

# V SYMPOSIUM ON PLANAR VECTOR FIELDS

Isolated invariant closed curves in planar piecewise  
linear systems with no sliding regions

Victoriano Carmona, Fernando Fernández-Sánchez

Dpto. Matemática Aplicada II & IMUS  
Universidad de Sevilla



Douglas D. Novaes

Departamento de Matemática & IMECC  
Universidade Estadual de Campinas



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# Planar Piecewise Linear System (PWL) with Two Zones

$$\dot{x} = \begin{cases} A_L x + b_L, & \text{if } x_1 < 0, \\ A_R x + b_R, & \text{if } x_1 > 0, \end{cases}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad A_{L,R} = \begin{pmatrix} a_{11}^{L,R} & a_{12}^{L,R} \\ a_{21}^{L,R} & a_{22}^{L,R} \end{pmatrix} \quad b_{L,R} = \begin{pmatrix} b_1^{L,R} \\ b_2^{L,R} \end{pmatrix}$$

POINCARÉ SECTION (Separation line)  $\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$

The main objective:

Prove the uniqueness of non-trivial isolated invariant closed curves and determine their stability as a simple function of the parameters of the system (with no sliding regions)

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$$b_{L,R} = \begin{pmatrix} b_1^{L,R} \\ b_2^{L,R} \end{pmatrix}$$

• If  $a_{12}^L a_{12}^R \leq 0$ ,

then the flow system does not cross twice the **Poincaré section**

$$\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$$

Since system is piecewise linear, this condition precludes the existence non-trivial isolated invariant closed curves.

Thus, we assume that  $a_{12}^L a_{12}^R > 0$

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## Liénard Canonical Form

- If  $a_{12}^L a_{12}^R > 0$ , then there exists a homeomorphism preserving  $\Sigma$  such that the system can be transformed into the Liénard canonical form

$$\begin{cases} \dot{x} = T_L x - y \\ \dot{y} = D_L x - a_L \end{cases} \text{ for } x < 0, \quad \begin{cases} \dot{x} = T_R x - y + b \\ \dot{y} = D_R x - a_R \end{cases} \text{ for } x > 0,$$

$$a_{L,R} = a_{12}^{L,R} b_2^{L,R} - a_{22}^{L,R} b_1^{L,R}$$

$$T_{L,R} = \text{tr}(A^{L,R})$$

$$D_{L,R} = \det(A^{L,R})$$

$$b = \frac{a_{12}^L}{a_{12}^R} b_1^R - b_1^L$$

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

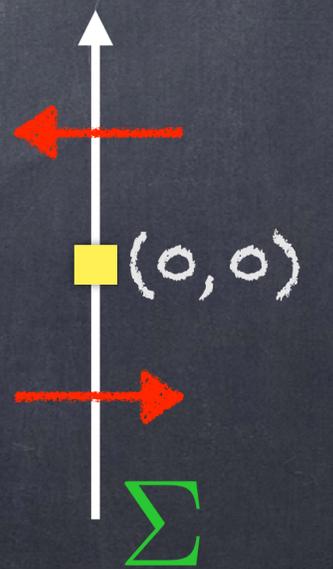
## Liénard Canonical Form

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$$a_{L,R} = a_{12}^{L,R} b_2^{L,R} - a_{22}^{L,R} b_1^{L,R} \quad T_{L,R} = \text{tr}(A^{L,R}) \quad D_{L,R} = \det(A^{L,R}) \quad b = \frac{a_{12}^L}{a_{12}^R} b_1^R - b_1^L$$

The system has no sliding regions if and only if  $b = 0$

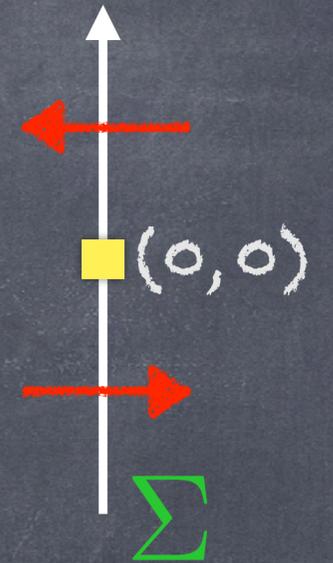
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- The origin is the unique point in  $\Sigma$  where the flow is tangent.

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

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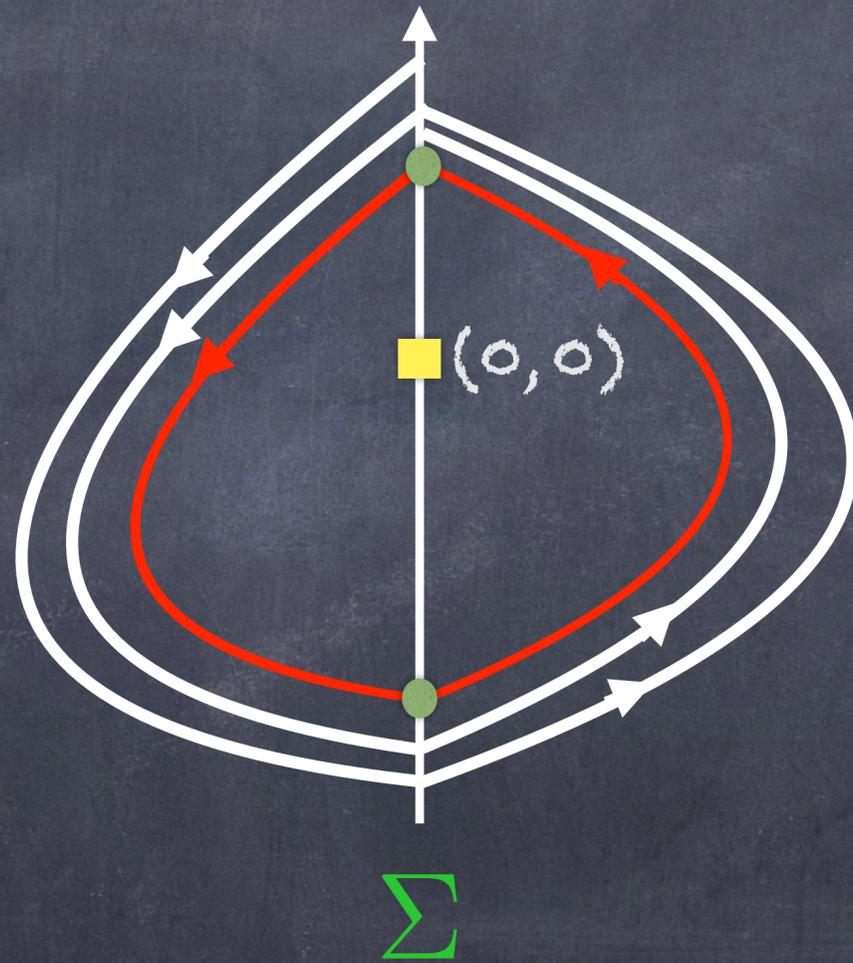
## Non-trivial isolated invariant closed curves:

- Limit cycles: (crossing) isolated periodic orbits.
- Homoclinic connections.
- Two heteroclinic connections consisting a heteroclinic cycle.
- A fold-fold connection.

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Non-trivial isolated invariant closed curves:

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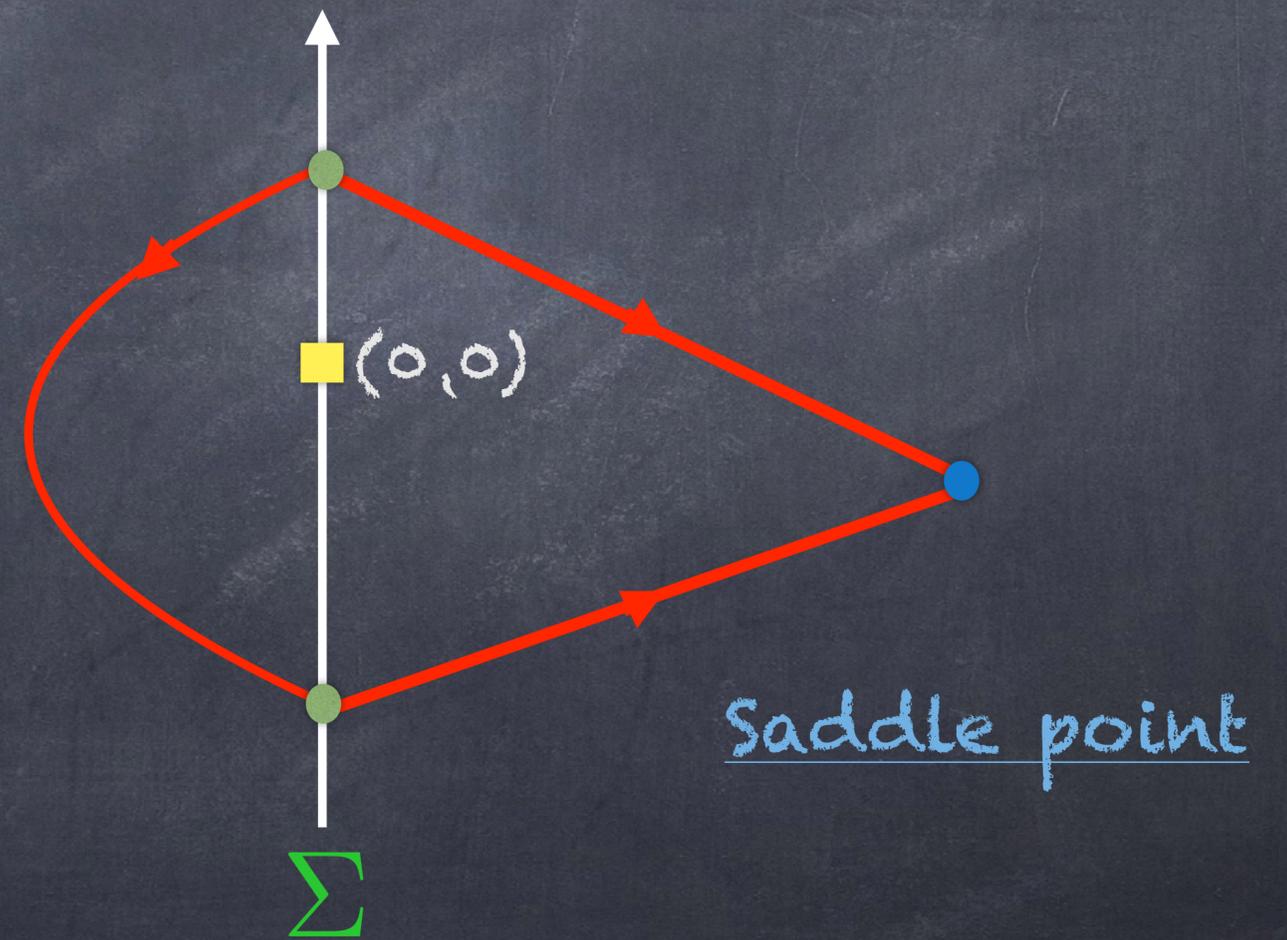
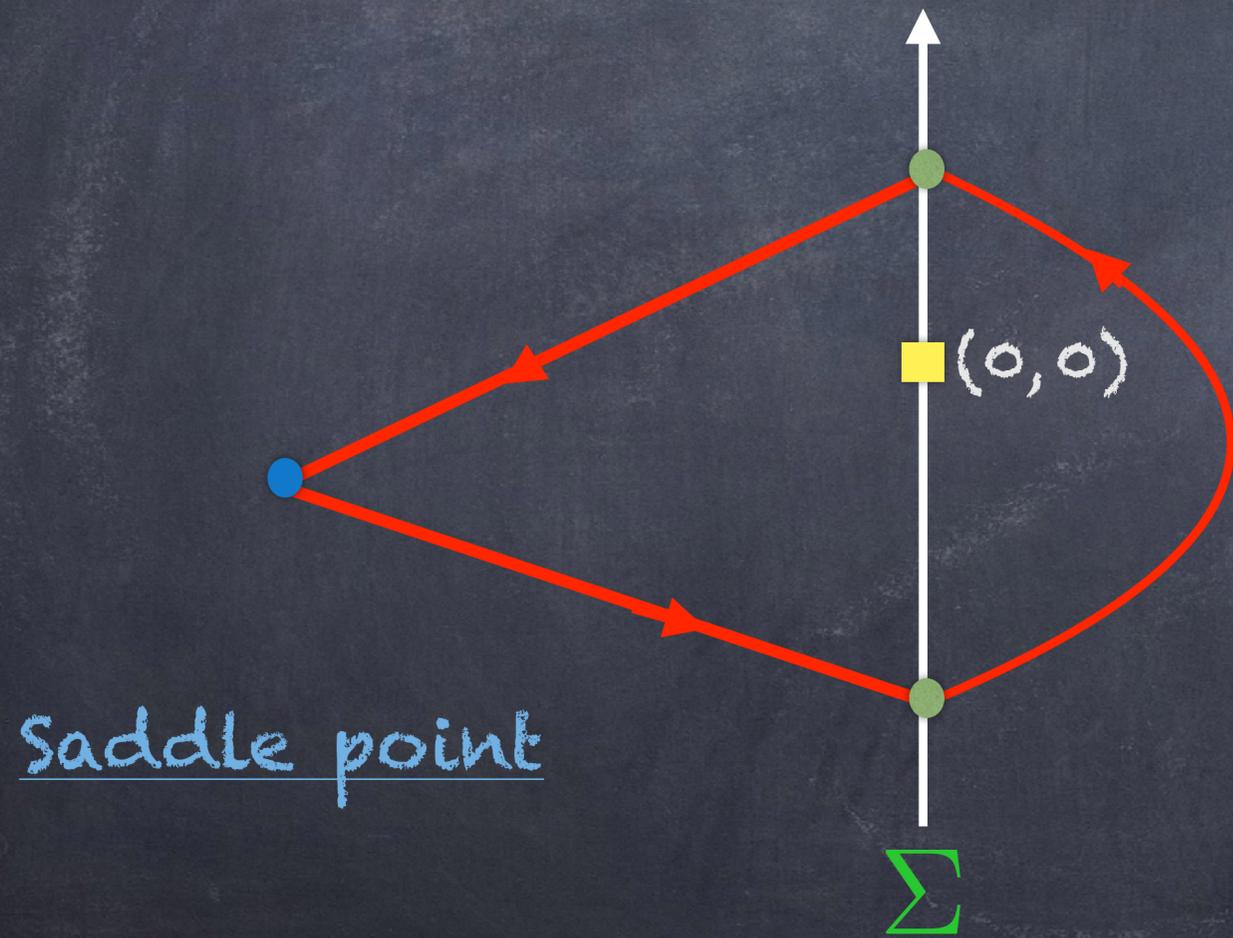
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# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

## Non-trivial isolated invariant closed curves:

- Homoclinic connections.

- Homoclinic connection associated to an equilibrium point outside  $\Sigma$ .



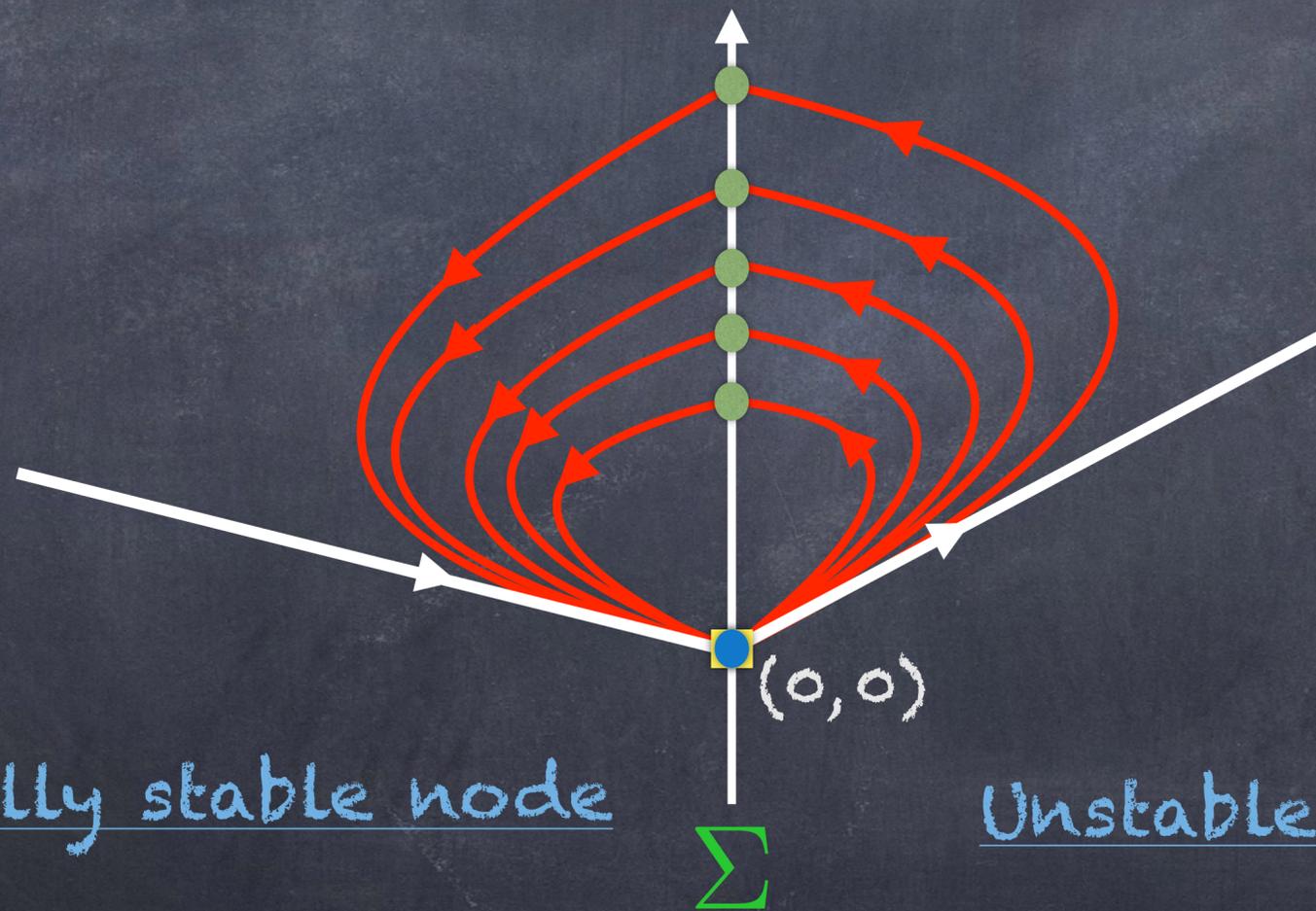
Saddle homoclinic.

