On convergence of normal form transformations

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(the talk is based on joint works with Tatjana Petek and Sebastian Walcher)

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The talk is based on the recent works:

- T. Petek, V.G. Romanovski, On Computation of Poincaré-Dulac Normal Forms, J. Differential Equations, vol. 425, 2026.
- V. G. Romanovski, S. Walcher, On Convergence of Normal Form Transformations, https://arxiv.org/abs/2510.00925.

Poincaré-Dulac normal form

$$\dot{x} = f(x), \quad f(x) = Ax + \sum_{j=2}^{\infty} f_j(x), \quad x = (x_1, \dots, x_n)^T, \qquad x \in \mathbb{C}^n, \quad f_j(x), \tag{1}$$

$f_j(x)$ is an n-dimensional vector valued homogeneous polynomial of degree j,
$\lambda = (\lambda_1, \dots, \lambda_n)$ – eigenvalues of A, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ (or $\alpha \in \mathbb{Z}^n$)

$$\langle \lambda, \alpha \rangle = \sum_{i=1}^{n} \alpha_i \lambda_i, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n$$

and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. x *is a column vector* and x^α *is a monomial.* Normalization:

$$x = y + \tilde{h}(y) = y + \sum_{j=2}^{\infty} \tilde{h}_j(y). \tag{2}$$

$$\dot{y} = Ay + \tilde{g}(y), \quad \tilde{g}(y) = \sum_{j=2}^{\infty} \tilde{g}_j(y), \qquad \tilde{g}_j(y) \in \mathcal{V}_j^n, \qquad j = 2, 3, \dots . \tag{3}$$

y^α in the k-th entry is a *resonant* monomial if it is of the form $g^{(\alpha)} y^\alpha e_k$ with $\langle \lambda, \alpha \rangle - \lambda_k = 0$.
System (3) is in the *Poincaré–Dulac normal form* (in the normal form) if $\tilde{g}(y)$ contains only resonant terms.

Normalizing transformation can be divergent!

Definition 1 (Poincaré domain)

Let $\lambda_1, \ldots, \lambda_n$ *be the eigenvalues of the linear part A. We say that the spectrum is in the Poincaré domain if there exists a line through the origin such that all eigenvalues are strictly on one side of this line. In particular, the convex hull of* $\{\lambda_1, \ldots, \lambda_n\}$ *does not contain* 0*.*

Theorem 1 (Poincaré-Dulac)

Consider the analytic system

$$\dot{x} = Ax + \sum_{s=2}^{\infty} f_s(x),$$

where A has eigenvalues $\lambda_1, \ldots, \lambda_n$*.*
If the spectrum is in the Poincaré domain (no small denominators), then there exists a convergent analytic transformation to the Poincaré–Dulac normal form.

- Siegel (1952): if there exist $C > 0$ and $\nu > 0$ such that for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| > 1$ and for all $k \in \{1, \dots, n\}$

$$\left|\sum_{i=1}^{n} \alpha_i \lambda_i - \lambda_k\right| \geq C|\alpha|^{-\nu}, \tag{4}$$

then there exists a convergent transformation of system (1) to its normal form.

- Siegel (1952): if there exist $C > 0$ and $\nu > 0$ such that for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| > 1$ and for all $k \in \{1, \dots, n\}$

$$\left| \sum_{i=1}^{n} \alpha_i \lambda_i - \lambda_k \right| \geq C|\alpha|^{-\nu}, \tag{4}$$

  then there exists a convergent transformation of system (1) to its normal form.

- An essential further step in the investigation of the convergence of normalizing transformation is due to Pliss (1965):

  - if for system (1):

    (i) the nonzero elements among the $\sum_{j=1}^{n} \alpha_j \lambda_j - \lambda_k$ satisfy condition (4),

    (ii) some formal normal form of system (1) is linear,

  then there exists a convergent transformation to normal form.

- We say that the system satisfies Siegel-Pliss condition if the *nonzero* elements among the $\sum_{j=1}^{n} \alpha_j \lambda_j - \lambda_k = \langle \alpha, \lambda \rangle - \lambda_k$ satisfy condition (4).

