result.

Corollary 3.1

Let $K \subset K(\eta) \subset L$, with $K = \mathbb{C}(x_1, \ldots, x_n)$, where

- η is transcendental over K
- L is an algebraic extension of $K(\eta)$.

If

- $\eta, H_1, H_2 \in C(L, \mathscr{X})$, and
- H_1 and H_2 are functionally independent,

then

- either $\mathscr{X}$ has two funct. indep. rational first integrals,
- or $\exists$ an $\widetilde{H}_1 \in C(K, \mathscr{X})$ such that η and $\widetilde{H}_1$ are two funct. indep. first integrals of $\mathscr{X}$.

Up to now, we have proved Theorem 3.1

- for $n = 2$ and $K \subset K(t) \subset L$ satisfying $\mathscr{X}(t) = 0$.

Next we study the case $\mathscr{X}(t) \neq 0$.

Proposition 3.3

Consider the tower of the elementary field extension

- $K \subset K(t) \subset L$ with t transcendental
- L algebraic extension of $K(t)$.

Assume that

- H_1 and H_2 are two funct. indep. first integrals of $\mathscr{X}$ in L.
- $\mathscr{X}(t) \neq 0$

Then Theorem 3.1 holds.

Proof: We adopt the same notations as above

Case 2.1. $t = \ln v$. Since

- $Q_j, j = 1, \ldots, r$, are monic and $\mathscr{X}(t) = \mathscr{X}(v)/v$,

we have for each $Q \in \{Q_1, \ldots, Q_r\}$ via (5)

$$
\begin{aligned}
\mathscr{X}(Q) = & \left[\mathscr{X}(A_{q-1}) + q \frac{\mathscr{X}(v)}{v} \right] t^{q-1} \\
& + \left[\mathscr{X}(A_{q-2}) + (q-1)A_{q-1} \frac{\mathscr{X}(v)}{v} \right] t^{q-1} \\
& + \ldots + \left[\mathscr{X}(A_1) + 2A_2 \frac{\mathscr{X}(v)}{v} \right] t + \mathscr{X}(A_0) + A_1 \frac{\mathscr{X}(v)}{v}.
\end{aligned}
\tag{7}
$$

Now $v \in K$ implies

$$\deg \mathscr{X}(Q_j(t)) \leq q_j - 1 < q_j = \deg(Q_j(t)).$$

- By $Q_j | \mathscr{X}(Q_j) \implies \mathscr{X}(Q_j) = 0 \implies \mathscr{X}(h_a) = 0$

Similarly, one has

- $\mathscr{X}(R_i) = 0$, $i = 1, \ldots, l$, and $\mathscr{X}(h_b) = 0$.

If $\{Q,R\} = \{h_a, h_b\}$, we are done.

If $\{Q,R\} \neq \{h_a, h_b\}$, by (7) gives

$$0 = \mathscr{X}(Q) = \left(\mathscr{X}(A_{q-1}) + q\frac{\mathscr{X}(v)}{v} \right) t^{q-1} + \cdots + \mathscr{X}(A_0) + A_1 \frac{\mathscr{X}(v)}{v}. \tag{8}$$

Since t is transcendental over K, by (8) gives

$$\mathscr{X}(A_{q-1}) + q\frac{\mathscr{X}(v)}{v} = 0.$$

Furthermore, by $0 \neq \mathscr{X}(t) = \mathscr{X}(v)/v$ yields

- $\mathscr{X}(A_{q-1}) \neq 0 \implies A_{q-1} \in K \setminus \mathbb{C}$, nonconstant rational

Now, we take

- $\widetilde{t} = A_{q-1} + q\ln v = A_{q-1} + qt$.

By Proposition 2.1

- $K(\widetilde{t}) = K(t)$, and $\widetilde{t}$ is transcendental over K.

Since $\mathscr{X}(\widetilde{t}) = 0$, we are in the previous case.

By Corollary 3.1, $\exists$ an $\widetilde{A} \in K$ such that

- $\widetilde{A}$ is funct. indep. of $\widetilde{t}$ and is a first integral of $\mathscr{X}$.