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

## Non-trivial isolated invariant closed curves:

### • Homoclinic connections.

- Homoclinic connection associated to an equilibrium point in  $\Sigma$ .



The system has a continuum of homoclinic orbits. These invariant closed curves are not isolated.

Asymptotically stable node

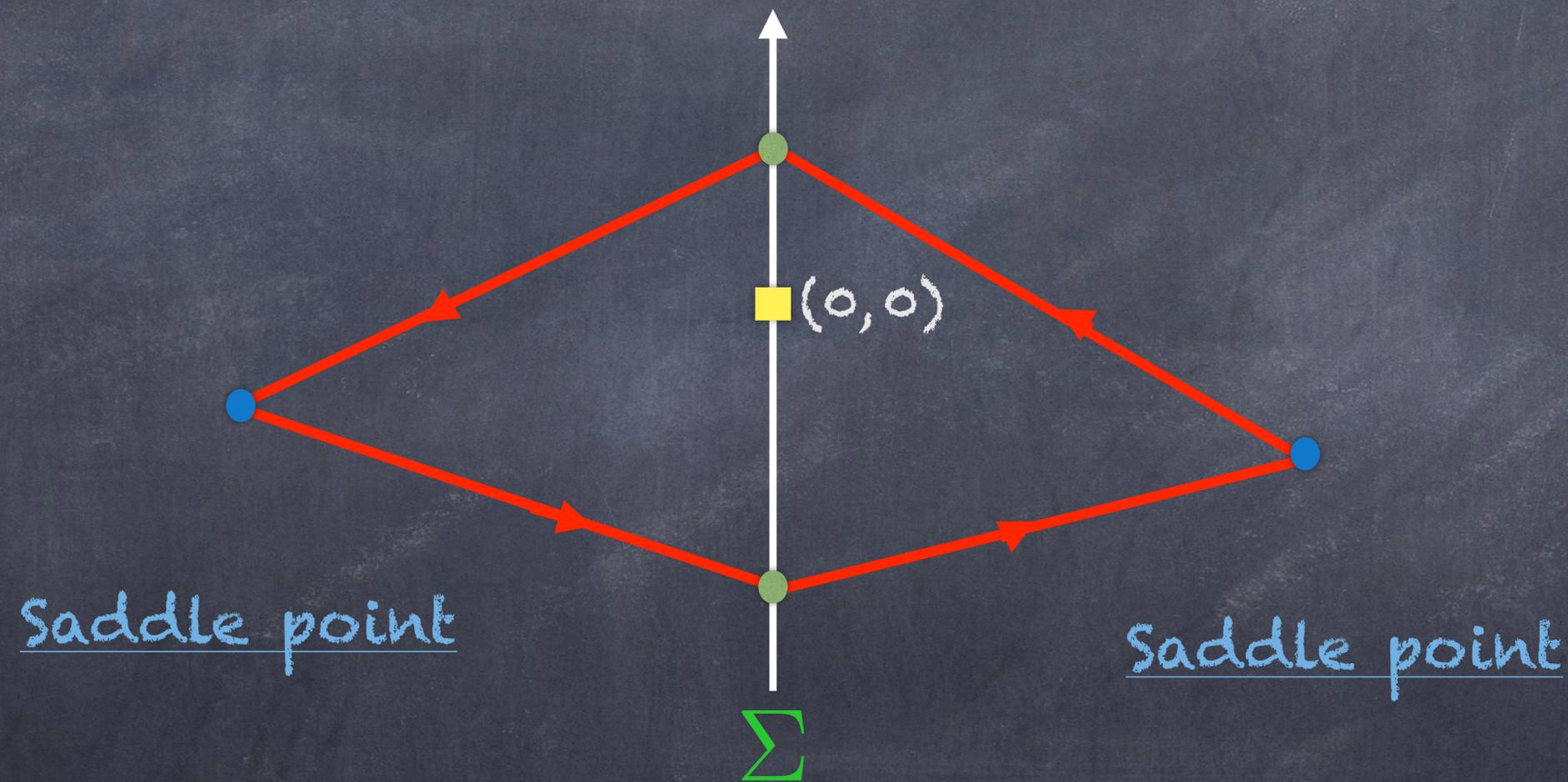
Unstable Node

Node homoclinic

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

Non-trivial isolated invariant closed curves:

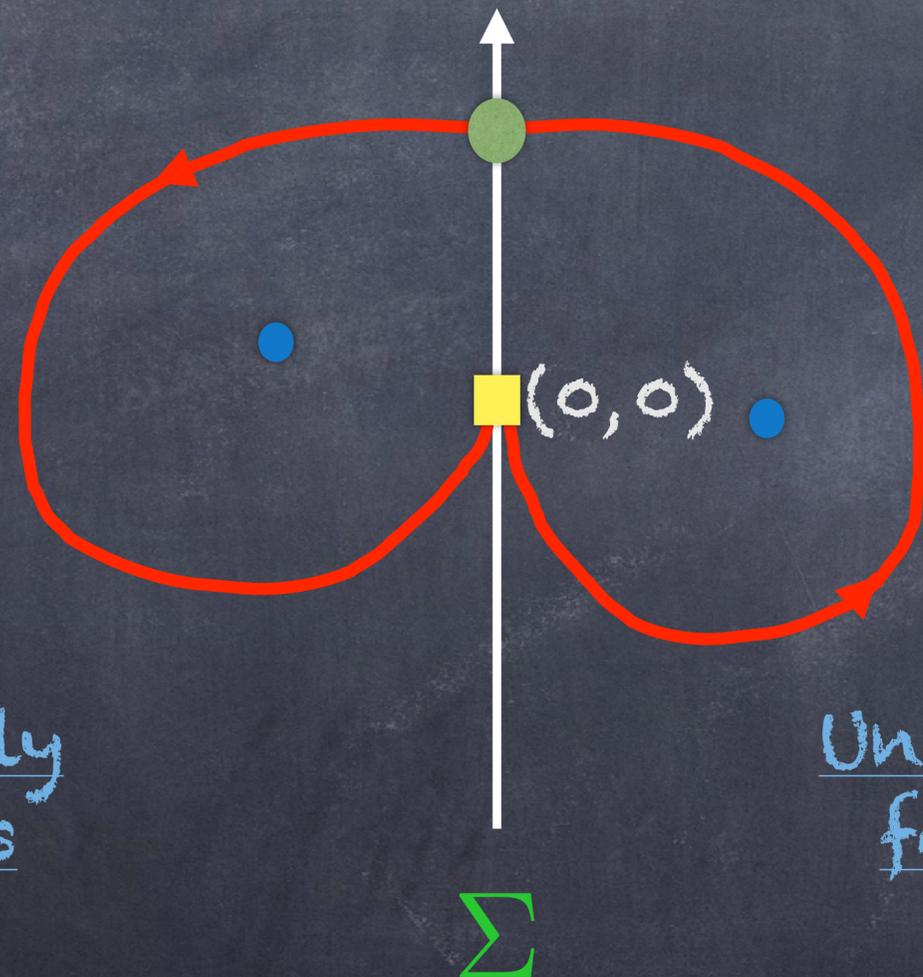
- Two heteroclinic connections consisting a heteroclinic cycle.



# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

Non-trivial isolated invariant closed curves:

- A fold-fold connection.

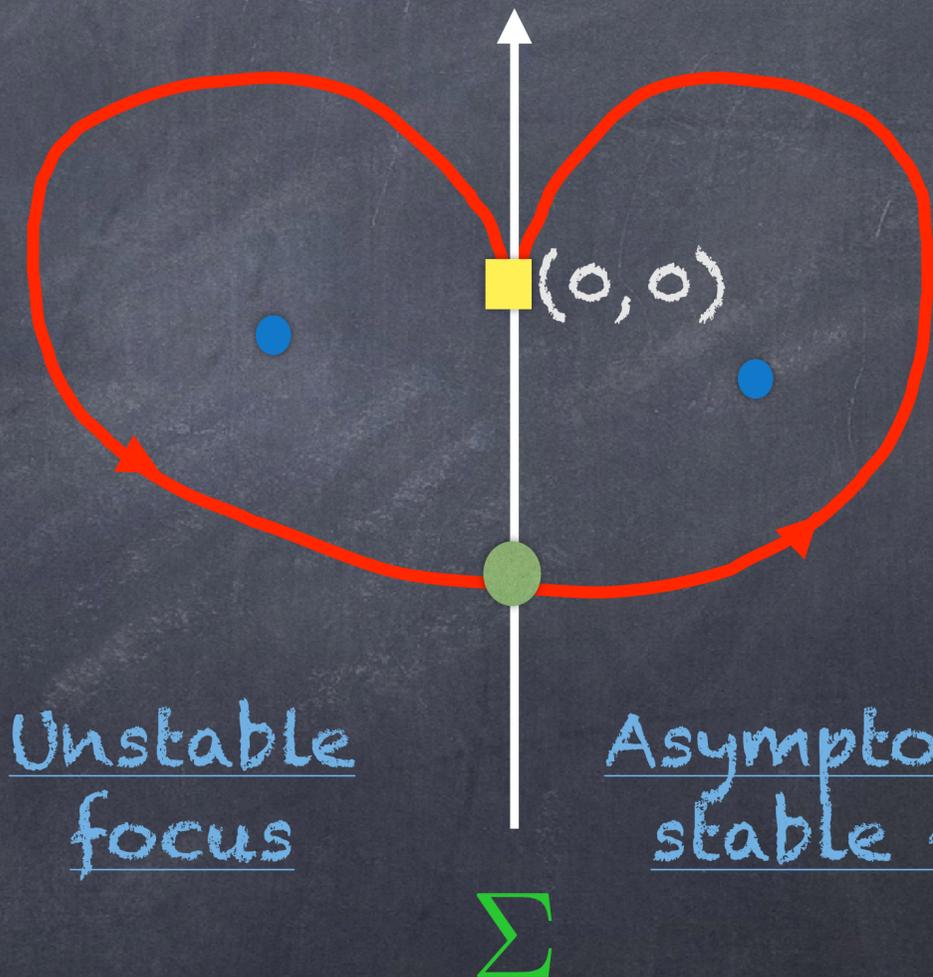


Asymptotically  
stable focus

Unstable  
focus

Unstable  
focus

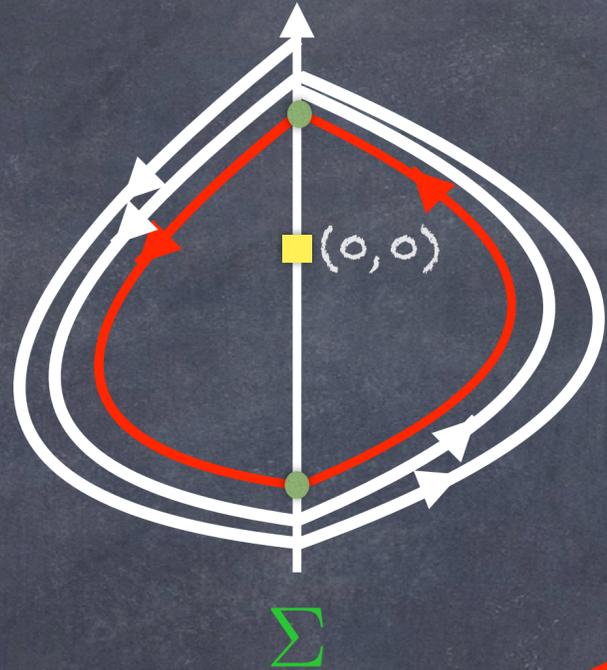
Asymptotically  
stable focus



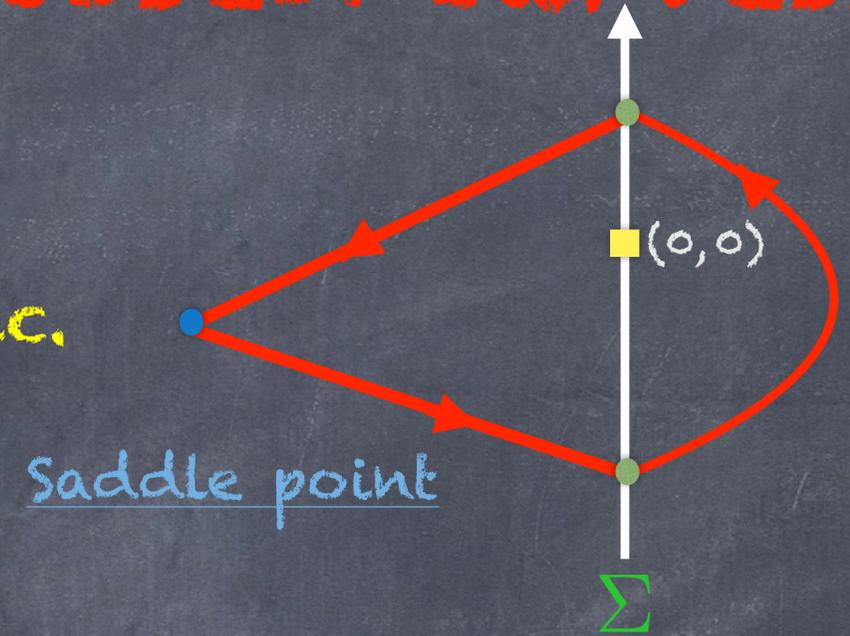
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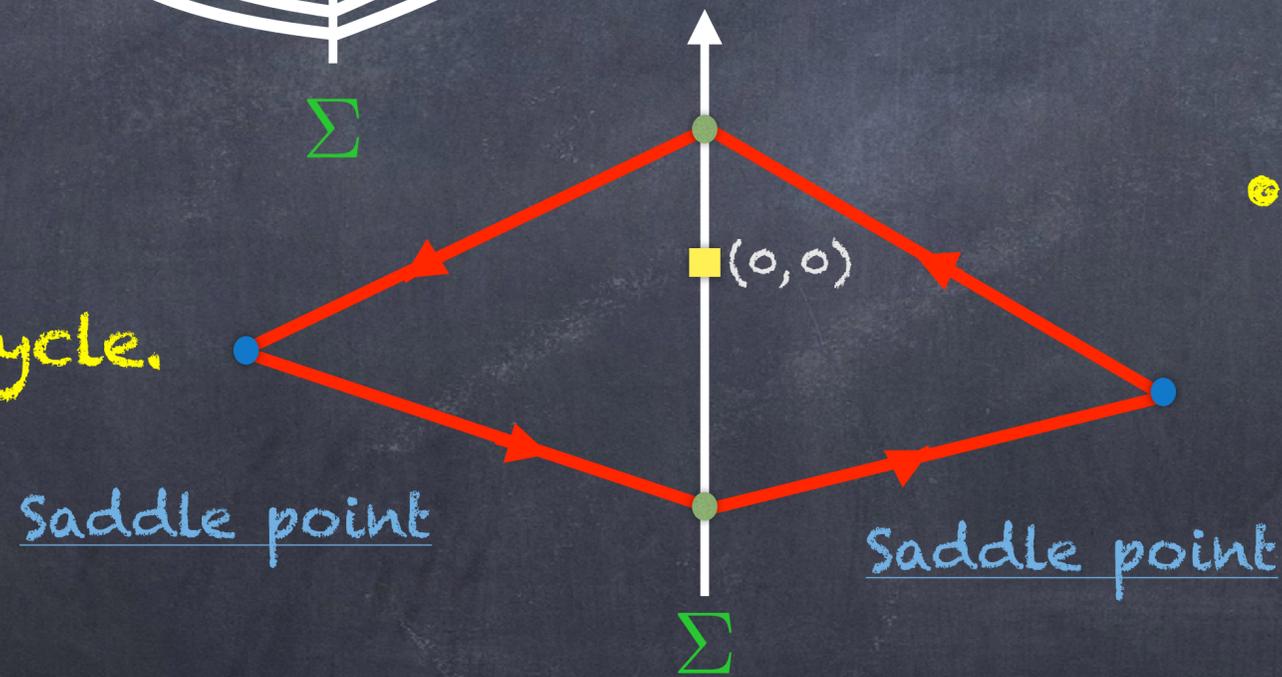
• Limit cycle.