Among the most important contributions to the theory of ODEs in 20th century are the results of Alexander Brjuno (Bruno) on convergence of normal forms.
A. D. Brjuno, The analytical form of differential equations. *Trans. Mosc. Math. Soc.*, 25:131–248 (1972).

He gave two conditions sufficient for convergence of the normalizing transformation.

Bruno's Condition ω (properties of eigenvalues): for $\omega_\ell = \inf\{|\langle\alpha,\lambda\rangle - \lambda_j|\}$ over all $\alpha \in \mathbb{Z}_+^n$ and $j \in \{1,\dots,n\}$ for which $\langle\alpha,\lambda\rangle - \lambda_j \neq 0$ and $2 \leq |\alpha| \leq 2^\ell$, it holds $-\sum 2^{-\ell} \ln \omega_\ell < +\infty$.

The other one is condition A, which consists of conditions A_2, A_1' and A_1'' (the shape of the normal form).
Bruno's Simplified Condition A. The normal form satisfies *Simplified Condition A* if it can be written in the special form

$$\dot{x} = \big(1 + \alpha(x)\big) Ax,$$

where $\alpha(x)$ is a formal scalar power series with $\alpha(0) = 0$.

Bruno's Theorem

If the the analytic system

$$\dot{x} = Ax + F(x), \qquad F(x) = O(\|x\|^2), \quad x \in \mathbb{C}^n, \tag{1}$$

satisfies Conditions A and ω then it admits a convergent transformation to a normal form.

- In the case of 1-dim germs of holomorphic diffeomomorphisms J.-C. Yoccoz proved that Bruno's condition ω is is also a necessary condition for convergence [Yoccoz, Jean-Christophe. Théorème de Siegel, nombres de Bruno et polynômes quadratiques. Petits diviseurs en dimension 1. Astérisque No. 231 (1995), 3-88].
- The results obtained in this article were among the reasons to award Yoccoz a Fields Medal in 1994.

The idea of the proof of Bruno's Theorem (Newton's method)

One constructs the transformation to the NF $x = y + h(x)$ using iterations:

- Let
$$\dot{x} = Ax + F(x), \tag{5}$$
coincides with the normal form up to terms of order m. Then $\exists\ x = y + h_m(y)$, where the terms of $h_m(x)$ have the order between $m+1$ and $2m$, which transforms (5) to a system in the normal form up to terms of order $2m$.
- One proves that for $\|x\| < \rho$ $\|h_m(x)\| < c\omega_{k+1}^{-\kappa}$, where $m = 2^k$, $c > 0$, $\kappa \leq 0$ depends only on the eigenvalues of A.
- Since the expansion of h_m starts from terms of order m on the k-th step of iterations the radius of convergence is reduced in $\sqrt[m]{c\omega_{k+1}^{-\kappa}}$.
- But due to Condition ω $\prod(c\omega_{k+1}^{-\kappa})^{\frac{-1}{2k}} < \infty \Longrightarrow$ the resulting transformation $x = y + h(x)$ has nonzero radius of convergence.

The above iteration procedure was proposed by V.A. Pliss [Differential Equations, **1** 153–161 (1965)], who applied it to the case when the normal form is linear, so the method could be called Bruno-Pliss method.

"Traditional representation" of vector field:

$$\dot{x} = f(x), \quad f(x) = Ax + \sum_{j=2}^{\infty} f_j(x), \qquad x \in \mathbb{C}^n, \tag{6}$$

$f_j(x)$ is an n-dimensional vector valued homogeneous polynomial of degree j.

Bruno [Soviet Math. Dokl. vol. 143, 1962] proposed to write system (6) in the form

$$\dot{x}_i = x_i \sum_{Q \in \mathbf{N}_i} f_{iQ} x^Q \qquad i = 1, \dots, n,$$

where $\mathbf{N}_i = \{Q = (q_1, \dots, q_n) \in \mathbb{Z}^n : q_i \geq -1, q_j \geq 0 \text{ if } j \neq i\}$, $x^Q = x_1^{q_1} x_2^{q_2} \dots x_n^{q_n}$.