In summary,

- $\widetilde{A}$ and $\widetilde{t} = A_{q-1} + q\ln v$ are the desired funct. indep. first integrals of $\mathscr{X}$.

Case 2.2. $t = e^v$. Since Q_j is monic and $\mathscr{X}(t)/t = \mathscr{X}(v)$,

- by $\mathscr{X}(t) \neq 0$ gives $\deg Q_j = \deg \mathscr{X}(Q_j)$.

Therefore,

- $Q_j | \mathscr{X}(Q_j), j = 1, \ldots, r$, implies $\mathscr{X}(Q_j) = \mathscr{X}(v) q_j Q_j$.

If $\{a, b\} = \{h_a, h_b\}$, we are done.

For $\{a, b\} \neq \{h_a, h_b\}$, assume without loss that $a \neq h_a$.

Recall

- (3) for the factorization of a and b
- all the Q_j's are irreducible polynomials in $K[t]$.

Case 2.2.1. t divides all Q_j's. Then

- $a = h_a t^\ell$ for some non-zero integer ℓ.

Since $\mathscr{X}(t) \neq 0$, one has $h_a \notin C(K, \mathscr{X})$.

Take $\widetilde{t} = h_a e^{\ell v}$.

$\widetilde{t}$ transcendental over K implies that

- $K(\widetilde{t})(\subseteq K(t)) \subset L$ is an algebraic extension.

Now $\mathscr{X}(\widetilde{t}) = 0$ and Corollary 3.1 verifies that

- either $\exists$ an $\widetilde{A} \in K$ such that $\widetilde{A}$ is funct. indep. with $\widetilde{t}$ are two funct. indep. first integrals of $\mathscr{X}$,

- or two funct. indep. rational first integrals of $\mathscr{X}$.

The proof is completed.

Case 2.2.2. t cannot divide all Q_j's.

Assume t is not a factor of Q_1.

Set $\deg Q_1 = q_1 \geq 1$, and

$$Q_1 = t^{q_1} + A_{q_1-1}t^{q_1-1} + \cdots + A_1 t + A_0, \tag{9}$$

where $A_0, \ldots, A_{q_1-1} \in K$, with $A_0 \neq 0$ by the assumption.

One can show that

- A_0 is a nonconstant rational function.

Then

- $\mathscr{X}(\widetilde{t}) = 0$, with $\widetilde{t} = A_0^{-1} e^{q_1 v}$.

Again applying Corollary 3.1 yields

- either $\mathscr{X}$ has a rational first integral $\widetilde{A}$, funct. indep. of $\widetilde{t}$,
- or $\mathscr{X}$ has two funct. indep. rational first integrals.

The proposition follows.

Combining with Propositions 3.1–3.3,

- we have completed the proof of Theorem 3.1

for elementary field extension $K \subset K(t) \subset L$, with

- $k = 2$ funct. independ. first integrals in L of $\mathscr{X}$.

Case 3 : $k > 2$. Next we turn to prove Theorem 3.1

- for elementary field extension $K \subset K(t) \subset L$

with $k > 2$ functionally independent first integrals in L.

Combining with Propositions 3.1–3.3,

- we have completed the proof of Theorem 3.1

for elementary field extension $K \subset K(t) \subset L$, with

- $k = 2$ funct. independ. first integrals in L of $\mathscr{X}$.

Case $3 : k > 2$. Next we turn to prove Theorem 3.1

- for elementary field extension $K \subset K(t) \subset L$

with $k > 2$ functionally independent first integrals in L.

Definition: Let

- $\Theta := \{f_1(x), \cdots, f_s(x)\}$ be a set of smooth functions defined on $\mathbb{C}^n$ except perhaps a zero L–measure subset.