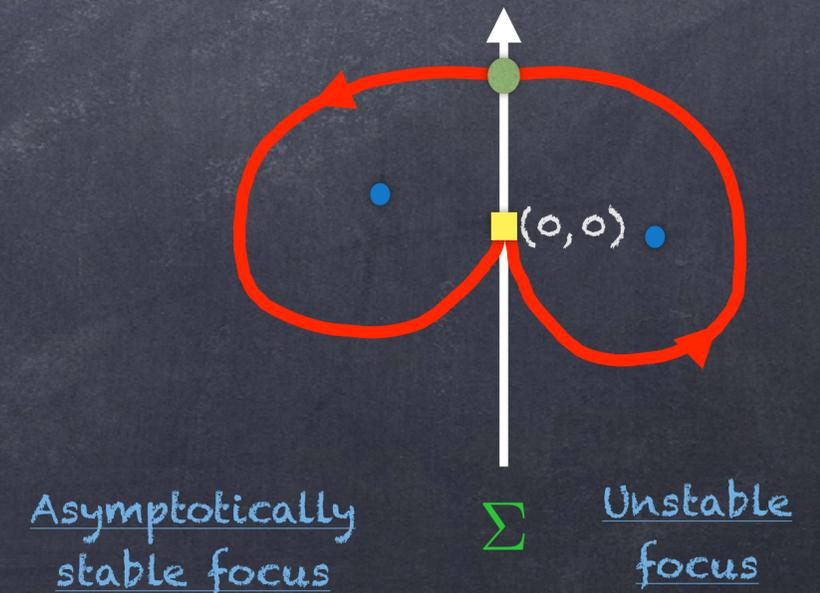
• Saddle homoclinic.



• Heteroclinic cycle.



• A fold-fold connection.



# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

$$\begin{cases} \dot{x} = T_L x - y \\ \dot{y} = D_L x - a_L \end{cases} \text{ for } x < 0, \quad \begin{cases} \dot{x} = T_R x - y \\ \dot{y} = D_R x - a_R \end{cases} \text{ for } x > 0.$$

## Limit cycles

We know:

- When the system is continuous ( $a_L = a_R$ ), the maximum number of crossing limit cycles is 1.

E. Freire, E. Ponce, F. Rodrigo, and F. Torres. Bifurcation sets of continuous piecewise linear systems with two zones. *Internat. J. Bifur. Chaos*, 1998. (Case-by-case study).

V. C., F. Fernández-Sánchez and Douglas D. Novaes. A new simple proof for Lum-Chua's conjecture, *Nonlinear Anal-Hybrid*, 2021. (No case-by-case study).

- In the general case, the maximum number of crossing limit cycles is also 1.

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Here, the value  $\xi_0 = a_R T_L - a_L T_R$  is essential

Theorem: Suppose that the system has a limit cycle. Then, it is unique, hyperbolic and  $\xi_0 \neq 0$ . Moreover,

- If  $\xi_0 < 0$ , then the limit cycle is asymptotically stable.
- If  $\xi_0 > 0$ , then the limit cycle is unstable.

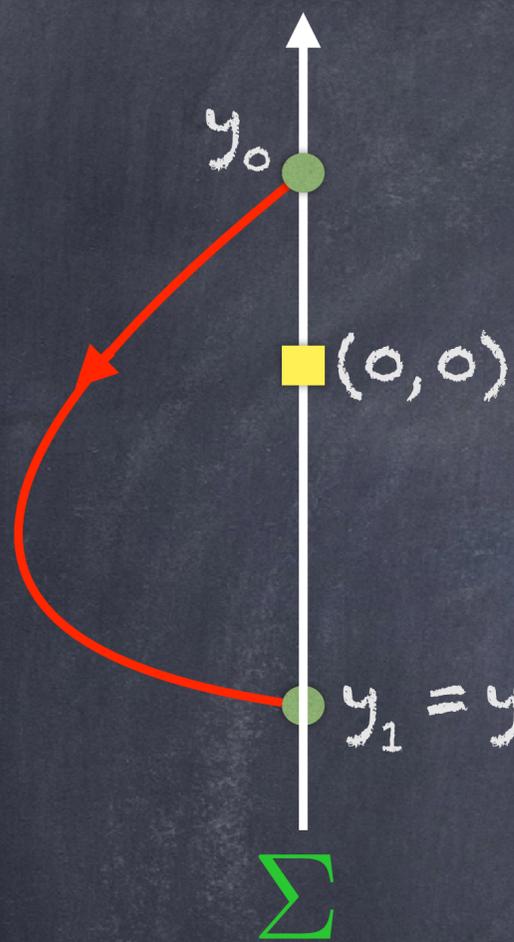
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## The most important and useful tools

- (Manageable) Expression of the Poincaré half-maps and their first derivatives.
- (Manageable) Expressions of the displacement function and its first derivative.

# Poincaré Half-Maps: an integral characterization



We consider the linear system 
$$\begin{cases} \dot{x} = Tx - y \\ \dot{y} = Dx - a \end{cases}$$

The function

$$V(x, y) = D^2x^2 - DTxy + Dy^2 + a(T^2 - 2D)x - aTy + a^2$$

is an inverse integrating factor (IIF) for the linear system.

This means that the vector field

$$G(x, y) = \left( -\frac{Dx - a}{V(x, y)}, \frac{Tx - y}{V(x, y)} \right) \quad V \neq 0$$

- is orthogonal to the flow of the linear system
- and conservative in each connected component of  $V \neq 0$ .

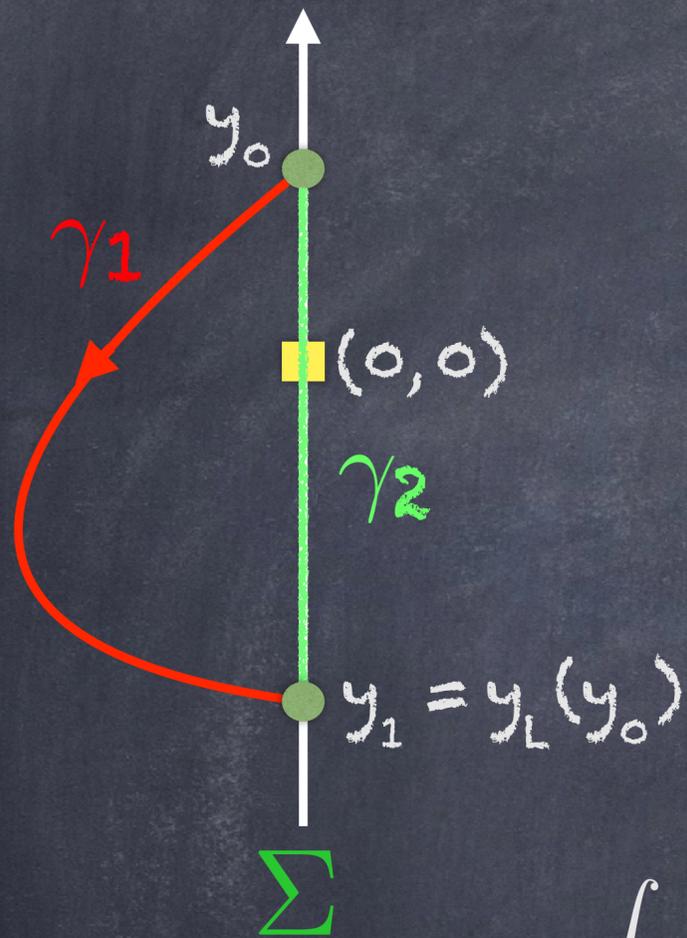
Giacomini, J. Llibre, H. M. Viano. Nonlinearity 9 (1996).

L.R. Berrone, H. Giacomini. Qual. Theory Dyn. Syst. 1 (2000).

I.A. García, M. Grau. Qual. Theory Dyn. Syst. 9 (2010).

I.A. García, D.S. Shafer. JDE 217 (2005).

# Poincaré Half-Maps: an integral characterization



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$$G(x, y) = \begin{pmatrix} -\frac{Dx - a}{V(x, y)}, \frac{Tx - y}{V(x, y)} \end{pmatrix}$$

$$V(x, y) = D^2x^2 - DTxy + Dy^2 + a(T^2 - 2D)x - aTy + a^2$$

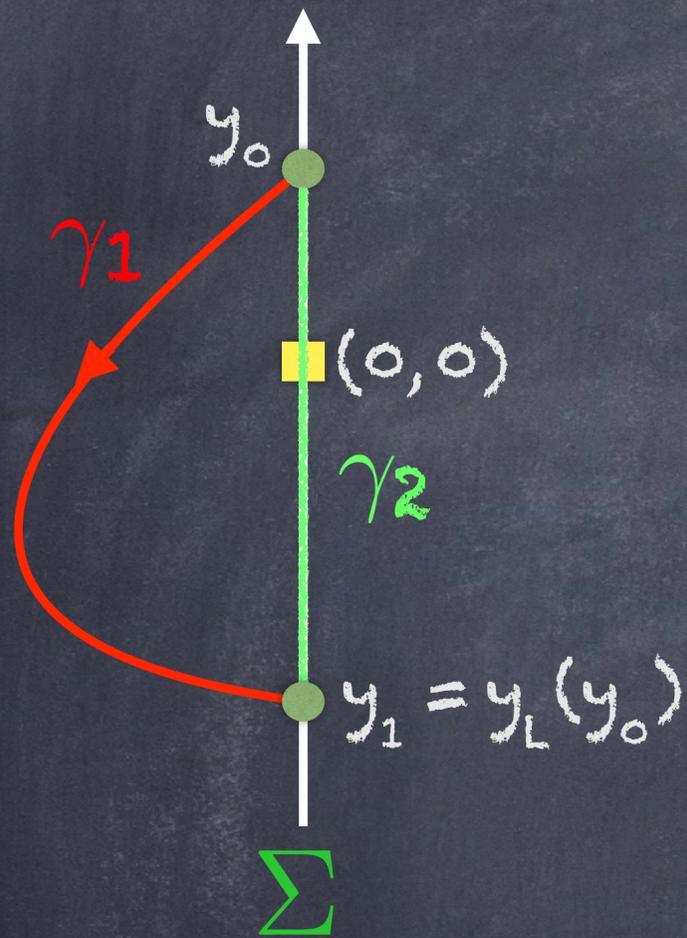
$V^{-1}(\{0\})$  contains the equilibria and the linear invariant manifolds of the linear system

Suppose that the equilibria are not located in  $\text{int}(\gamma_1 \cup \gamma_2)$

$$\int_{\gamma_1 \cup \gamma_2} G(x, y) \cdot dr = 0 \quad (\text{conservative})$$

$$\int_{\gamma_1 \cup \gamma_2} G(x, y) \cdot dr = \int_{\gamma_1} G(x, y) \cdot dr + \int_{\gamma_2} G(x, y) \cdot dr = \int_{\gamma_2} G(x, y) \cdot dr \quad (\text{orthogonal})$$

# Poincaré Half-Maps: an integral characterization



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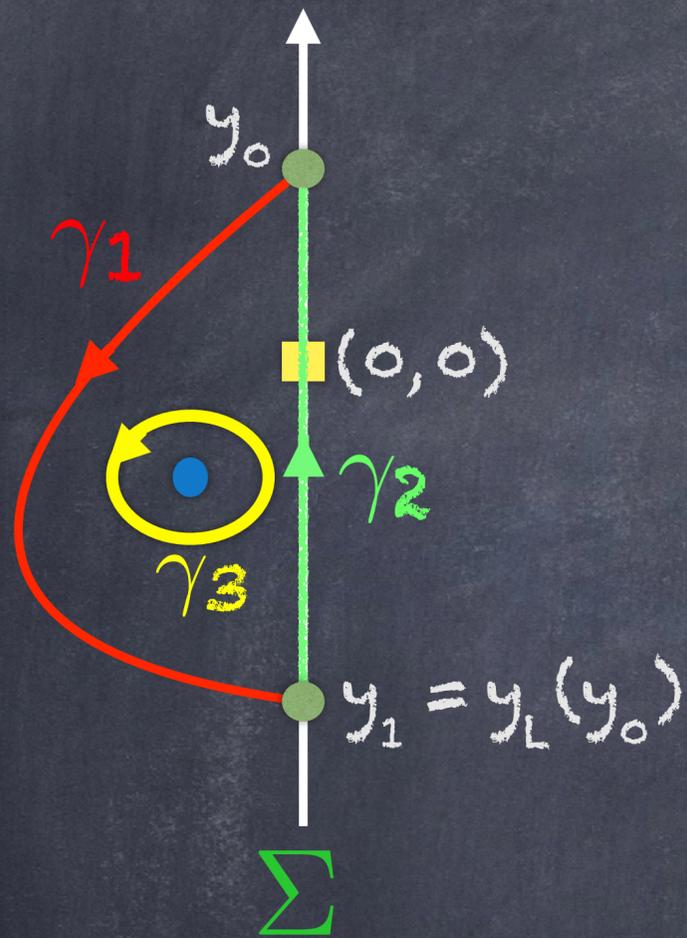
Suppose that the equilibrium is not located in  $\text{int}(\gamma_1 \cup \gamma_2)$

$$\int_{\gamma_1 \cup \gamma_2} G(x, y) \cdot dr = 0$$

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$$0 = \int_{\gamma_2} G(x, y) \cdot dr = \int_{y_L(y_0)}^{y_0} \frac{-y}{V(0, y)} dy$$

# Poincaré Half-Maps: an integral characterization



$$\begin{cases} \dot{x} = Tx - y \\ \dot{y} = Dx - a \end{cases} \quad G(x, y) = \begin{pmatrix} -\frac{Dx - a}{V(x, y)}, \frac{Tx - y}{V(x, y)} \end{pmatrix}$$

$$V(x, y) = D^2x^2 - DTx y + Dy^2 + a(T^2 - 2D)x - aTy + a^2$$

Suppose that the equilibrium is located in  $\text{int}(\gamma_1 \cup \gamma_2)$

$$\int_{\gamma_1 \cup \gamma_2} G(x, y) \cdot dr = \int_{\gamma_3} G(x, y) \cdot dr \quad (\text{conservative})$$

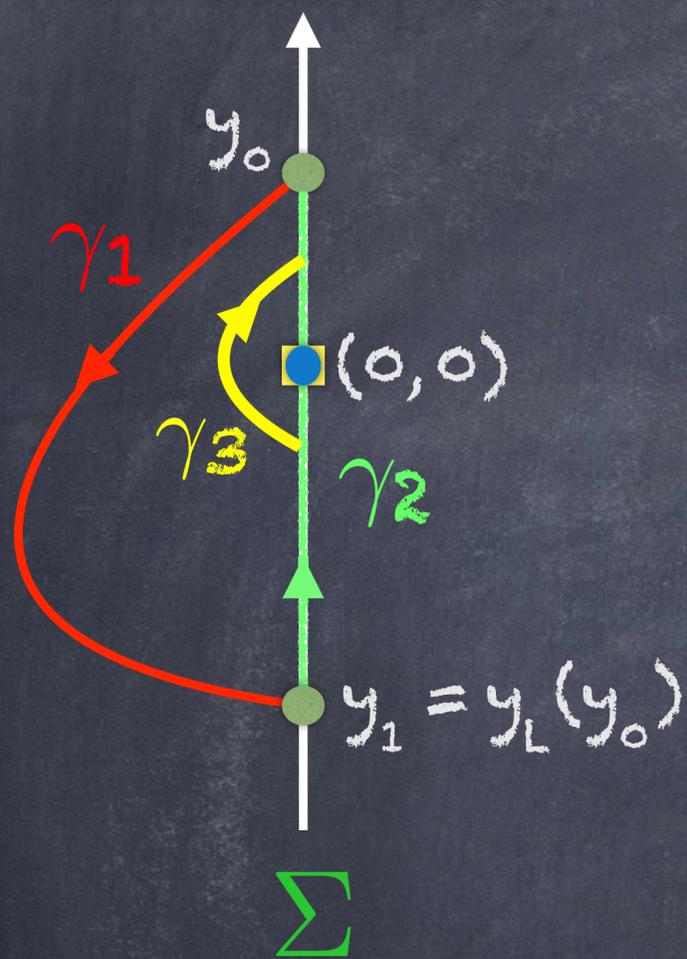
$$\int_{\gamma_3} G(x, y) dr = 2\pi \left( D_R \sqrt{4D_R - T_R^2} \right)^{-1} \quad (\text{a direct computation})$$

$$\int_{\gamma_1 \cup \gamma_2} G(x, y) \cdot dr = \int_{\gamma_1} G(x, y) \cdot dr + \int_{\gamma_2} G(x, y) \cdot dr = \int_{\gamma_2} G(x, y) \cdot dr$$

$$\int_{y_L(y_0)}^{y_0} \frac{-y}{V(0, y)} dy = 2\pi \left( D_R \sqrt{4D_R - T_R^2} \right)^{-1}$$

# Poincaré Half-Maps: an integral characterization

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$$V(x, y) = D^2x^2 - DTxy + Dy^2 + a(T^2 - 2D)x - aTy + a^2$$

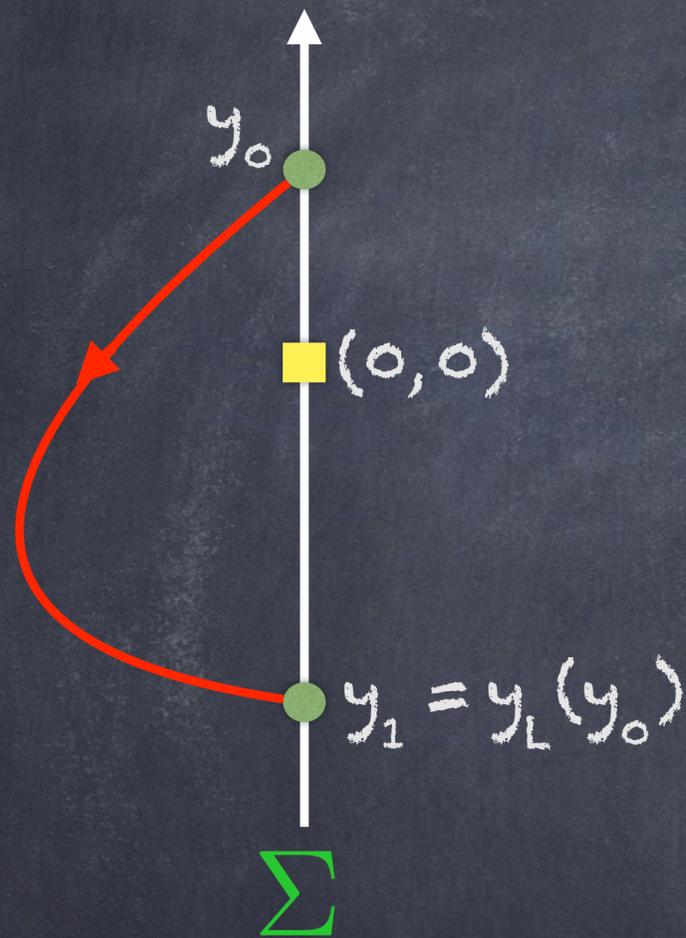
Suppose that the equilibrium point  $(0,0)$  is located on  $\Sigma$

$$\text{PV} \left\{ \int_{y_L(y_0)}^{y_0} \frac{-y}{V(0, y)} dy \right\} = \pi \left( D_R \sqrt{4D_R - T_R^2} \right)^{-1}$$

PV system stands for the Cauchy Principal Value at the origin.