- It seems that, since then, most mathematicians have regarded the above notation as rather unfortunate, cumbersome, or unclear.
- However, the idea behind this notation is a fundamental part of Bruno's method. It is not merely a notation – it provides **a different way to represent vector fields**.

We improve Bruno's notation:

$$\text{Hadamard product of } a = \begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ \dots \\ b_n \end{pmatrix}, \quad a \odot b = \begin{pmatrix} a_1 b_1 \\ \dots \\ a_n b_n \end{pmatrix}.$$

$\mathbf{N}_i = \{Q = (q_1, \dots, q_n) \in \mathbb{Z}^n : q_i \geq -1, q_j \geq 0 \text{ if } j \neq i\}$ and $\mathbf{N} = \mathbf{N}_1 \cup \dots \cup \mathbf{N}_n$.

- We call the set $\mathbf{N}$ Bruno's lattice.

Any system

$$\dot{x} = f(x), \quad f(x) = Ax + \sum_{j=2}^{\infty} f_j(x), \qquad x \in \mathbb{C}^n,$$

can be written in the form

$$\dot{x} = \sum_{Q \in \mathbf{N}} (x \odot F_Q) x^Q. \tag{7}$$

where $F_Q = (F_Q^{(1)}, \dots, F_Q^{(n)})^\top \in \mathbb{C}^n$, $x^Q = x_1^{q_1} x_2^{q_2} \dots x_n^{q_n}$.

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We call (7) Bruno's vector fields.

- Bruno's vector fields forms a vector space denoted by $\mathcal{F}$.

- For any two vector fields $U(x), V(x) \in \mathbb{C}^n$

$$[U(x), V(x)] := (DV(x))U(x) - (DU(x))V(x).$$

- This Lie bracket does not preserve the structure of Bruno's vector fields.
- One possibility to overcome this difficulty is to work in the ring of invariants [L. Stolovitch, Singular complete integrability. IHES Publ. Math. 91:133–210 (2000)] (requires a very sophisticated machinery).
- In [T. Petek, V. R., JDE, vol. 425, 2026] we introduced the Lie bracket which preserves the structure of Bruno's vector fields.

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We speak about different representation of the same object:

- One can describe the same point in $\mathbb{R}^2$ using the Cartesian coordinates or using the polar coordinates.
- One can work with vector fields in $\mathbb{C}^n$ written in the "traditional form" $\dot{x} = f(x) = \sum_{k=1}^{\infty} f_k(x)$ or in Bruno's form $\dot{x} = \sum_Q (x \odot F_Q)x^Q$.

We call $(x \odot F_Q)x^Q$ a term.

Using the Lie algebra structure on Bruno's vector fields in [V.R. and S. Walcher, On convergence of normal forms,https://arxiv.org/abs/2510.00925] we

- give simple proofs of theorems of Bruno on convergence of normal forms;
- give *drastically* simpler proofs of some theorems of Stolovitch about simultaneous normalization of a family of commuting vector fields [L. Stolovitch, Singular complete integrability. IHES Publ. Math. 91:133–210 (2000)];
- give some results about the interrelation of convergence of normal forms and integrability of the system.

In the rest of my talk I am going to:

- explain how to introduce the Lie algebra structure on Bruno's vector fields;
- introduce lattice (discrete) vector fields;
- present a normalization algorithm;
- explain the main ideas of Bruno's method.