We call $\{f_{k_1}, \cdots, f_{k_\ell}\} \subset \Theta$

- a maximal functionally independent group of Θ,

if

- $f_{k_1}, \cdots, f_{k_\ell}$ are functionally independent on $\mathbb{C}^n$, i.e.
$$\mathrm{d}f_{k_1} \wedge \cdots \wedge \mathrm{d}f_{k_\ell} \neq 0, \text{ except ....}$$
- any other $f \in \Theta$ satisfies $\mathrm{d}f \wedge \mathrm{d}f_{k_1} \wedge \cdots \wedge \mathrm{d}f_{k_\ell} = 0$ except .... .

Proposition 3.4

Consider the diff. field extension $K \subset K(t) \subset L$, where

- t is transcendental over K, and
- $L/K(t)$ is algebraic extension.

Let H_1, H_2, ..., H_k ($k \geq 2$) be

- funct indepen. first integrals of $\mathscr{X}$ in L.

If $\mathscr{X}(t) = 0$, then Theorem 3.1 holds.

Proof. Let

$$H_j^{n_j} + a_{j,n_j-1}H_j^{n_j-1} + \cdots + a_{j,1}H_j + a_{j,0} = 0, \quad j \in \{1,\ldots,k\} \quad (10)$$

be the minimal algebraic equation of H_j with $j \in \{1,\ldots,k\}$, and

$$a_{j,0},\ldots,a_{j,n_j-1} \in K(t)$$

By Lemma 3.1, one gets that

$$\Theta_1 := \{a_{1,0},\ldots,a_{1,n_1-1},\ldots,a_{k,0},\ldots,a_{k,n_k-1}\} \subset C(K(t), \mathscr{X})$$

Denote by $a_1,\ldots,a_s$

- the the maximal funct. indep. group of Θ_1.

Then $k \leq s$.

Hereafter, for convenience, we take $s = k$ so that

$$da_1 \wedge \cdots \wedge da_k \neq 0.$$

Factorizing a_j's as

$$a_j = h_{a_j} \prod_{i=1}^{m_j} Q_{j,i}^{r_{j,i}}, \qquad (11)$$

where $h_{a_j} \in K$, and

- $Q_{j,1}, \ldots, Q_{j,m_j}$ are pairwise distinct irreducible polyn. in $K[t]$
- $r_{j,1}, \ldots, r_{j,m_j} \in \mathbb{Z} \setminus \{0\}$.

Substituting (11) into $\mathscr{X}(a_j) = 0$ yields

$$\prod_{i=1}^{m_j} Q_{j,i} \left(\frac{\mathscr{X}(h_{a_j})}{h_{a_j}} + \sum_{i=1}^{m_j} r_{j,i} \frac{\mathscr{X}(Q_{j,i})}{Q_{j,i}} \right) = 0, \quad j = 1, \ldots, k. \qquad (12)$$

The left hand sides in $K[t]$ forces that $Q_{j,i} | \mathscr{X}(Q_{j,i})$.

Set
$$\Theta_2 := \left\{ h_{a_j}, Q_{j,i} \,|\, j = 1, \ldots, k, \; i = 1, \ldots, m_j \right\}.$$

Denote by $b_1, \ldots, b_{\widetilde{s}}$

- the maximal funct. indep. group of Θ_2.

Then $\widetilde{s} \geq k$.

For simplicity, set $\widetilde{s} = k$.

Since $\mathscr{X}(t) = 0$ and $Q_{j,i} \in K[t]$ is monic, one gets

- $\Theta_2 \subset C(K(t), \mathscr{X})$, and so

- $b_1, \ldots, b_k$ are funct. indep. first integrals of $\mathscr{X}$

If $\{b_1, \ldots, b_k\} = \{h_{a_1}, \ldots, h_{a_k}\}$, we are done.