# Poincaré Half-Maps: an integral characterization

The forward Poincaré half-map  $y_1 = y_L(y_0)$



$$y_L : I_L \subset [0, +\infty) \rightarrow (-\infty, 0]$$

$$\text{PV} \left\{ \int_{y_L(y_0)}^{y_0} \frac{-y}{W_L(y)} dy \right\} = c_L T_L, \text{ for } y_0 \in I_L,$$

$$W_L(y) = D_L y^2 - a_L T_L y + a_L^2 \geq 0$$

(i)  $c_L = 0$  if  $a_L > 0$

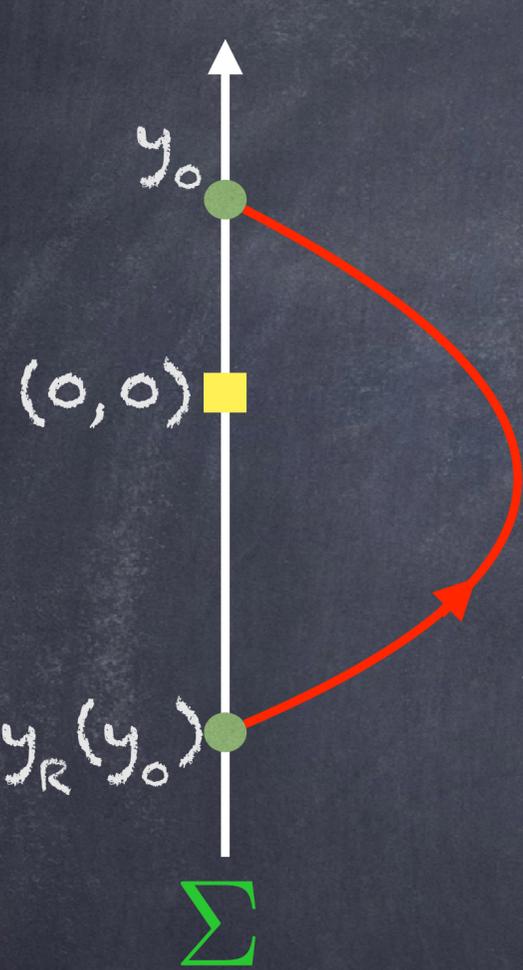
(ii)  $c_L = \pi \left( D_L \sqrt{4D_L - T_L^2} \right)^{-1} \in \mathbb{R}$  if  $a_L = 0$

(iii)  $c_L = 2\pi \left( D_L \sqrt{4D_L - T_L^2} \right)^{-1} \in \mathbb{R}$  if  $a_L < 0$

PV system stands for the Cauchy Principal Value at the origin.

# Poincaré Half-Maps: an integral characterization

## The backward Poincaré half-map



$$y_R : \mathbb{I}_R \subset [0, +\infty) \rightarrow (-\infty, 0]$$

$$\text{PV} \left\{ \int_{y_R(y_0)}^{y_0} \frac{-y}{W_R(y)} dy \right\} = -c_R T_R, \quad \text{for } y_0 \in \mathbb{I}_R,$$

$$W_R(y) = D_R y^2 - a_R T_R y + a_R^2 \geq 0$$

(i)  $c_R = 0$  if  $a_R < 0$

(ii)  $c_R = \pi \left( D_R \sqrt{4D_R - T_R^2} \right)^{-1} \in \mathbb{R}$  if  $a_R = 0$

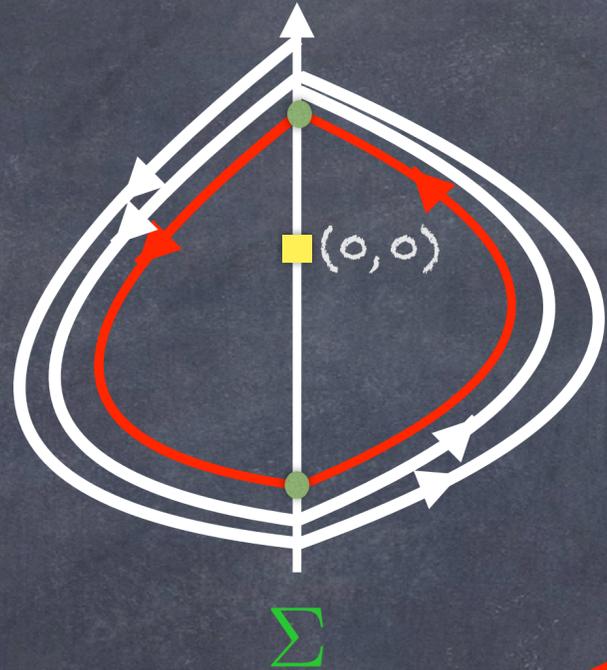
(iii)  $c_R = 2\pi \left( D_R \sqrt{4D_R - T_R^2} \right)^{-1} \in \mathbb{R}$  if  $a_R > 0$

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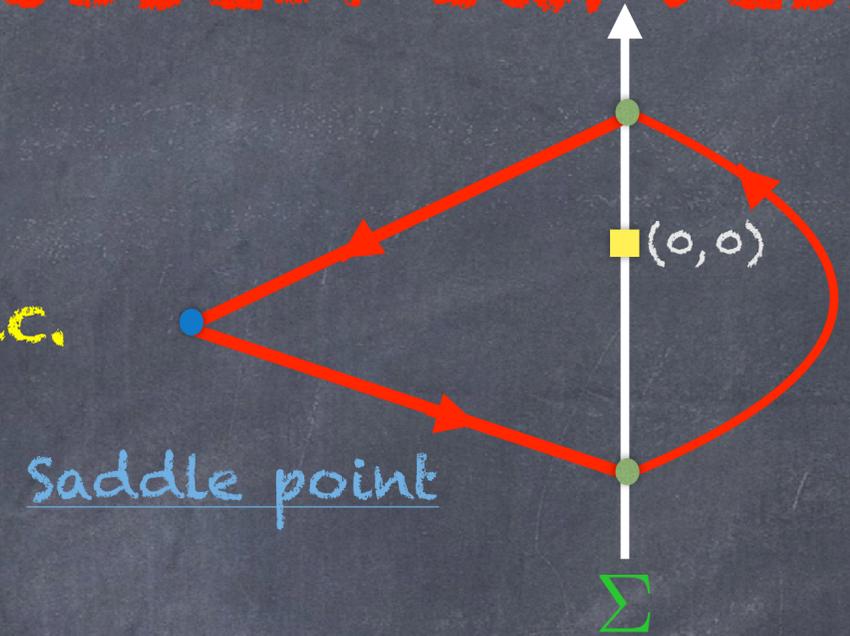
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## Non-trivial isolated invariant closed curves:

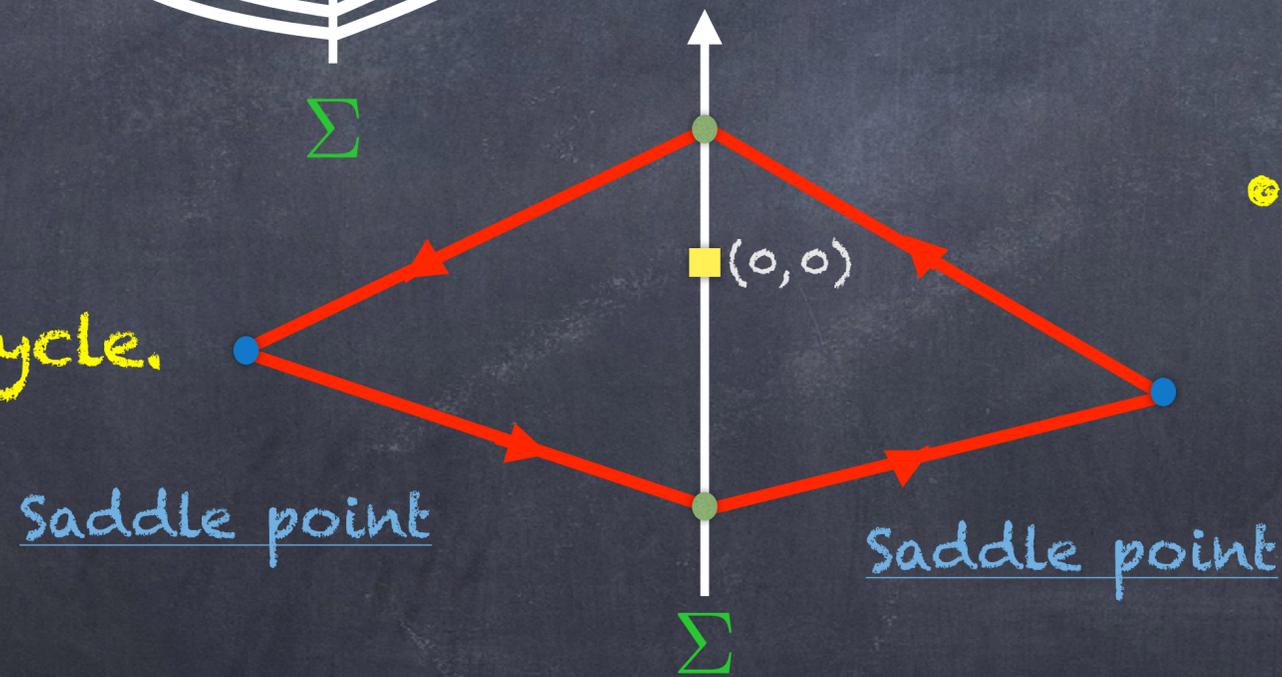
• Limit cycle.



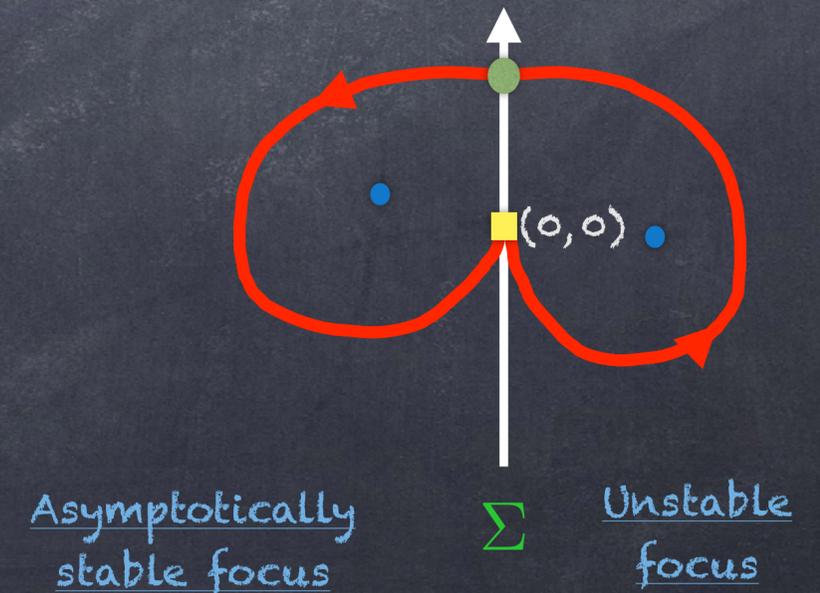
• Saddle homoclinic.



• Heteroclinic cycle.



• A fold-fold connection.



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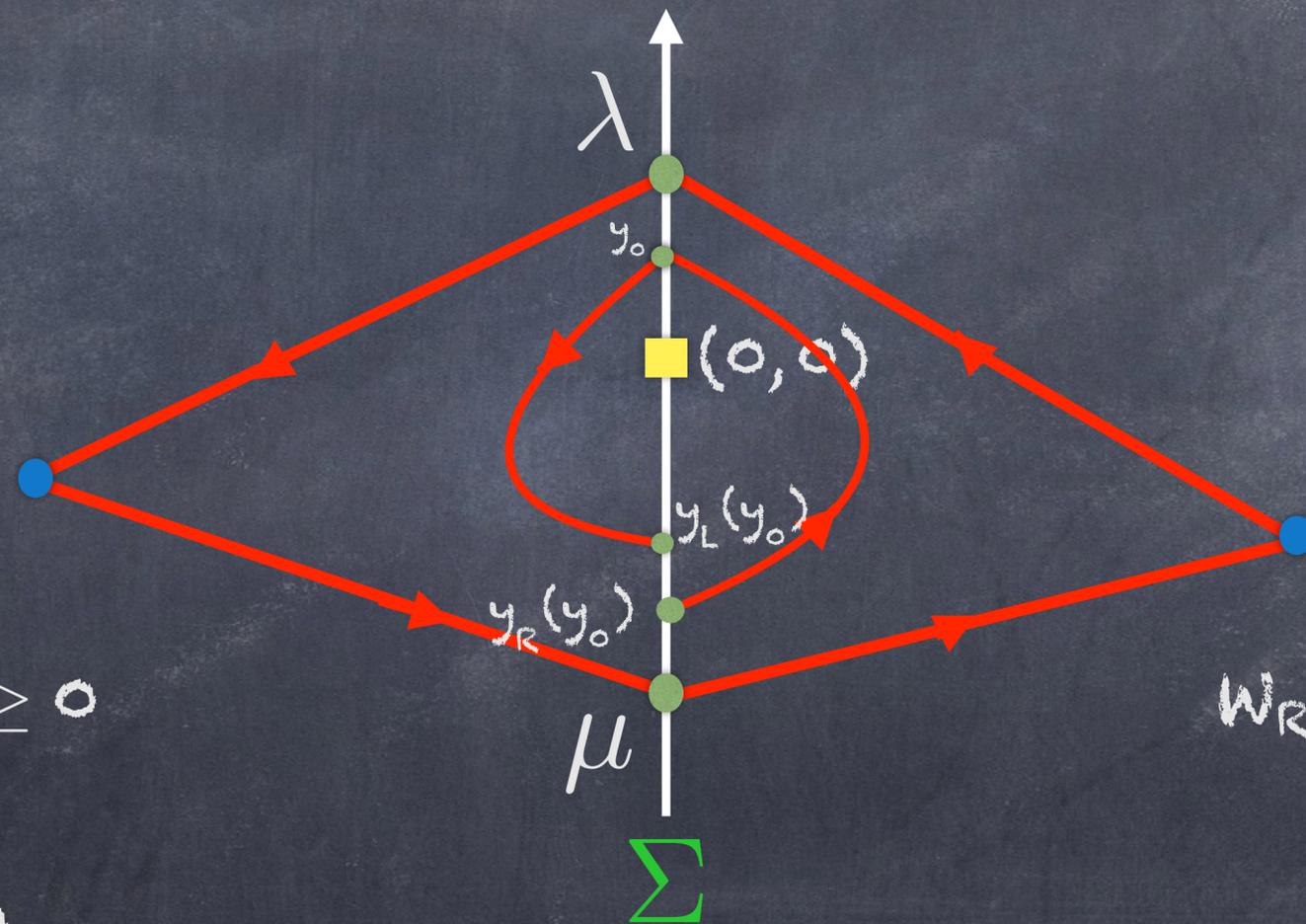
- Two heteroclinic connections consisting a heteroclinic cycle.