For any vector fields $U(x), V(x)$

$$[U(x), V(x)] := (DV(x))U(x) - (DU(x))V(x). \tag{8}$$

Lemma 1 (T. Petek & V.R., JDE, 2026)

If $U(x) = (x \odot U_Q)x^Q$ *and* $V(x) = (x \odot V_P)x^P$, *where* $Q, P \in \mathbf{N}$, $U_Q = (U_Q^{(1)}, \ldots, U_Q^{(n)})^\top \in \mathbb{C}^n$ *and* $V_P = (V_P^{(1)}, \ldots, V_P^{(n)})^\top \in \mathbb{C}^n$, *then*

$$[U(x), V(x)] = (x \odot W_{Q+P})x^{Q+P}, \tag{9}$$

with

$$W_{Q+P} = \langle P, U_Q\rangle V_P - \langle Q, V_P\rangle U_Q. \tag{10}$$

Recall:

$$\langle P, U_Q\rangle = \sum_{i=1}^{n} P_i U_Q^{(i)}, \quad \langle Q, V_P\rangle = \sum_{i=1}^{n} Q_i V_P^{(i)}.$$

With this operation the vector space of Bruno's vector fields becomes a Lie algebra, which we denote by $\mathcal{M}$. We grade $\mathcal{M}$ using $\mathbb{Z}^n$ defining the homogeneous components $\mathcal{M}_Q$ $(Q \in \mathbb{Z}^n)$ by

$$\mathcal{M}_Q = \begin{cases} 0, \text{ if } Q \notin \mathbf{N} \\ (x \odot F)x^Q, F \in \mathbb{C}^n \text{ if } Q \in \mathbf{N} \end{cases}$$

Then

$$[\mathcal{M}_P, \mathcal{M}_Q] \subset \mathcal{M}_{P+Q}. \tag{11}$$

The n-tuple Q is called the *degree* of term

$$(x \odot F_Q)x^Q$$

.

For $Q \neq \bar{0}$ the term

$$(x \odot F_Q)x^Q \tag{12}$$

of (7) is called *resonant* if

$$\langle Q, \lambda \rangle = 0. \tag{13}$$

Cf.

$$\langle Q, \lambda \rangle - \lambda_k = 0.$$

Assume that $\mathbb{Z}^n$ is ordered using the lexicographic term order with $x_1 > \cdots > x_n$ and vector field (7) is written accordingly to this order. With this ordering $\mathcal{M}$ is a graded Lie algebra which we denote as $\mathcal{B}_{lex}$.

Definition 2

We say that the vector field $\dot{x} = \sum_Q (x \odot F_Q) x^Q$ of $\mathcal{B}_{lex}$ is in the normal form up to degree P ($P \in \mathbf{N}$ – degree is a vector!!) if for all $Q \leq P$ $F_Q = \bar{0}$ if Q is not resonant.

Assume that the matrix A is diagonal with the eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\lambda = (\lambda_1, \ldots, \lambda_n)^\top$.

Theorem 2 (T. Petek & V.R., JDE, 2026; the simplest algorithm to compute PD normal form)

Assume that the vector field

$$\dot{x} = \sum_Q (x \odot F_Q) x^Q. \tag{14}$$

of $\mathcal{B}_{lex}$ is in the normal form up to degree Q. Let $S \in \mathbf{N}$, $Q < S$, $F_S \neq \bar{0}$ and there is no $T \in \mathbf{N}$ such that $Q < T < S$ and $F_T \neq \bar{0}$.
1) If $\langle S, \lambda \rangle = 0$, then (14) is in the normal form up to degree S.
2) If $\langle S, \lambda \rangle \neq 0$, let

$$h(x) = \frac{1}{\langle S, \lambda \rangle} (x \odot F_S) x^S \tag{15}$$

and $\mathcal{H}(x) = \exp(h(x))$. Then performing the substitution $y = \mathcal{H}(x)$ and changing y to x we obtain from (14) the vector field of the form (14) which is in the normal form up to degree S.

Discrete (lattice) vector fields

$\mathbf{N}_i = \{Q = (q_1, \dots, q_n) \in \mathbb{Z}^n : q_i \geq -1, q_j \geq 0 \text{ if } j \neq i\}$ and $\mathbf{N} = \mathbf{N}_1 \cup \dots \cup \mathbf{N}_n$.
For $P \in \mathbf{N}$, $F, G \in \mathbb{C}^n$ and $c \in \mathbb{C}$ we define