If $\{b_1,\ldots,b_k\} \neq \{h_{a_1},\ldots,h_{a_k}\}$, set $\deg b_1 \geq 1$, and

$$b_j = t^{\alpha_j} + B_{j,\alpha_j-1}t^{\alpha_j-1} + \ldots + B_{j,1}t + B_{j,0}, \quad j = 1,\ldots,k. \qquad (13)$$

Taking $\mathscr{X}$ acting on their both sides gives

$$\{B_{1,0},\ldots,B_{1,\alpha_1-1},\ldots,B_{k,0},\ldots,B_{k,\alpha_k-1}\} \subset C(K,\mathscr{X}),$$

Set

$$\Theta_3 := \{t, B_{1,0},\ldots,B_{1,\alpha_1-1},\ldots,B_{k,0},\ldots,B_{k,\alpha_k-1}\}.$$

Let $\{t, E_1,\ldots,E_{s'-1}\}$ be

- the maximal funct. indep. group of Θ_3.

Then $s' \geq k$, and so

- $t, E_1,\ldots,E_{k-1}$ are k funct. indep. first integrals of $\mathscr{X}$

The proof is completed.

Summarizing the proof of Proposition 3.4 reaches

Corollary 3.2

Let $K \subset K(\eta) \subset L$, with

- $K = \mathbb{C}(x_1, \ldots, x_n)$, η transcendental over K, and
- L an algebraic extension of $K(\eta)$.

Assume that

- η is a first integral of the vector field $\mathscr{X}$.

If $H_1, \ldots, H_k \in C(L, \mathscr{X})$ $(k \geq 2)$ are funct. independent, then

- either $\mathscr{X}$ has k funct. independent rational first integrals,
- or there exist $\widetilde{H}_1, \ldots \widetilde{H}_{k-1}$ in $C(K, \mathscr{X})$ such that
  $\eta, \widetilde{H}_1, \ldots \widetilde{H}_{k-1}$ are funct. independent.

Now turn to the case $\mathscr{X}(t) \neq 0$.

Proposition 3.5

Consider the elementary field extension $K \subset K(t) \subset L$ with

- t transcendental over K and
- L an algebraic extension of $K(t)$.

Let $H_1, H_2, \ldots, H_k$ ($k \geq 2$) be

- the functionally independent first integrals of $\mathscr{X}$ in L.

If $\mathscr{X}(t) \neq 0$, then Theorem 3.1 holds.

Proof. With the notations as above, see (11), set

$$q_{j,i} := \deg Q_{j,i} \text{ for } j = 1, \dots, k, \, i = 1, \dots, m_j.$$

Case 3.1. $t = \ln v$. $Q_{j,i}$'s monic and $\mathscr{X}(t) = \mathscr{X}(v)/v$ force

$$\deg Q_{j,i} > q_{j,i} - 1 \geq \deg \mathscr{X}(Q_{j,i}).$$

Since $Q_{j,i}$, $\mathscr{X}(Q_{j,i}) \in K[t]$ and $Q_{j,i} | \mathscr{X}(Q_{j,i})$, it follows

- $\mathscr{X}(Q_{j,i}) = 0$ for all $j = 1, \dots, k$, $i = 1, \dots, m_j$.

Then, the equality (12) induces that

- $\{h_{a_1}, \dots, h_{a_k}\} \subset C(K, \mathscr{X})$,
- and so $\{b_1, \dots, b_k\} \subset C(K(t), \mathscr{X})$.

If $\{b_1, \dots, b_k\} = \{h_{a_1}, \dots, h_{a_k}\}$, we are done.

If $\{b_1, \ldots, b_k\} \neq \{h_{a_1}, \ldots, h_{a_k}\}$, set $\deg b_1 \geq 1$.

By $\mathcal{X}(b_1) = 0$ yields

$$0 = \mathcal{X}(b_1) = (\mathcal{X}(B_{1,\alpha_1-1}) + \alpha_1 \mathcal{X}(v)/v) t^{\alpha_1-1} \\ + \ldots + (\mathcal{X}(B_{1,0}) + B_{1,1} \mathcal{X}(v)/v). \tag{14}$$

The transcendence of t induces that

$\mathcal{X}(B_{1,\alpha_1-1} + \alpha_1 \ln v) = 0.$

Set $\widetilde{t} = B_{1,\alpha_1-1} + \alpha_1 \ln v$. Then

- $\widetilde{t}$ is transcendental over K, and $K(\widetilde{t})(=K(t)) \subset L$.
- L is also an algebraic extension of $K(\widetilde{t})$.