$$\int_{y_L(y_0)}^{y_0} \frac{-y}{W_L(y)} dy = 0$$

Saddle point

$$W_L(y) = D_L y^2 - a_L T_L y + a_L^2 \geq 0$$

$$W_L(\mu) = W_L(\lambda) = 0$$



$$\int_{y_R(y_0)}^{y_0} \frac{-y}{W_R(y)} dy = 0$$

Saddle point

$$W_R(y) = D_R y^2 - a_R T_R y + a_R^2 \geq 0$$

$$W_R(\mu) = W_R(\lambda) = 0$$

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

## Non-trivial isolated invariant closed curves:

- Two heteroclinic connections consisting a heteroclinic cycle.

$$W_L(y) = D_L y^2 - a_L T_L y + a_L^2 \geq 0$$

$$\int_{y_L(y_0)}^{y_0} \frac{-y}{W_L(y)} dy = 0$$

$$W_L(y) = D_L y^2 - a_L T_L y + a_L^2 \geq 0$$

$$W_L(\mu) = W_L(\lambda) = 0$$

$$W_R(y) = D_R y^2 - a_R T_R y + a_R^2 \geq 0$$

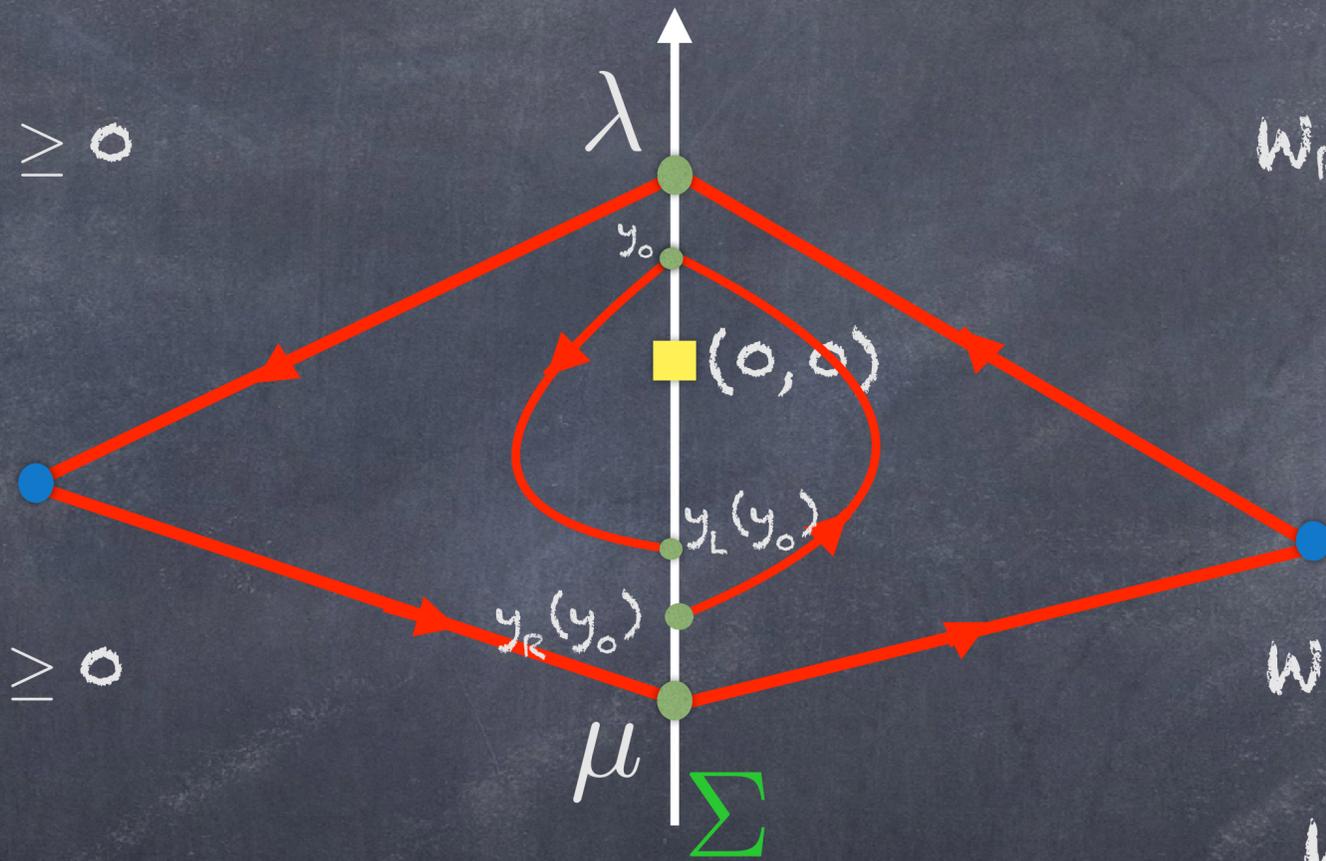
$$\int_{y_R(y_0)}^{y_0} \frac{-y}{W_R(y)} dy = 0$$

$$W_R(y) = D_R y^2 - a_R T_R y + a_R^2 \geq 0$$

$$W_R(\mu) = W_R(\lambda) = 0$$

$\{W_L(y), W_R(y)\}$  is linearly dependent

$$y_L(y_0) = y_R(y_0) \quad \forall y_0 \in (\mu, \lambda)$$

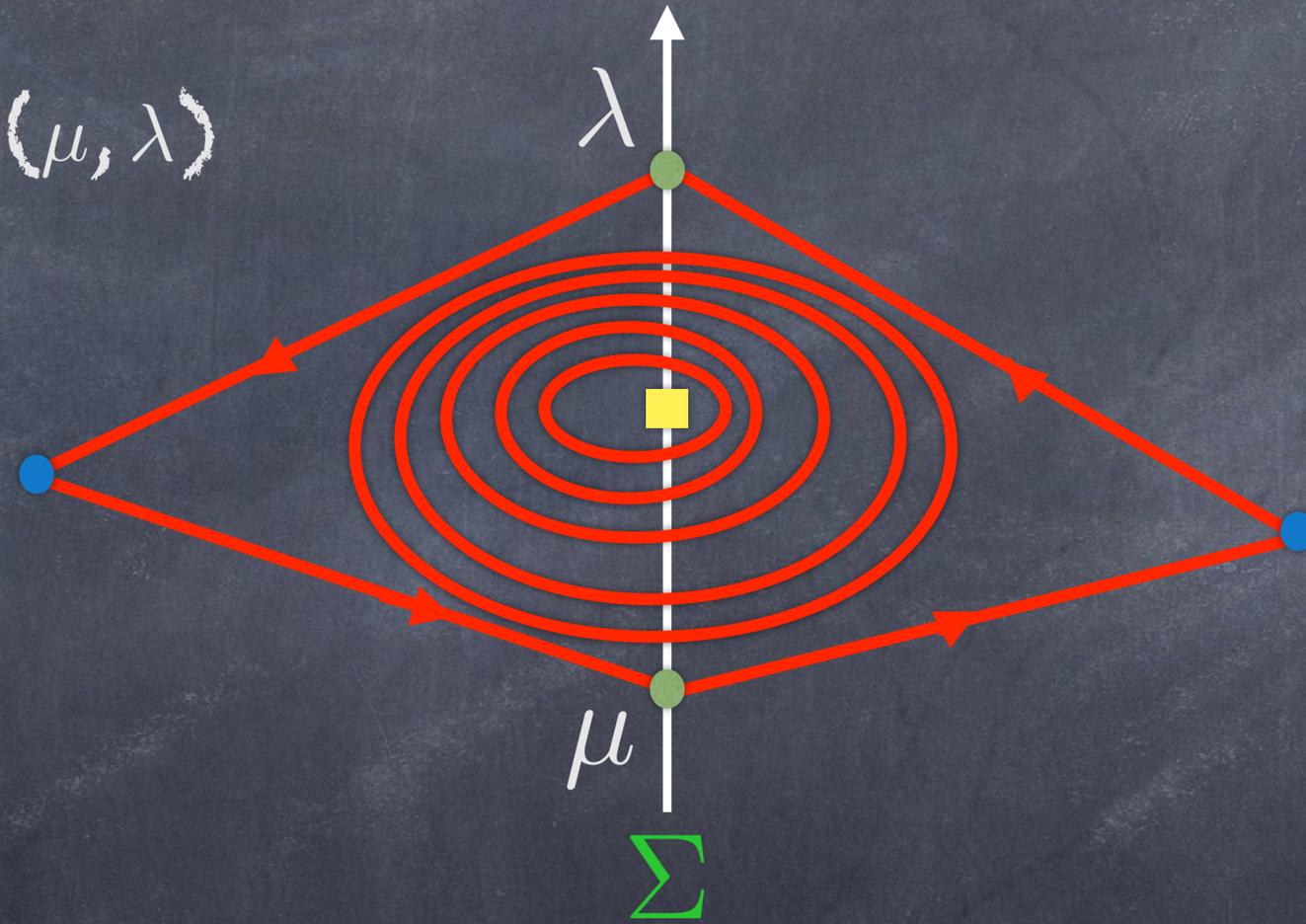


# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

Non-trivial isolated invariant closed curves:

- Two heteroclinic connections consisting a heteroclinic cycle.

$$y_L(y_0) = y_R(y_0) \quad \forall y_0 \in (\mu, \lambda)$$

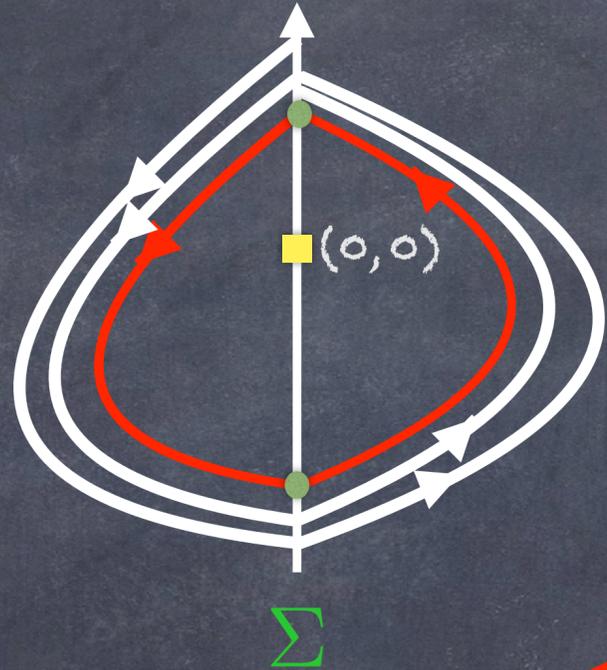


The system has a continuum of crossing periodic orbits (that is, a **crossing period annulus**) and the heteroclinic cycle is not an isolated invariant closed curve.

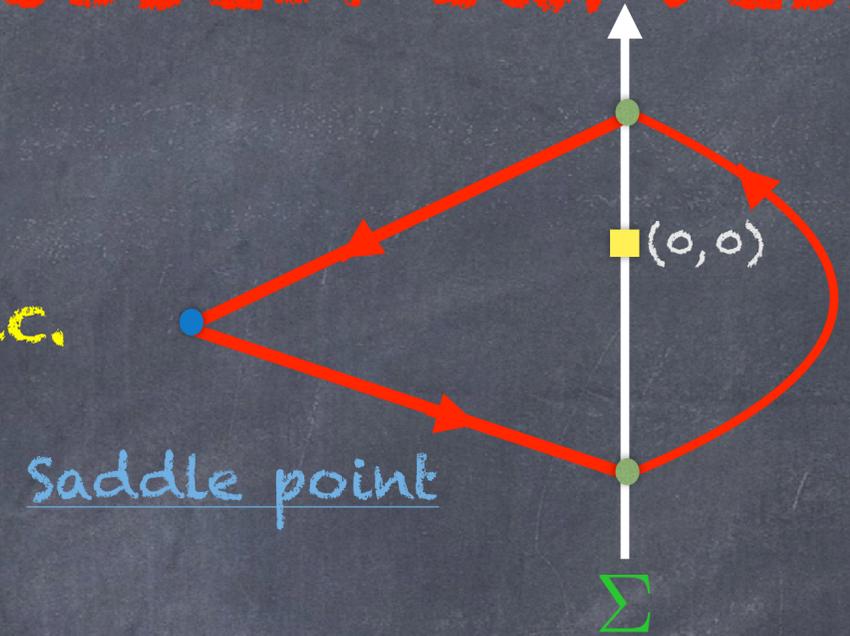
# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

## Non-trivial isolated invariant closed curves:

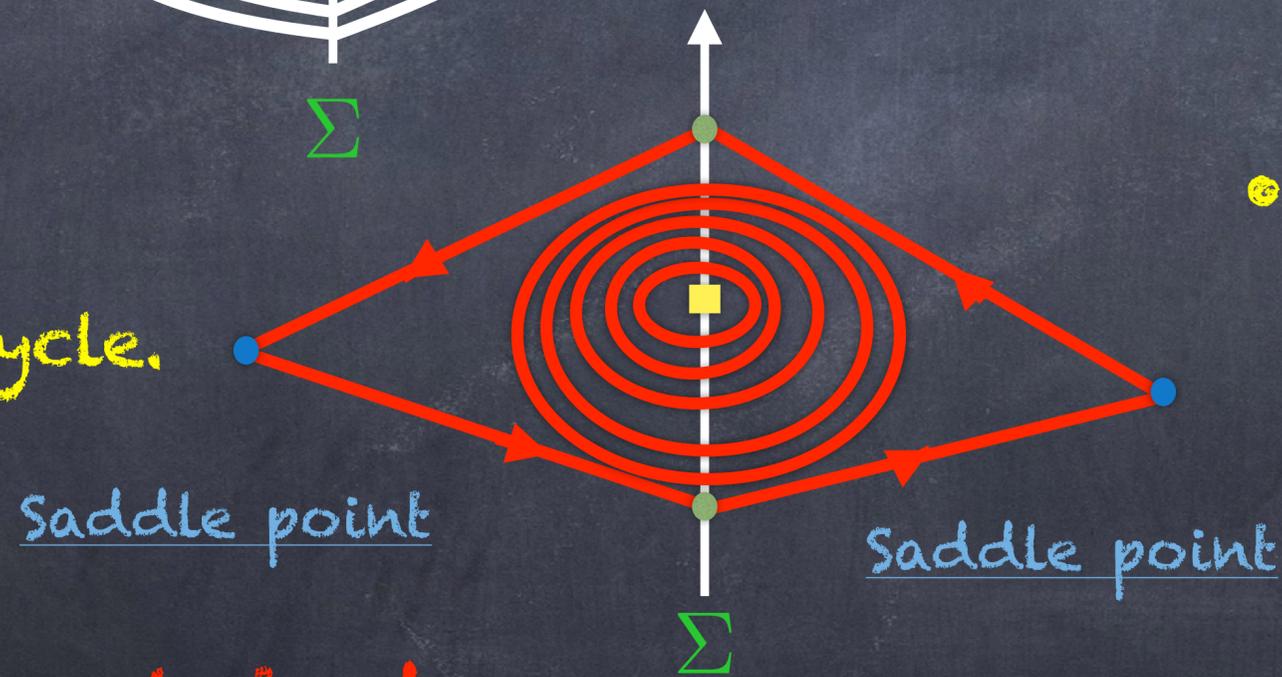
• Limit cycle.



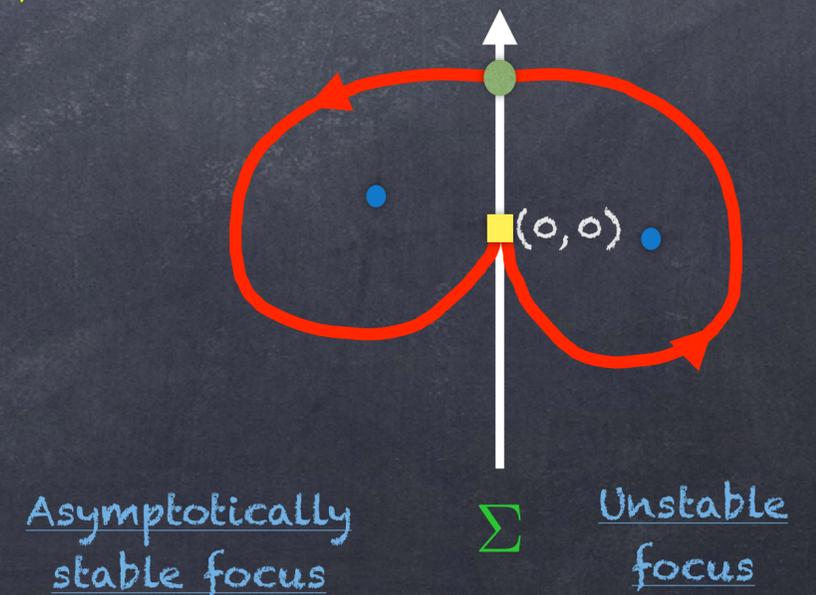
• Saddle homoclinic.



• Heteroclinic cycle.



• A fold-fold connection.