$$c(F, P) = (cF, P), \qquad (F, P) + (G, P) = (F + G, P).$$

We will write $(F)_P$ instead of (F, P), then

$$(F)_P + (G)_P = (F + G)_P, \qquad c(F)_P = (cF)_P.$$

It is convenient to think about $(F)_P$ as the vector F assigned to the element P of the lattice Bruno's lattice $\mathbf{N} \subset \mathbb{Z}^n$.
Let $\mathcal{A}_{\mathbf{N}}$ be the set of all formal series of the form

$$\sum_{P \in \mathbf{N}} (F)_P.$$

$\mathcal{A}_{\mathbf{N}}$ is a vector space which is isomorphic to $\mathcal{F}$ by the map

$$T(\sum_{P \in \mathbf{N}} (F_P)_P) = \sum_{P \in \mathbf{N}} (x \odot F_P) x^P. \tag{16}$$

Since $\mathcal{A}_{\mathbf{N}}$ is also a Lie algebra with the Lie bracket defined by

$$[(U)_Q, (V)_P] = (\langle P, U \rangle V - \langle Q, V \rangle U)_{Q+P} \tag{17}$$

T is an isomorphism of Lie algebras $\mathcal{M}$ and $\mathcal{A}_{\mathbf{N}}$.

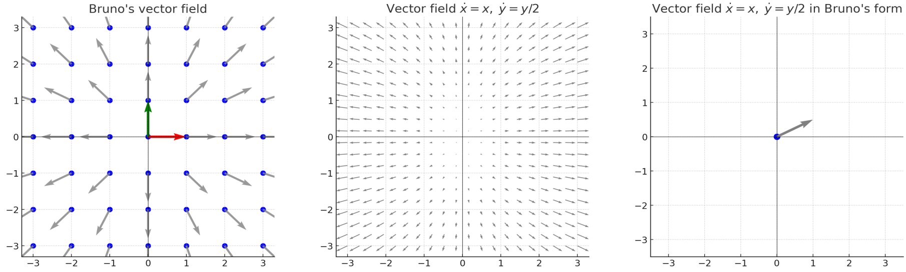
Let $\mathcal{A}_{\mathbf{N}}$ be the set of all formal series of the form $\sum_{P\in\mathbf{N}}(F)_P$. $\mathcal{A}_{\mathbf{N}}$ is a vector space which is isomorphic to $\mathcal{F}$ by the map

$$T\Big(\sum_{P\in\mathbf{N}}(F_P)_P\Big) = \sum_{P\in\mathbf{N}}(x\odot F_P)x^P. \tag{18}$$

Since $\mathcal{A}_{\mathbf{N}}$ is also a Lie algebra with the Lie bracket defined by

$$[(U)_Q,(V)_P] = (\langle P,U\rangle V - \langle Q,V\rangle U)_{Q+P} \tag{19}$$

we see that T is also an isomorphism of Lie algebras $\mathcal{M}$ and $\mathcal{A}_{\mathbf{N}}$.
$\dot{x}=x,\ \dot{y}=y/2 \quad\Longleftrightarrow \binom{\dot{x}}{\dot{y}} = \binom{x}{y}\odot\binom{1}{1/2}x^0y^0$ that is $Q=(0,0)$, $F_Q=\binom{1}{1/2}$.



Algorithm

Choose $Q \in \mathbf{N}$, $|Q| \geq 1$.

Set

$$\alpha := (\lambda)_{\bar{0}} + \sum_{i=1}^{m} (F_{Q_i})_{Q_i} \quad (\dot{x} = \lambda \odot x + \sum_{i=1}^{m} (x \odot F_{Q_i}) x^{Q_i}),$$

where $Q_1 < Q_2 < \cdots < Q_m = Q$.

FOR $i = 1$ TO m DO

IF $\langle Q_i, \lambda \rangle \neq 0$ THEN

BEGIN

$h := \frac{F_{Q_i}}{\langle Q_i, \lambda \rangle}$;

$\beta := \alpha + \sum_{i=1}^{m} \frac{1}{i!} \mathrm{ad}\,_h^i \alpha$;

take as α the part of β consisting of terms of degree less than or equal to Q.