Applying Corollary 3.2 can complete the proof.

Case 3.2. $t = e^v$.

- $Q_{j,i}$'s monic and $\mathscr{X}(t)/t = \mathscr{X}(v) \implies \deg Q_{j,i} = \deg \mathscr{X}(Q_{j,i})$.
- $Q_{j,i} | \mathscr{X}(Q_{j,i}) \implies q_{j,i} \mathscr{X}(v) Q_{j,i} = \mathscr{X}(Q_{j,i})$.

If $\{a_1, \ldots, a_k\} = \{h_{a_1}, \ldots, h_{a_k}\}$, we are done.

If $\{a_1,\ldots,a_k\} \neq \{h_{a_1},\ldots,h_{a_k}\}$, set $a_1 \neq h_{a_1}$.

Assume: t divides $Q_{1,i}$ for all i, then

- $a_1 = h_{a_1} t^\ell$, for some $\ell \in \mathbb{Z} \setminus \{0\}$.

By $\mathscr{X}(t) \neq 0$ forces $\mathscr{X}(h_{a_1}) \neq 0$.

Taking $\widetilde{t} = a_1 = h_{a_1} e^{\ell v}$, then

- $\widetilde{t}$ is a transcendental first integral of $\mathscr{X}$,
- so is $\ln\widetilde{t} = \ell v + \ln h_{a_1}$.
- $K(t)$ is algebraic over $K(\widetilde{t})$, and so is L.

The remaining proof follows from Corollary 3.2.

If some $Q_{1,j}$ cannot be divided by t, set $\deg Q_{1,1} = q_{1,1} \geq 1$,

- t is not a factor of $Q_{1,1}$ and

$$Q_{1,1} = t^{q_{1,1}} + A_{q_{1,1}} t^{q_{1,1}-1} + \cdots + A_1 t + A_0, \qquad (15)$$

where $A_{q_{1,1}}, \ldots, A_0 \in K$, with A_0 nonconstant.

Taking $\widetilde{t} = A_0^{-1} e^{q_{1,1}v}$, then

- $\widetilde{t}$ is a transcendental first integral of $\mathscr{X}$,

- and $K(t)$ is algebraic over $K(\widetilde{t})$, and so is L.

Replacing η in Corollary 3.2 by $\widetilde{t}$ reaches the result.

It completes the proof of the proposition. □

Up to now we have proved Theorem 3.1 for

- $K \subset K(t) \subset L$ with t transcendental over K and
- L an algebraic extension of $K(t)$.

That is, Theorem 3.1 holds for $\text{tr}[L : K] = 1$

♠ $\mathrm{tr}[L : K] = m - 1$ for $2 \le m \in \mathbb{N}$.

To apply induction, we assume that

- Theorem 3.1 holds

♠ $\mathrm{tr}[L:K] = m$. The elementary field extension

- L/K can be written in the **tower** form

$$K_0 := K \subset K_1 := K(t_1) \subset K_2 := K_1(t_2) \subset \cdots \subset K_N := K_{N-1}(t_N) \subset L,$$

where

- t_j is either algebraic or transcendental over K_{j-1} for $j = 1, \ldots, N$,
- L is algebraic over $K_{N-1}(t_N)$ ($N \geq m$).

Note that

- if the last one is $K_N(t_{N+1})$ with transc. element t_{N+1} of K_N, one can can add $L = K_N(t_{N+1})$ to the tower

♠ $\mathrm{tr}[L:K] = m$. The elementary field extension

- L/K can be written in the **tower** form

$$K_0 := K \subset K_1 := K(t_1) \subset K_2 := K_1(t_2) \subset \cdots \subset K_N := K_{N-1}(t_N) \subset L,$$

where

- t_j is either algebraic or transcendental over K_{j-1} for $j = 1, \ldots, N$,

- L is algebraic over $K_{N-1}(t_N)$ $(N \geq m)$.