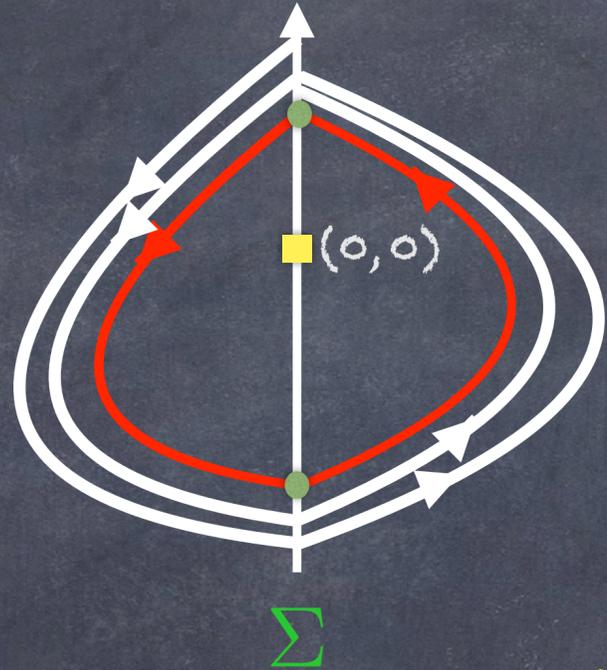


Non-isolated

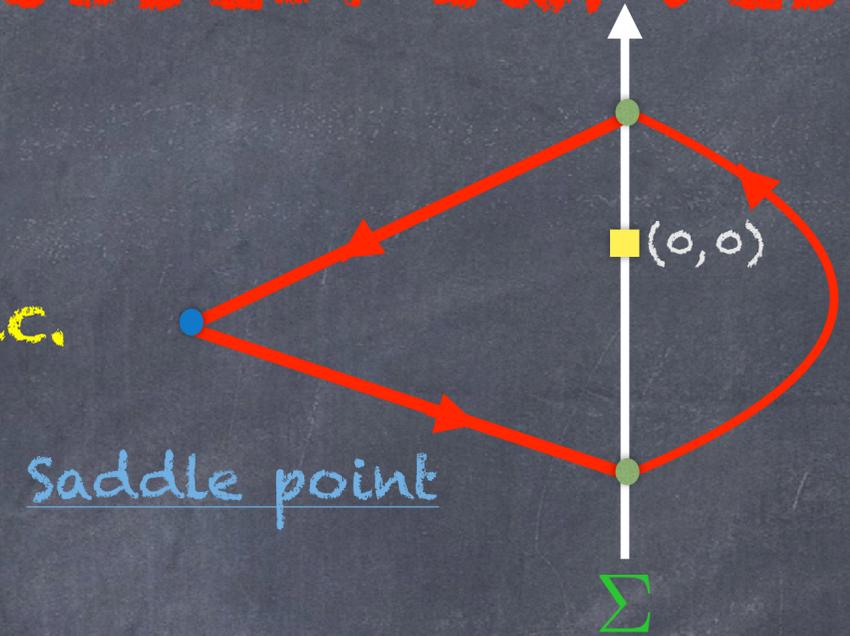
# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

Non-trivial isolated invariant closed curves:

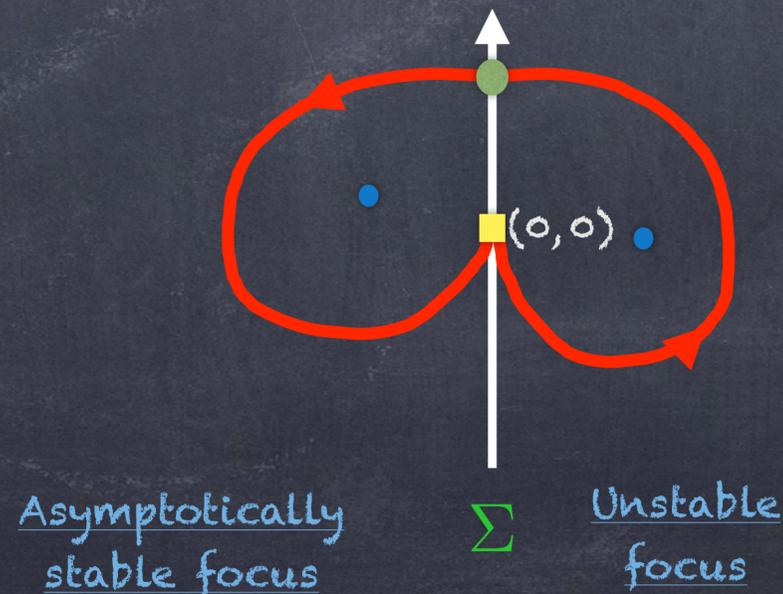
• Limit cycle.



• Saddle homoclinic.



• A fold-fold connection.



# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

## A important detail

$$W_L(y) = D_L y^2 - a_L T_L y + a_L^2 \geq 0$$

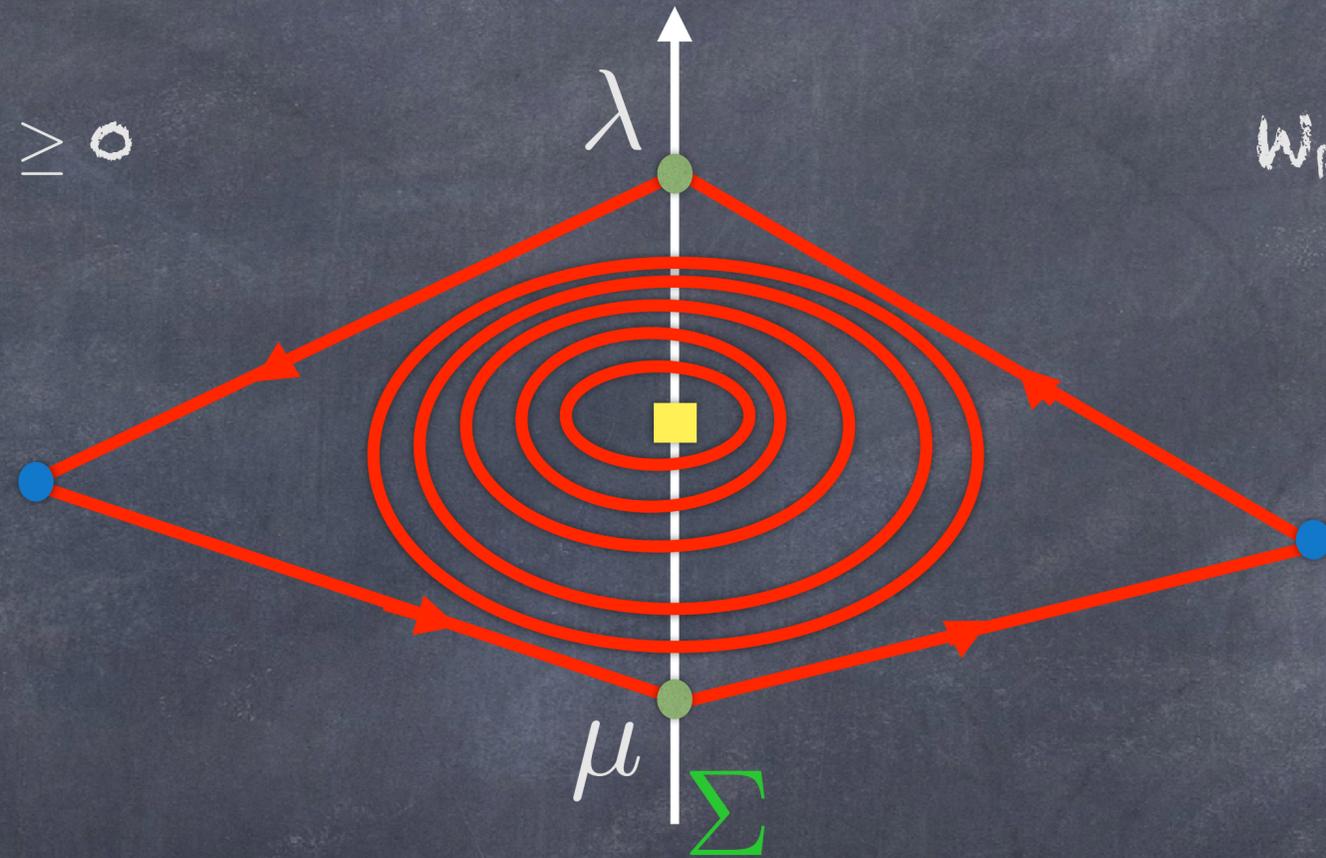
$$\int_{y_L(y_0)}^{y_0} \frac{-y}{W_L(y)} dy = 0$$

$$W_L(\mu) = W_L(\lambda) = 0$$

$$W_R(y) = D_R y^2 - a_R T_R y + a_R^2 \geq 0$$

$$\int_{y_R(y_0)}^{y_0} \frac{-y}{W_R(y)} dy = 0$$

$$W_R(\mu) = W_R(\lambda) = 0$$



The system has a crossing period annulus.

$\{W_L(y), W_R(y)\}$  is linearly dependent  $\Leftrightarrow \xi_0 = \xi_\infty = 0$

$$\xi_0 = a_R T_L - a_L T_R$$

$$\xi_\infty = T_L^2 D_R - T_R^2 D_L$$

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

$$\begin{cases} \dot{x} = T_L x - y \\ \dot{y} = D_L x - a_L \end{cases} \text{ for } x < 0, \quad \begin{cases} \dot{x} = T_R x - y \\ \dot{y} = D_R x - a_R \end{cases} \text{ for } x > 0.$$

## A important detail

Existence of both Poincaré half-maps  $\iff \begin{cases} a_L \leq 0 \text{ and } 4D_L - T_L^2 > 0, \text{ or } a_L > 0; \\ a_R \geq 0 \text{ and } 4D_R - T_R^2 > 0, \text{ or } a_R < 0. \end{cases}$

Green Th.

Existence a crossing invariant closed curve  $\implies \text{sgn}(T_R) = -\text{sgn}(T_L)$

### Theorem:

Under this hypotheses, the system has a crossing period annulus  $\iff \xi_0 = \xi_\infty = 0$

$$\xi_0 = a_R T_L - a_L T_R$$

$$\xi_\infty = T_L^2 D_R - T_R^2 D_L$$

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

## A important detail

Existence of both Poincaré half-maps  $\Leftrightarrow \begin{cases} a_L \leq 0 \text{ and } 4D_L - T_L^2 > 0, \text{ or } a_L > 0; \\ a_R \geq 0 \text{ and } 4D_R - T_R^2 > 0, \text{ or } a_R < 0. \end{cases}$

Existence a crossing invariant closed curve  $\Rightarrow \text{sgn}(T_R) = -\text{sgn}(T_L)$

### Theorem:

Under this hypotheses, the system has a crossing period annulus  $\Leftrightarrow \xi_0 = \xi_\infty = 0$

$$\xi_0 = a_R T_L - a_L T_R$$

$$\xi_\infty = T_L^2 D_R - T_R^2 D_L$$

### Remainder:

Theorem: Suppose that the system has a limit cycle. Then, it is unique, hyperbolic and  $\xi_0 \neq 0$ . Moreover,

- If  $\xi_0 < 0$ , then the limit cycle is asymptotically stable.
- If  $\xi_0 > 0$ , then the limit cycle is unstable.

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

## The stability of the monodromic singularities

$$\xi_0 = a_R T_L - a_L T_R$$

$$\xi_\infty = T_L^2 D_R - T_R^2 D_L$$

### Theorem:

Suppose that  $a_L^2 + a_R^2 \neq 0$ ,  $T_L T_R < 0$  and the system admits a unique monodromic singularity. Denote  $c_2 = a_L^2 D_R - a_R^2 D_L$ .

- If  $\xi_0 > 0$  or  $\xi_0 = 0$  and  $T_L c_2 < 0$ , then the singularity is asymptotically stable.
- If  $\xi_0 < 0$  or  $\xi_0 = 0$  and  $T_L c_2 > 0$ , then the singularity is unstable.
- If  $\xi_0 = c_2 = 0$ , then  $\xi_0 = \xi_\infty = 0$  and the system has a crossing period annulus.

### Proof:

$$y_L(y_0) = -y_0 - \frac{2T_L y_0^2}{3a_L} - \frac{4T_L^2 y_0^3}{9a_L^2} + \frac{2(9D_L T_L - 22T_L^3) y_0^4}{135a_L^3} + o(y_0^5)$$

$$y_R(y_0) = -y_0 - \frac{2T_R y_0^2}{3a_R} - \frac{4T_R^2 y_0^3}{9a_R^2} + \frac{2(9D_R T_R - 22T_R^3) y_0^4}{135a_R^3} + o(y_0^5)$$

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

## The stability of the infinity

$$\xi_0 = a_R T_L - a_L T_R$$

$$\xi_\infty = T_L^2 D_R - T_R^2 D_L$$

### Theorem:

Suppose that  $4D_L - T_L^2 > 0$ ,  $4D_R - T_R^2 > 0$  and  $T_L T_R < 0$ . Then, the next items hold.

- If  $T_L \xi_\infty > 0$ , then the infinity is attracting.
- If  $T_L \xi_\infty < 0$ , then the infinity is repelling.

### Corollary:

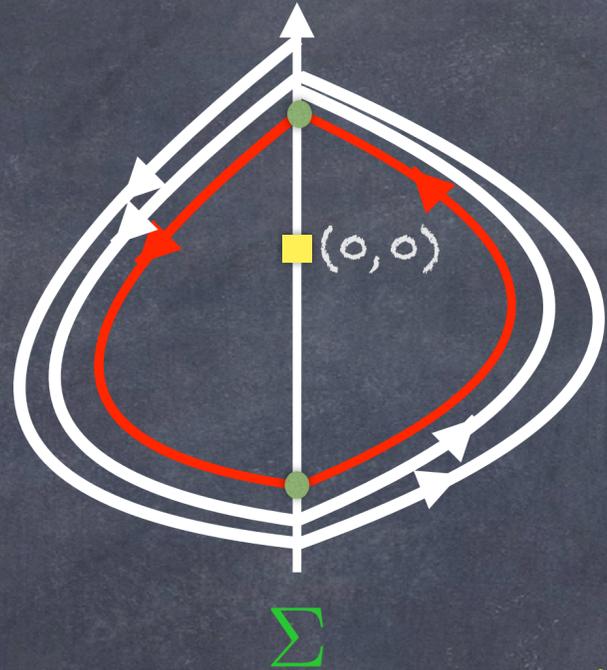
Under hypotheses of the above theorems, if  $\xi_0 T_L \xi_\infty > 0$ , then the system has a **crossing limit cycle**.

V. C., F. Fernández-Sánchez and Douglas D. Novaes. Uniqueness and stability of limit cycles in planar piecewise linear differential systems without sliding region, Commun. Nonlinear Sci. Numer. Simul., 2023. (No case-by-case study).

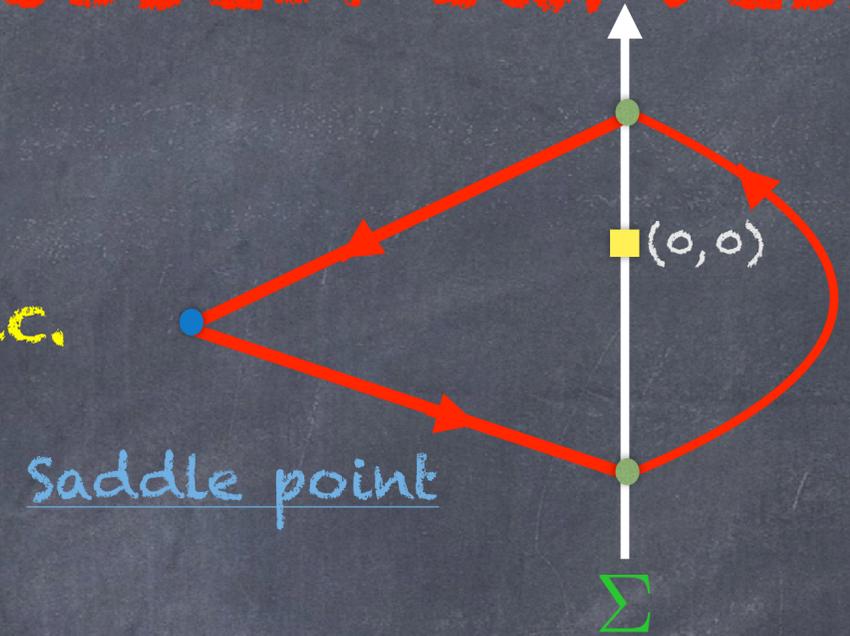
# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

## Non-trivial isolated invariant closed curves:

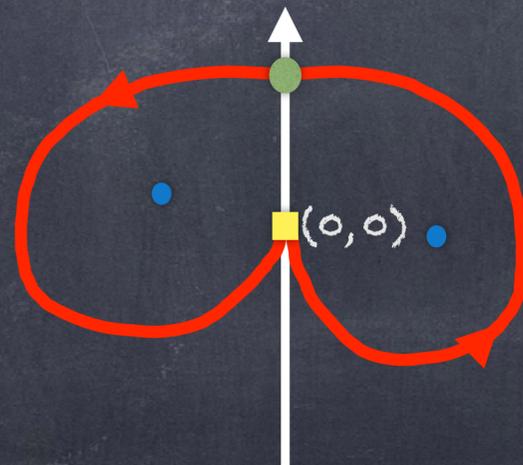
• Limit cycle.