END

Example

Bruno's form: $\dot{x} = \sum_Q (x \odot F_Q) x^Q$.

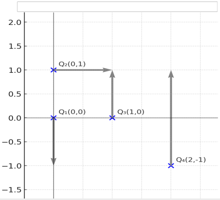
$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= -x_2 + 2x_1^2 + x_1 x_2 \end{aligned} \tag{20}$$

Let $x = (x_1, x_2)^\top$. Then system (20) is written in Bruno's form as

$$\dot{x} = x \odot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + x \odot \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_2 + x \odot \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_1 + x \odot \begin{pmatrix} 0 \\ 2 \end{pmatrix} x_1^2 x_2^{-1} =$$

$$x \odot \begin{pmatrix} 0 \\ -1 \end{pmatrix} x_1^0 x_2^0 + x \odot \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_1^0 x_2^1 + x \odot \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_1^1 x_2^0 + x \odot \begin{pmatrix} 0 \\ 2 \end{pmatrix} x_1^2 x_2^{-1}.$$

$$Q_1 = (0,0),\ Q_2 = (0,1),\ Q_3 = (1,0),\ Q_4 = (2,-1)$$



Example. Computation of the normal form of the system

$$\begin{aligned}\dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= -x_2 + 2x_1^2 + x_1 x_2\end{aligned} \tag{21}$$

up to order 3 (that is, we have to compute terms of the normal form of total degree 3 or less in variables x_1, x_2).
Let $x = (x_1, x_2)^\top$. In form (7):

$$\dot{x} = x \odot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + x \odot \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_2 + x \odot \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_1 + x \odot \begin{pmatrix} 0 \\ 2 \end{pmatrix} x_1^2 x_2^{-1} = \tag{22}$$

$$x \odot \begin{pmatrix} 0 \\ -1 \end{pmatrix} x_1^0 x_2^0 + x \odot \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_1^0 x_2^1 + x \odot \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_1^1 x_2^0 + x \odot \begin{pmatrix} 0 \\ 2 \end{pmatrix} x_1^2 x_2^{-1}.$$

Applying to (22) the isomorphism T^{-1} (which sends x to 1) we obtain the lattice vector field

$$\alpha = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{(0,0)} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{(0,1)} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{(1,0)} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}_{(2,-1)}.$$

The 2-tuples Q_i appearing in Algorithm are

$$(0,1) < (0,2) < (1,0) < (1,1) < (2,-1) < (2,0).$$

Now we go through the loop in the algorithm. In this case $m = 6$. Below the symbol $\sim$ means that the terms larger than $(2,0)$ are neglected.
For $i = 1$, we have $h = \binom{-1}{0}_{(0,1)}$.

$$\beta = \alpha + [h, \alpha] + \frac{1}{2}[h, [h, \alpha]] + \cdots \sim \alpha + \begin{pmatrix} -1 \\ 0 \end{pmatrix}_{(0,1)} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}_{(1,1)} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}_{(2,0)} \sim$$
$$\begin{pmatrix} 0 \\ -1 \end{pmatrix}_{(0,0)} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{(1,0)} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}_{(1,1)} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}_{(2,-1)} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}_{(2,0)},$$

and we set as α the sum in the second line of the above display.
For $i = 2$, the corresponding term $(0,2)$ is the zero term, so α stays unchanged.
For $i = 3$, the term $(1,0)$ is resonant, so α does not change.

For $i = 4$, $h = \binom{-1}{1}_{(1,1)}$, so

$$\beta = \alpha + [h, \alpha] + \cdots \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{(0,0)} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{(1,0)} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}_{(2,-1)} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}_{(2,0)},$$

and we set as α the obtained expression.
For $i = 5$, we have $h = \binom{0}{2}_{(2,-1)}$, so

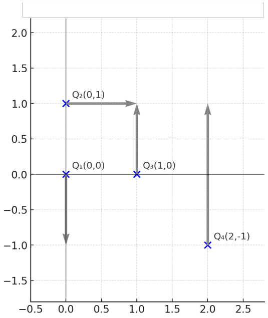
$$\alpha + [h, \alpha] + \cdots \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{(0,0)} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{(1,0)} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}_{(2,0)}.$$