Note that

- if the last one is $K_N(t_{N+1})$ with transc. element t_{N+1} of K_N, one can can add $L = K_N(t_{N+1})$ to the tower

For convenience, assume that

- t_1 is transcendental over K.

Combining with Lemma 2.1, one can show

- $\text{tr}[L : K(t_1)] = m - 1$.

Applying induction:

- replacing K with $K(t_1)$
- using $\text{tr}[L : K(t_1)] = m - 1$, there exist

$$\widetilde{H}_j = \widetilde{w}_{0,j} + \sum_{i=1}^{\widetilde{m}_j} \widetilde{c_{i,j}} \ln \widetilde{w}_{i,j}, \quad j = 1, \ldots, k,$$

with $\widetilde{w}_{0,j}, \ldots, \widetilde{w}_{\widetilde{m}_j,j} \in \overline{K(t_1)} \cap L$, and $\widetilde{c}_{0,j}, \ldots, \widetilde{c}_{\widetilde{m}_j,j} \in \mathbb{C}$, such that

- $\widetilde{H}_1, \ldots, \widetilde{H}_k$ are k funct. indep. first integrals of $\mathscr{X}$.

Recall that $\overline{K(t_1)}$ is the algebraic closure of $K(t_1)$.

Set

$$\widetilde{K} := K(t_1, \widetilde{w}_{0,1}, \ldots, \widetilde{w}_{\widetilde{m}_k, k}).$$

Obviously

- $\widetilde{K} \subset L$, and
- $K \subset K(t_1) \subset \widetilde{K}$ is an elementary extension.
- $\widetilde{K}/K(t_1)$ is algebraic extension.

By Lemma 2.1

- $\mathrm{tr}[\widetilde{K} : K] = \mathrm{tr}[\widetilde{K} : K(t_1)] + \mathrm{tr}[K(t_1) : K] = 1$,

  where we have used $\mathrm{tr}[\widetilde{K} : K(t_1)] = 0$.

Applying the result in $\mathrm{tr}[L:K]=1$ to $\widetilde{H}_1,\ldots,\widetilde{H}_k$ reaches that

- there exist

$$\overline{H}_j = w_{0,j} + \sum_{i=1}^{m_j} c_{i,j}\ln w_{i,j}, \quad j=1,\ldots,k,$$

with

$- w_{0,j},\ldots,w_{m_j,j} \in \overline{K}\cap\widetilde{K} \subset \overline{K}\cap L,$

$- c_{0,j},\ldots,c_{m_j,j} \in \mathbb{C}, j=1,\ldots,k,$

such that

$\star$) $\mathcal{X}$ has the k funct. indep. first integrals $\overline{H}_1,\ldots,\overline{H}_k$,
   which are of the desired form.

Theorem 3.1 follows. $\qquad\qquad\square$

Figure: **Yellow Mountain/Anhui**:

Having proved Theorem 3.1 also verifies Theorem 1.1

With this theorem in hand, we can prove Theorem 1.2:

Recall Theorem 1.2

If the vector field $\mathscr{X}$ is elementary integrable, i.e. it has

$n - 1$ functionally independent elementary first integrals,

then $\mathscr{X}$ has

- either a rational first integral in $\mathbb{C}(x)$

- or a rational Jacobian multiplier in $\mathbb{C}(x)$.

Having proved Theorem 3.1 also verifies Theorem 1.1

With this theorem in hand, we can prove Theorem 1.2:

Recall Theorem 1.2

If the vector field $\mathscr{X}$ is elementary integrable, i.e. it has

$n - 1$ functionally independent elementary first integrals,

then $\mathscr{X}$ has

- either a rational first integral in $\mathbb{C}(x)$
- or a rational Jacobian multiplier in $\mathbb{C}(x)$.