• Saddle homoclinic.



• A fold-fold connection.



Asymptotically  
stable focus

Σ

Unstable  
focus

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

Non-trivial isolated invariant closed curves:

$$\xi_0 = a_{RT_L} - a_{LT_R}$$

## Theorem (main):

The system has at most one isolated non-trivial invariant closed curve. In addition, if such an invariant closed curve exists, it is either a limit cycle, or a saddle homoclinic connection, or a fold-fold connection and  $\xi_0 \neq 0$ .

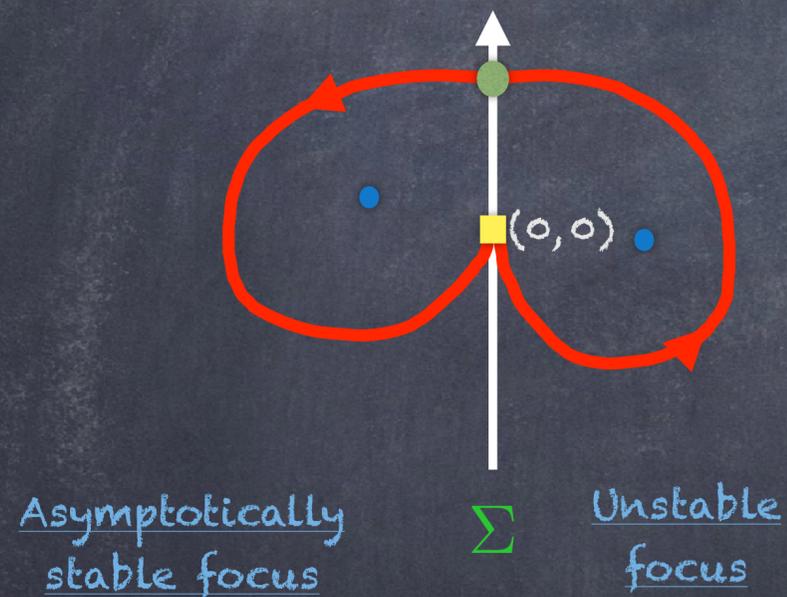
Moreover,

- If  $\xi_0 < 0$ , then the invariant closed curve is attracting.
- If  $\xi_0 > 0$ , then the invariant closed curve is repelling.

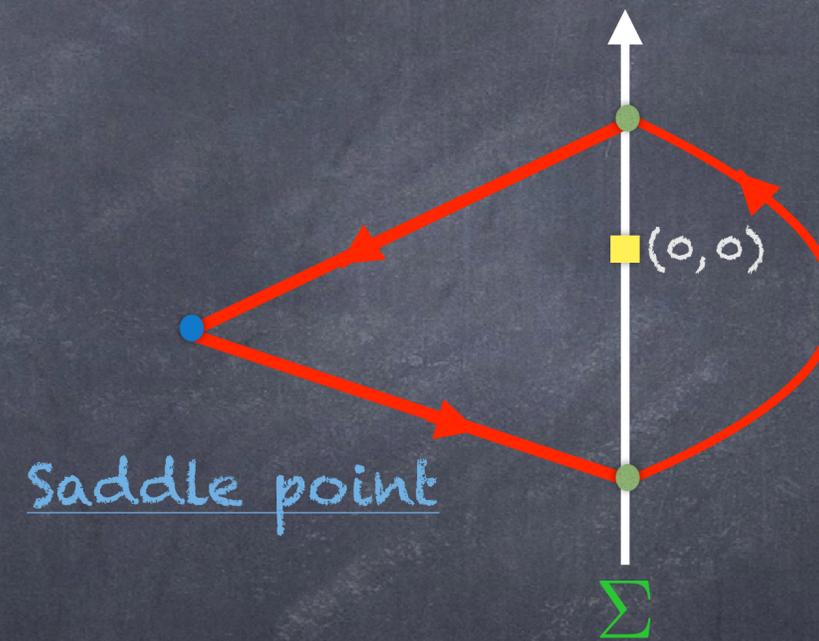
# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

## Non-trivial isolated invariant closed curves:

- A fold-fold connection.



- Saddle homoclinic.



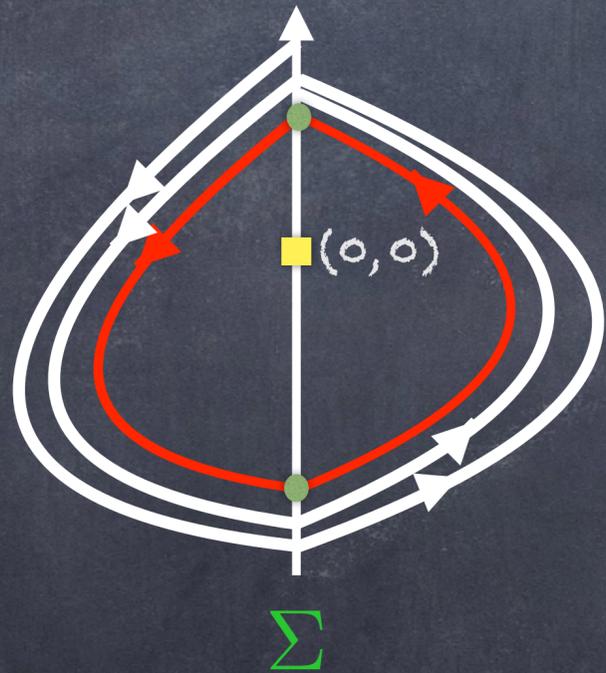
It is direct to see that the saddle homoclinic and the fold-fold connection can not coexist.

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

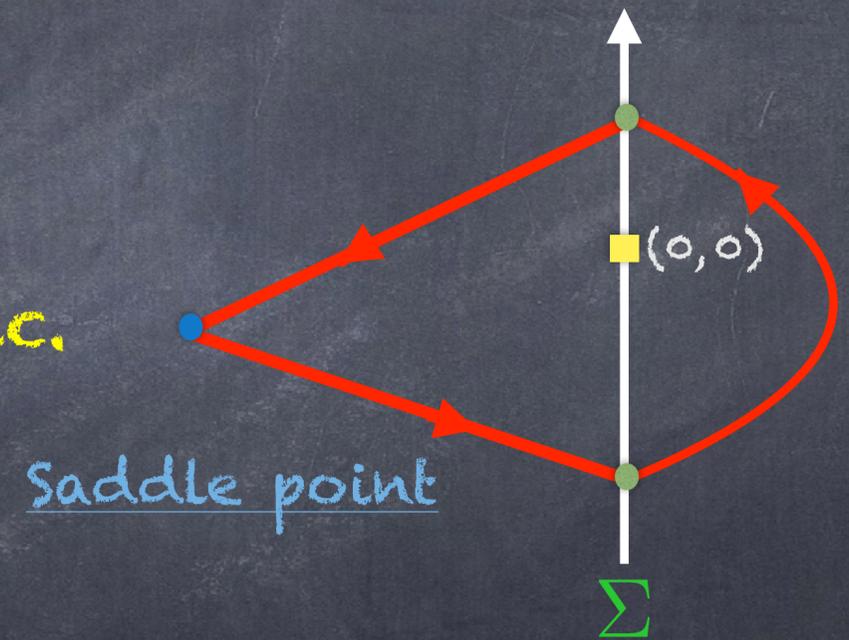
## Non-trivial isolated invariant closed curves:

Now, we will prove that the saddle homoclinic and the limit cycle can not coexist.

• Limit cycle.

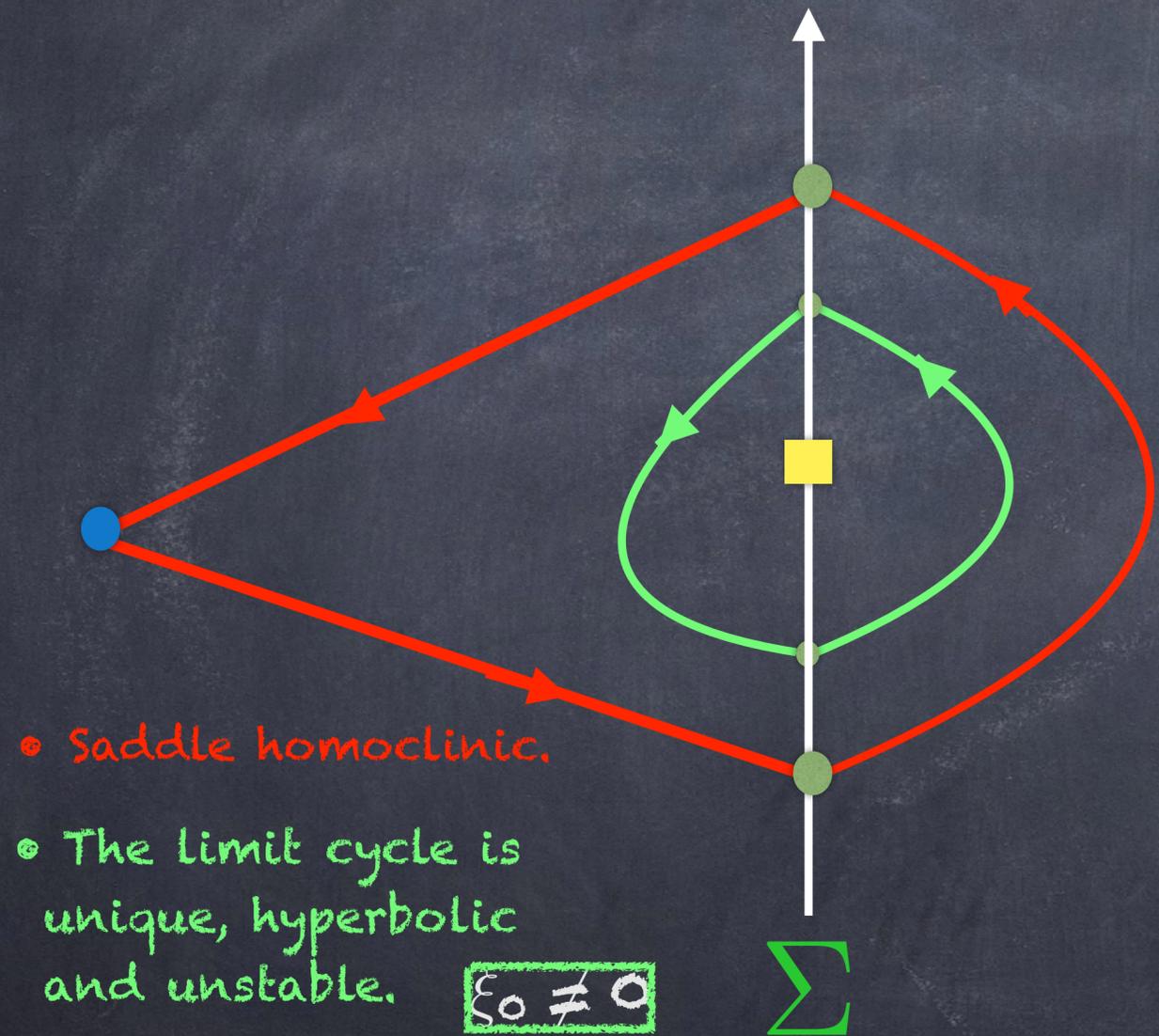


• Saddle homoclinic.

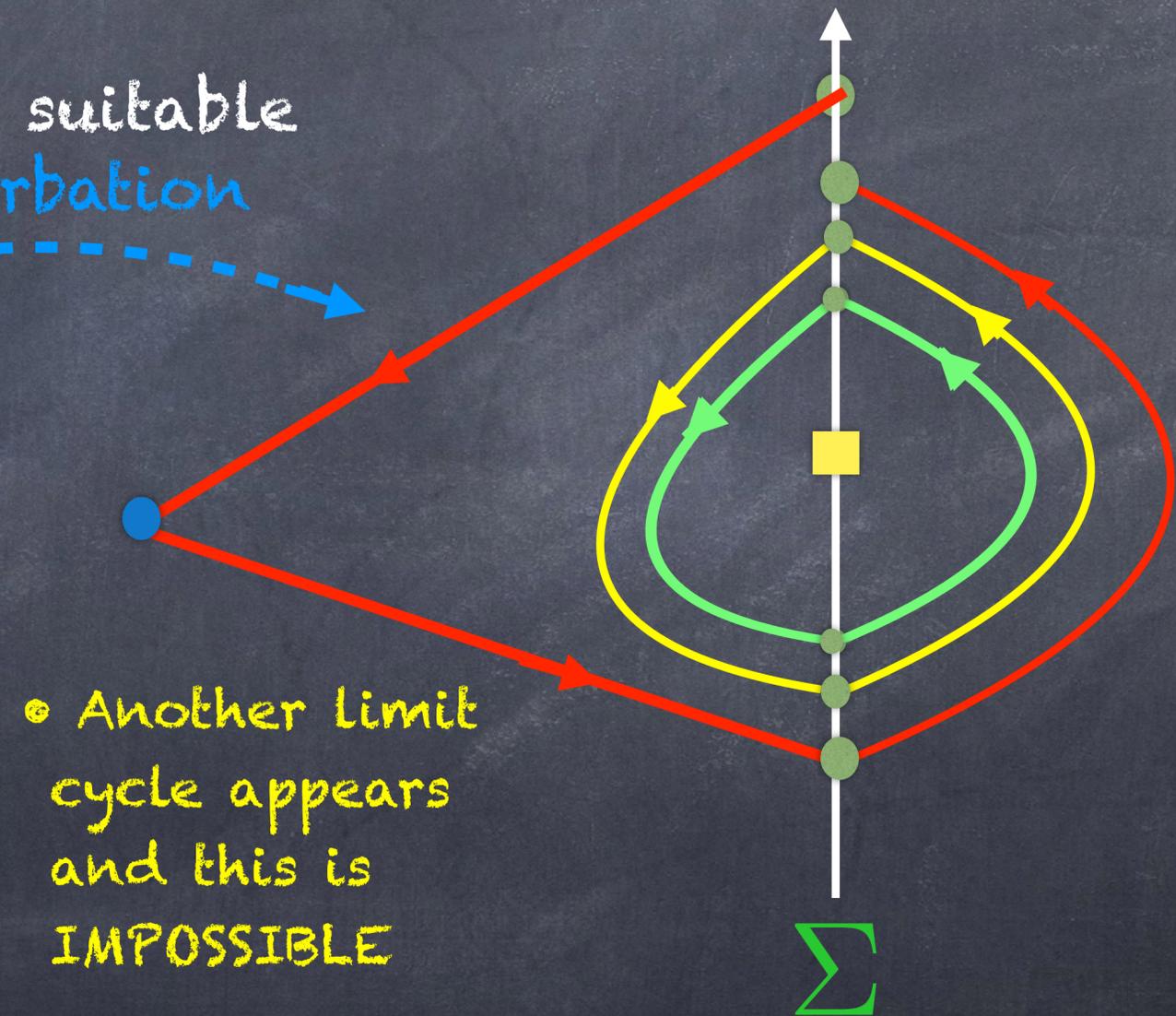
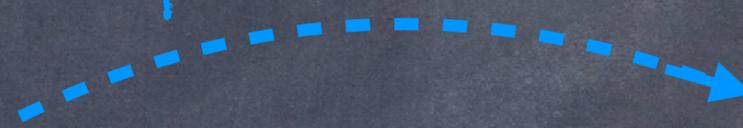


# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

Admit that there exists a system with a **saddle homoclinic** and a **limit cycle**.  
Suppose that the **limit cycle is unstable** (an similar reasoning can be done when the limit cycle is asymptotically stable)