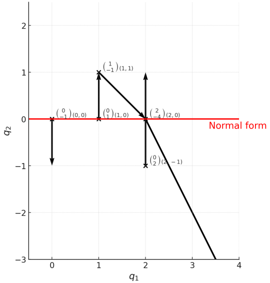
Applying the map (18) to the latter lattice vector field we obtain the system

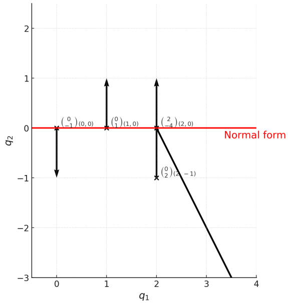
$$\dot{x} = x \odot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + x \odot \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_1 + x \odot \begin{pmatrix} 2 \\ -4 \end{pmatrix} x_1^2,$$

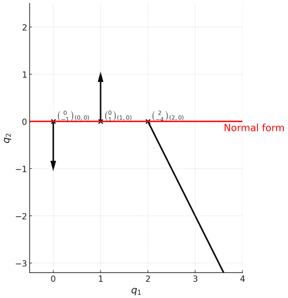
or, in the traditional form,

$$\begin{aligned} \dot{x}_1 &= 2x_1^3, \\ \dot{x}_2 &= -x_2 + x_1 x_2 - 4x_1^2 x_2. \end{aligned}$$









Normal form lattice: $\langle Q, \lambda \rangle = 0 \Rightarrow q_1 0 + q_2(-1) = 0 \Rightarrow q_2 = 0.$

The method of Bruno (or Bruno-Pliss method) is based on

Proposition 1

Assume that system (1) *coincides with the normal form up to terms of order* $m-1$, *that is,*

$$\dot{x} = Ax + G^{(m)}(x) + F(x), \quad Ax = \lambda \odot x, \tag{23}$$

where $1 \leq \mathit{ord}\ \ G^{(m)}(x) \leq m-1, \quad m \leq \mathit{ord}\ \ F(x)$ *and* $G^{(m)}(x)$ *contains only the resonant terms. There is* h_m *such that* $m \leq \operatorname{ord} h_m \leq 2m-1$ *and the substitution*

$$x = y + h_m(y) \tag{24}$$

transforms (23) *into the system*

$$\dot{y} = G(y) + \widetilde{F}(y) \tag{25}$$

where $G(y) = \lambda \odot y + G^{(m)}(y) + \widetilde{G}(y)$ *and* $m \leq \operatorname{ord} \widetilde{G}(y) \leq 2m-1, \quad 2m \leq \operatorname{ord} \widetilde{F}(y)$ *and* $\widetilde{G}(y)$ *contains only resonant terms.*

The proof in [V. R. & S. Walcher, 2025] is different from ones by Bruno and Stolovitch and allows to obtain a simple proof of a theorem of Stolovitch about simultaneous normalization of a family of commuting vector fields.

- A. N. Kolmogorov (1954) applied Newton's method to the studies of dynamical systems $\Longrightarrow$ KAM theory
- V. A. Pliss (1965) introduced iterations $x = y + h_m(y)$ and used them to prove convergence in the case of the linear normal form $\dot{x} = Ax$.
- A.D. Bruno (1971) generalized the method to the case of nonlinear normal forms and got fundamental results on convergence of normal forms.

Bruno's approach:

Look for h_m in the form $h_m = \sum_\delta h_\delta$, where

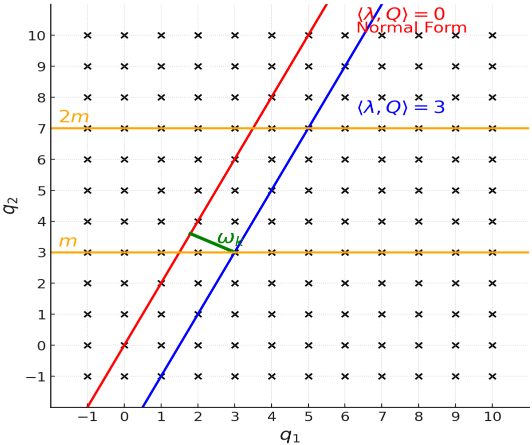
$$h_\delta(x) = \sum_{Q:\langle Q,\lambda\rangle=\delta} (x \odot H_Q)x^Q.$$

It turns out that h_δ are eigenfunctions of ad_A,

$$\mathrm{ad}_A h_\delta = \delta h_\delta,$$

and h_δ depends only on the term of the normal form up to order m.