Having proved Theorem 3.1 also verifies Theorem 1.1

With this theorem in hand, we can prove Theorem 1.2:

Recall Theorem 1.2

If the vector field $\mathscr{X}$ is elementary integrable, i.e. it has

$n - 1$ functionally independent elementary first integrals,

then $\mathscr{X}$ has

- either a rational first integral in $\mathbb{C}(x)$
- or a rational Jacobian multiplier in $\mathbb{C}(x)$.

To prove Theorem 1.2, we first characterize

- derivatives of algebraic elements.

Lemma 4.1

Consider the differential field (K, Δ) with
$$K = \mathbb{C}(x), \ x = (x_1, \ldots, x_n) \text{ and } \Delta = \{\partial_{x_1}, \ldots, \partial_{x_n}\}.$$
If ω is algebraic over K, i.e. $\omega \in \overline{K}$, then

- for any $\delta \in \Delta$,

$$\delta\omega \text{ is also algebraic over } K.$$

Proof of Theorem 1.2

If $\operatorname{div}\mathscr{X} = 0$, the theorem is trivial.

Consider $\operatorname{div}\mathscr{X} \not\equiv 0$. By definition,

- a Jacobian multiplier J of $\mathscr{X}$ satisfies $\operatorname{div}(J\mathscr{X}) \equiv 0$, i.e.

$$\mathscr{X}(J) = -J\operatorname{div}\mathscr{X}.$$

One gets from Theorem 1.1 that

- $\mathscr{X}$ has $n-1$ funct. indep. first integrals of the form

$$H_j = u_{0,j} + \sum_{i=1}^{n_j} c_{i,j}\ln u_{i,j}, \quad j = 1,\ldots,n-1, \tag{16}$$

where $u_{0,j},\ldots,u_{n_j,j} \in \overline{K} \cap L, j = 1,\ldots,n-1$.

Any derivation $\delta \in \Delta$ acting on (16) gives

$$\delta H_j = \delta u_{0,j} + \sum_{i=1}^{n_j} c_{i,j} \frac{\delta u_{i,j}}{u_{i,j}}.$$

Let

- F be the minimal algebraic extension containing

$$\{u_{0,j}, \ldots, u_{n_j,j} | \ j = 1, \ldots, n-1\}$$

  of $K = \mathbb{C}(x_1, \ldots, x_n)$.

By Lemma 4.1, it follows

- $\delta H_j \in F$ for all $\delta \in \Delta$.

By funct. indep. of $H_1, \ldots, H_{n-1}$, set

$$\Lambda(x) := \det(\partial_1 \mathscr{H}_1(x), \ldots, \partial_{n-1} \mathscr{H}_{n-1}(x)) \neq 0, \quad x \in \Omega,$$

where

$$\mathscr{H}(x) := (H_1(x), \ldots, H_{n-1}(x))^T,$$
$$\partial_i \mathscr{H}(x) := (\partial_i H_1(x), \ldots, \partial_i H_{n-1}(x))^T, \quad i = 1, \ldots, n.$$

Hereafter, $\partial_s := \partial_{x_s}$ and $\partial := (\partial_1, \ldots, \partial_n)$.

For $s = 1, \ldots, n-1$, define

$$\Lambda_s(x) := \det(\partial_1 \mathscr{H}(x), \ldots, \partial_{s-1} \mathscr{H}(x), \partial_n \mathscr{H}(x), \partial_{s+1} \mathscr{H}(x), \ldots, \partial_{n-1} \mathscr{H}(x)).$$

By funct. indep. of $H_1, \ldots, H_{n-1}$, set

$$\Lambda(x) := \det(\partial_1 \mathscr{H}_1(x), \ldots, \partial_{n-1} \mathscr{H}_{n-1}(x)) \neq 0, \quad x \in \Omega,$$

where

$$\mathscr{H}(x) := (H_1(x), \ldots, H_{n-1}(x))^T,$$
$$\partial_i \mathscr{H}(x) := (\partial_i H_1(x), \ldots, \partial_i H_{n-1}(x))^T, \quad i = 1, \ldots, n.$$

Hereafter, $\partial_s := \partial_{x_s}$ and $\partial := (\partial_1, \ldots, \partial_n)$.

For $s = 1, \ldots, n-1$, define

$$\Lambda_s(x) := \det(\partial_1 \mathscr{H}(x), \ldots, \partial_{s-1} \mathscr{H}(x), \partial_n \mathscr{H}(x), \partial_{s+1} \mathscr{H}(x), \ldots, \partial_{n-1} \mathscr{H}(x)).$$

Then

- $\Lambda, \Lambda_s \in F$
- $\partial_1 \Lambda_1 + \ldots + \partial_{n-1} \Lambda_{n-1} - \partial_n \Lambda = 0$

Solving $\mathscr{X}(H)(x) = 0$ by Cramer's rule gives

$$P_s(x_1, \ldots, x_n) = -\frac{\Lambda_s}{\Lambda} P_n(x_1, \ldots, x_n), \quad s = 1, \ldots, n-1.$$

Recall that P_s's are the components of $\mathscr{X}$

Set

$$h(x) := P_n/\Lambda = -P_s/\Lambda_s, \, s = 1, \ldots, n-1.$$

Then

- $h \in F$, and

$$\mathscr{X}(h) = P_1\partial_1 h + \ldots P_n\partial_n h = h\mathrm{div}(\mathscr{X}).$$

This shows that

- $J = h^{-1} \in F$ and
- it is a Jacobian multiplier of $\mathscr{X}$, because

$$\mathscr{X}(J) = -J\mathrm{div}(\mathscr{X}). \tag{17}$$

Let

- $\overline{F}^N$ be the normal closure of F, and
- $\mathscr{G}$ the Galois group formed by the automorphisms of $\overline{F}^N$ fixing K.

According to Lang[10] [Chap. VI, Thm. 1.8], proved by Artin,

- the group $\mathscr{G}$ is of finite order, denoted by N.
- $N \leq [F : K]$, degree of the algeb. extension of the field.

Rosenlicht[13] proved that

- the K–automorphisms of $\overline{F}^N$ commute with the derivations on $\overline{F}^N$.

So, for any $\sigma \in \mathscr{G}$, one gets from (17): $\mathscr{X}(J) = -J\mathrm{div}(\mathscr{X})$ that

$$\mathscr{X}(\sigma J) = -\sigma(J)\mathrm{div}\,\mathscr{X}.$$

This further shows that

$$\mathscr{X}(\widetilde{J}) = -\widetilde{J}\operatorname{div}\mathscr{X},$$

where

$$\widetilde{J} = \frac{1}{N}\sum_{\sigma \in \mathscr{G}}\sigma(J).$$

Note that for any $\sigma_0 \in \mathscr{G}$

$$\sigma_0(\widetilde{J}) = \frac{1}{N}\sigma_0\left(\sum_{\sigma \in \mathscr{G}}\sigma(J)\right) = \frac{1}{N}\sum_{\sigma \in \mathscr{G}}\sigma(J) = \widetilde{J}.$$

This shows

$$\widetilde{J} \in K = \mathbb{C}(x_1, \ldots, x_n).$$

Hence, $\widetilde{J}$ is a rational Jacobian multiplier of $\mathscr{X}$

Theorem 3.2 follows.

□

Figure: **Lleida**:

- ♠ Dating back to the Bronze Age period.
- ♠ In 1297, University of Lleida was founded
- ♠ International Friendship City with Hefei, Anhui, China

Figure: **Shanghai**:

♠ Around 6000 ago， West part of Shanghai was under sea
♠ Around 2000 ago， East part of Shanghai was under sea.
♠ In 1292， Shanghai County was founded.
♠ In 1896, Shanghai Jiao Tong University was founded

Figure: **Shanghai night**

♠ 25 million residences

♠ 21 metro lines (first one running in 1993, 15 lines in constructions)

♠ 13 Metro lines and 20 tunnels pass under the river

Figure: Welcome to Shanghai–SJTU!

谢　　谢!

Thanks for your attention!