- NB: Bruno did not worked with the adjoint operator.



For $m = 2^k$, ω_k is the "distance" to the normal form, but for the points P of the lattice with $m + 1 < order\ P < 2m$ ("smallest small denominator").

The essence of Bruno's method (with some our improvements):

- the blue line is invariant under the action of ad $_A$;
- the segment between the orange lines is computed using ad $_{G^{(m)}}$.

Main steps of the proof of convergence:

- h_δ satisfies the equation $(I - cD)h_\delta = F$, where c, D, F are known a scalar, matrix and vector, respectively.
- $(I - cD)h_\delta = F \implies$

$$h_\delta = (I - cD)^{-1} F = (I + cD + (cD)^2 + \dots) F.$$

- Condition A $\implies D^2 = 0$!!!
- Therefore, $h_\delta = (I + cD)F$, where D depends on the terms of the normal form up to degree m, F is a known vector.
- Estimate h_δ (and, therefore, h_m) in terms of ω_k $(m = 2^k)$ and check that the estimation yields Condition ω.

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Our approach (V. R. & S. Walcher, 2025) considerably simplifies the computation of h_δ (two first steps of the proof).

When Condition ω holds?

Theorem 3 (V.R. & S. Walcher, arxiv.org/abs/2510.00925, 2025)

1) If the Siegel-Pliss condition holds, then Condition ω is satisfied.
2) If all λ_i are algebraic numbers, then the Siegel-Pliss condition (and, therefore, Condition ω) is satisfied.

Proof. A complex number is algebraic when it is a root of a monic polynomial with rational coefficients, and an algebraic integer when it is a root of a monic polynomial with integer coefficients.
Let L be a finite Galois extension of $\mathbb{Q}$ that contains all λ_i, G be the Galois group of L, and set $|G| = \nu + 1$. Let all λ_i be algebraic integers. For any $Q = (q_1, \ldots, q_n)^\top \in \mathbb{Z}^n$, one has that $\langle Q, \lambda \rangle$ is an algebraic integer. We use the fact that the norm of an algebraic integer, i.e., the products of all its conjugates under the Galois group, is a rational integer. Thus,

$$\prod_{\sigma \in G} \left(\sum_{1 \le i \le n} q_i \sigma(\lambda_i) \right) \in \mathbb{Z}.$$

In case $\langle Q, \lambda \rangle \neq 0$ this implies

$$1 \leq |\langle Q, \lambda \rangle| \cdot \prod_{\sigma \in G \setminus \mathrm{id}} \left| \sum_i q_i \sigma(\lambda_i) \right|.$$

With

$$C^* := \max \{ |\sigma(\lambda_i)|;\ \sigma \in G,\ 1 \leq i \leq n \}$$

one has

$$\left| \sum_i q_i \sigma(\lambda_i) \right| \leq C^* \cdot |Q|$$

for every $\sigma \in G$, and therefore

$$1 \leq |\langle Q, \lambda \rangle| \cdot C^{*\nu} \cdot |Q|^\nu$$

for all $Q \in \mathbf{N}$. This is the assertion when all λ_i are algebraic integers. Passing to general algebraic numbers, there exists a natural number N such that all $N \cdot \lambda_i$ are algebraic integers. From the reasoning in part (ii) we then obtain

$$\frac{1}{N} \leq |\langle Q, \lambda \rangle| \cdot C^{*\nu} \cdot |Q|^\nu$$

for all $Q \in \mathbf{N}$. □

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Thank you for your attention!