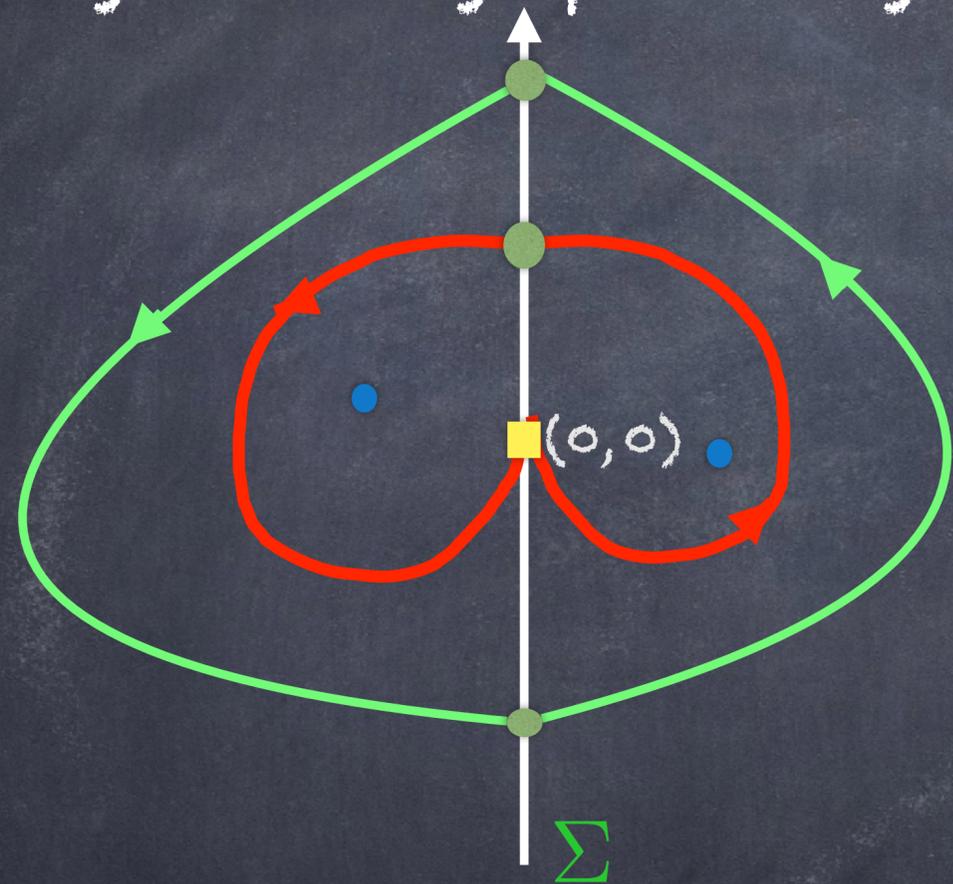


With a suitable perturbation



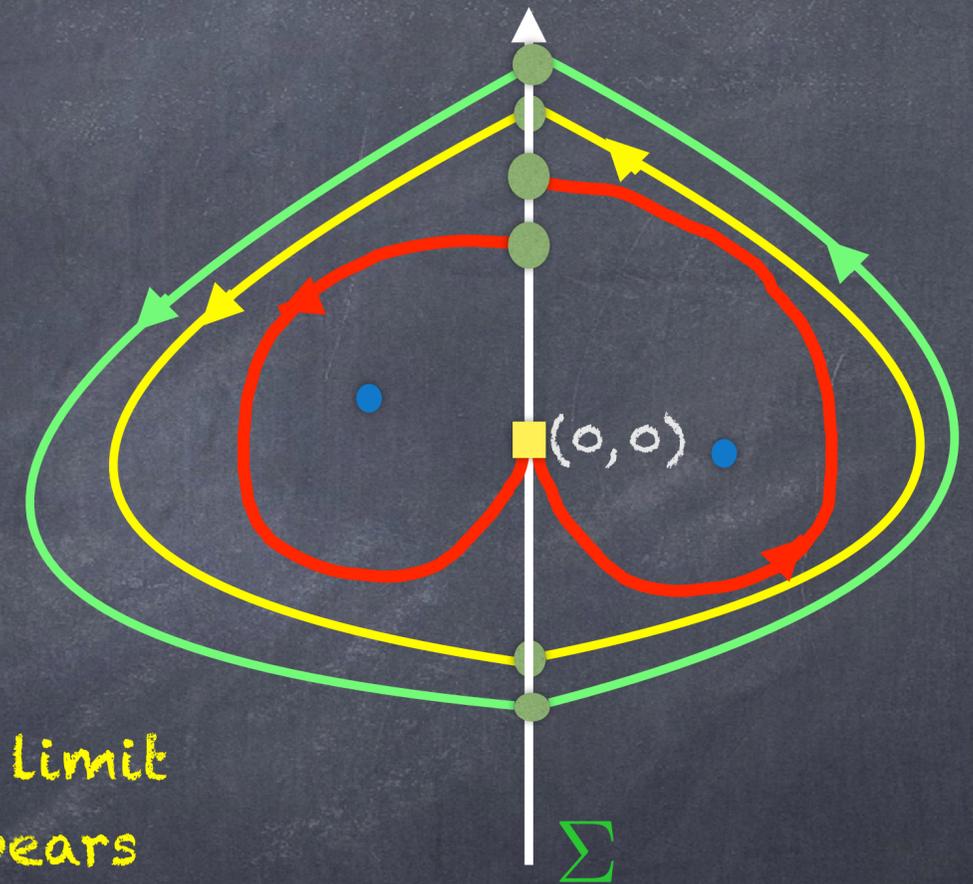
# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

Admit that there exists a system with a **fold-fold connection** and a **limit cycle**.  
Suppose that the **limit cycle is unstable** (an similar reasoning can be done when the limit cycle is asymptotically stable)



- Fold-fold connection
- The limit cycle is unique, hyperbolic and unstable.

With a suitable perturbation



- Another limit cycle appears and this is IMPOSSIBLE

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

Non-trivial isolated invariant closed curves:

$$\xi_0 = a_{RT_L} - a_{LT_R}$$

## Theorem (main):

The system has at most one isolated non-trivial invariant closed curve. In addition, if such an invariant closed curve exists, it is either a limit cycle, or a saddle homoclinic connection, or a fold-fold connection and  $\xi_0 \neq 0$ .

Moreover,

- If  $\xi_0 < 0$ , then the invariant closed curve is attracting.
- If  $\xi_0 > 0$ , then the invariant closed curve is repelling.

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions $\xi_0 = a_{RT_L} - a_{LT_R}$

Now, we will prove that the parameter that  $\xi_0 \neq 0$  when the system has a **isolated invariant closed curve**.

Remainder:

Theorem: Suppose that the system has a limit cycle. Then, it is unique, hyperbolic and  $\xi_0 \neq 0$ . Moreover,

- If  $\xi_0 < 0$ , then the limit cycle is asymptotically stable.
- If  $\xi_0 > 0$ , then the limit cycle is unstable.

If the **isolated invariant closed curve** is a limit cycle, then  $\xi_0 \neq 0$ .

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

Now, we will prove that the parameter that  $\xi_0 \neq 0$  when the system has a **isolated invariant closed curve**.

Next, we will show that if the value  $\xi_0 = 0$ , then the **system has no a isolated saddle homoclinic**.

Existence a crossing invariant closed curve  $\Rightarrow \text{sgn}(T_R) = -\text{sgn}(T_L)$   
**Green Th.**

If  $T_L = T_R = 0 \Rightarrow \xi_0 = \xi_\infty = 0 \Rightarrow$  Existence a crossing period annulus

$$\xi_\infty = T_L^2 D_R - T_R^2 D_L$$

$$\xi_0 = a_R T_L - a_L T_R$$

Existence of a **isolated invariant closed curve**  $\Rightarrow T_L T_R < 0$

$$\xi_0 = a_R T_L - a_L T_R = 0 \Rightarrow a_L = a_R \frac{T_L}{T_R} \Rightarrow \text{sgn}(a_L) = -\text{sgn}(a_R)$$

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

Now, we will prove that the parameter that  $\xi_0 \neq 0$  when the system has a **isolated invariant closed curve**.

Next, we will show that if the parameter  $\xi_0 = 0$ , then the **system has no a isolated saddle homoclinic**.

$$\xi_0 = a_R T_L - a_L T_R = 0 \Rightarrow a_L = a_R \frac{T_L}{T_R} \Rightarrow \text{sgn}(a_L) = -\text{sgn}(a_R) \quad \boxed{T_L T_R < 0}$$

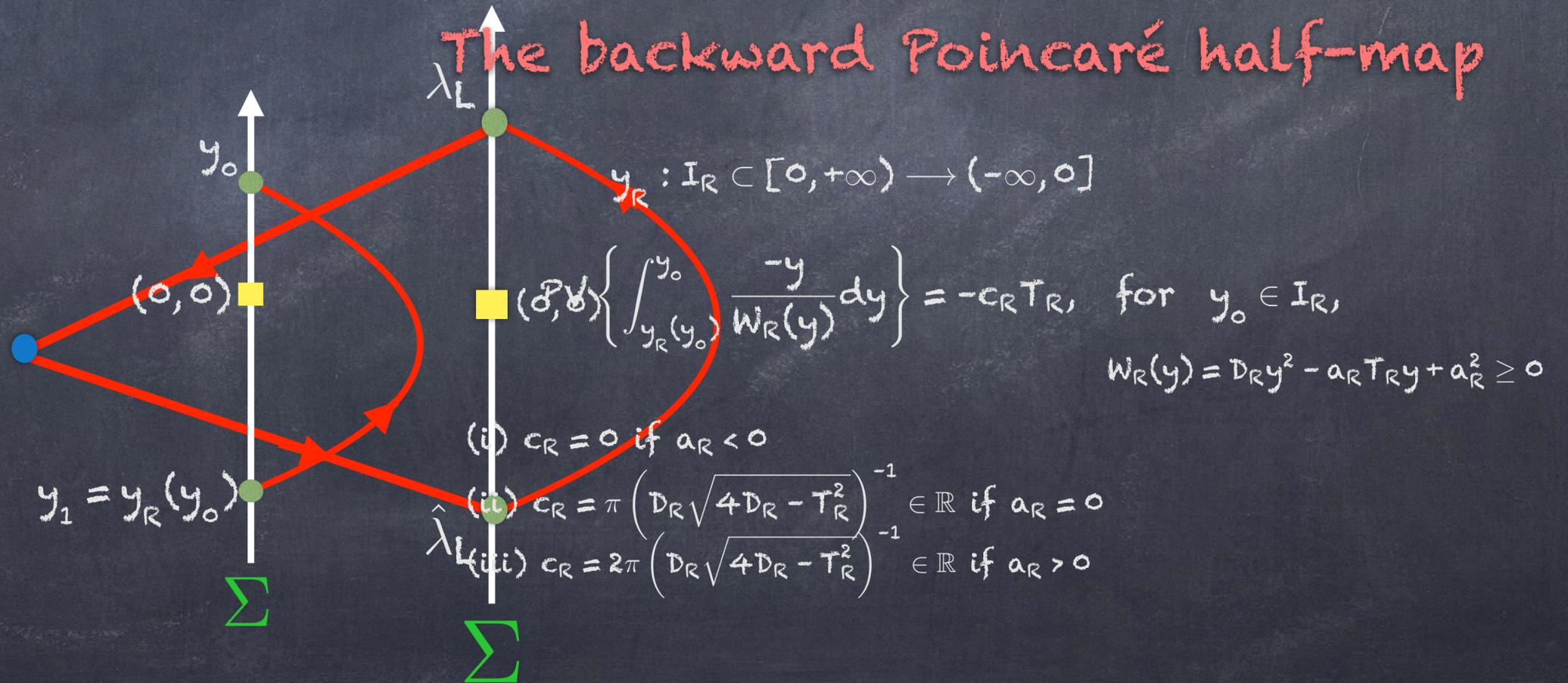
Saddle point in the left zone

$$a_L > 0, D_L < 0$$

$$\text{sgn}(a_L) = -\text{sgn}(a_R)$$

$$a_R < 0$$

**The backward Poincaré half-map**



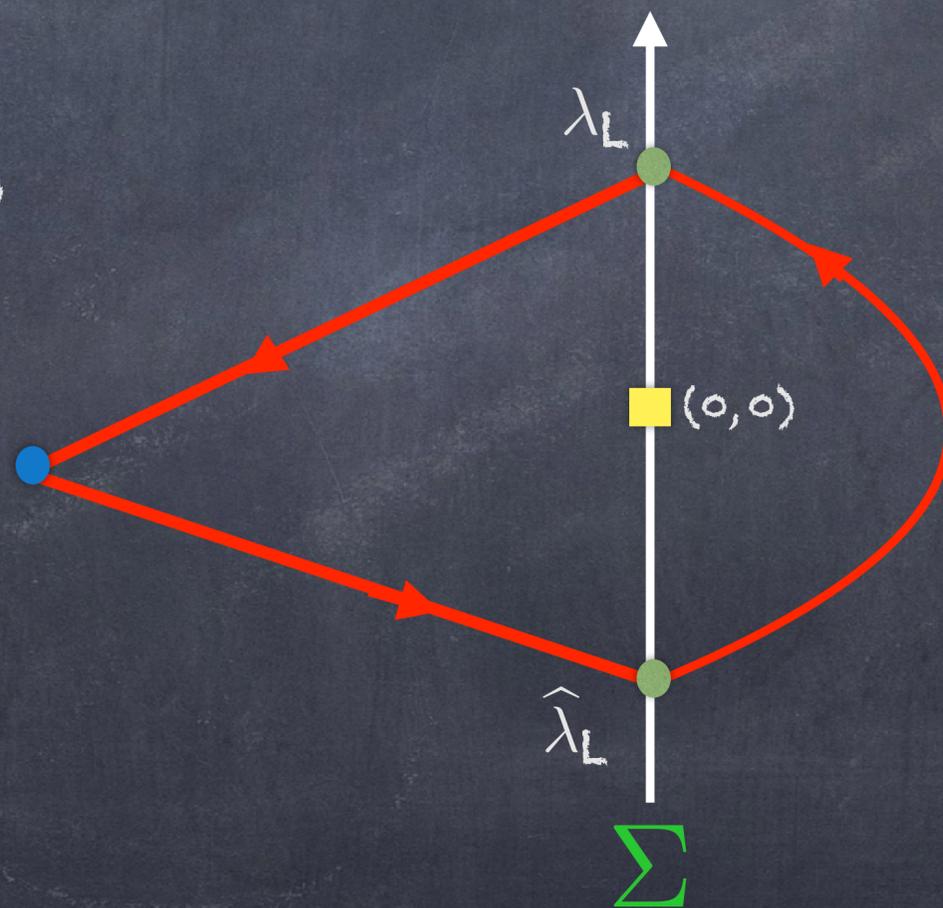
# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions $\xi_0 = a_R T_L - a_L T_R$

Now, we will prove that the parameter that  $\xi_0 \neq 0$  when the system has a **isolated invariant closed curve**.

Next, we see that if the parameter  $\xi_0 = 0$ , then the **system has no a isolated saddle homoclinic**.

$$\lambda_L = \frac{a_L \left( T_L - \sqrt{T_L^2 - 4D_L} \right)}{2D_L} > 0$$

$$\hat{\lambda}_L = \frac{a_L \left( T_L + \sqrt{T_L^2 - 4D_L} \right)}{2D_L} < 0$$



$$\int_{\hat{\lambda}_L}^{\lambda_L} \frac{-y}{W_R(y)} dy = 0$$

$$W_R(y) = D_R y^2 - a_R T_R y + a_R^2 \geq 0$$

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

$$\lambda_L = \frac{a_L \left( T_L - \sqrt{T_L^2 - 4D_L} \right)}{2D_L} > 0$$

$$\hat{\lambda}_L = \frac{a_L \left( T_L + \sqrt{T_L^2 - 4D_L} \right)}{2D_L} < 0$$

By using  $a_L = a_R \frac{T_L}{T_R}$ ,

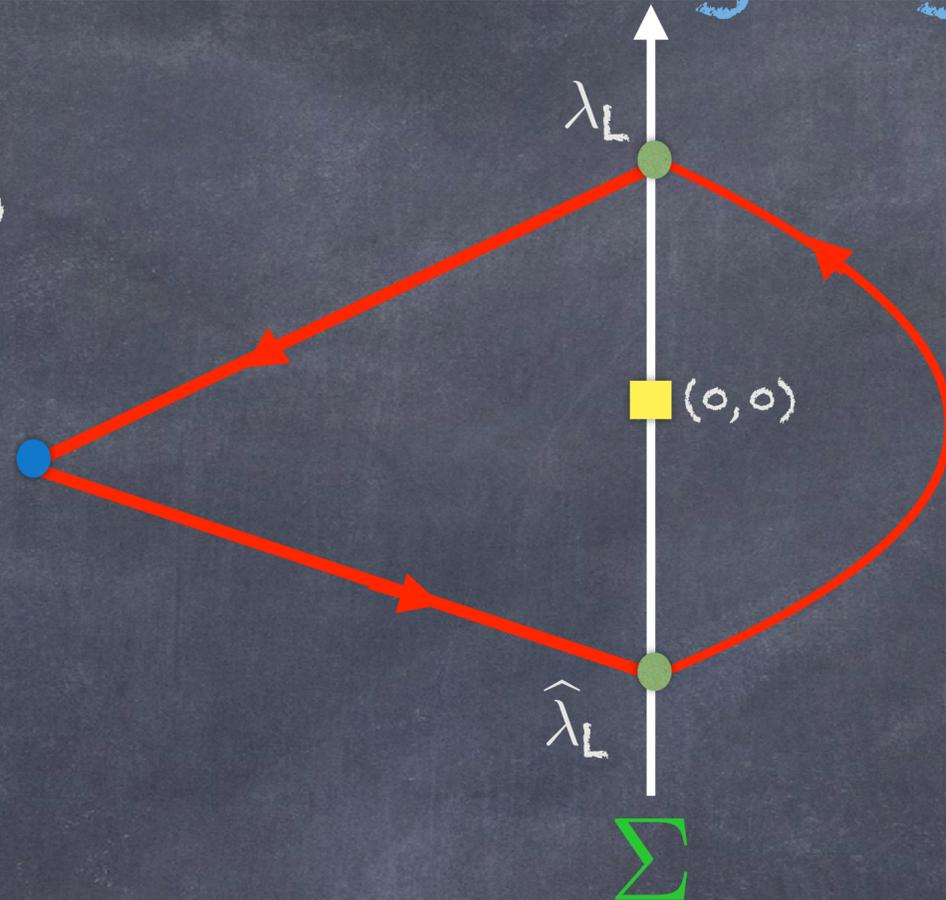
the change of variable  $y = \frac{y}{a_L}$  transforms the integral into the integral

$$H(D_L) = \int_{\hat{\mu}_L(D_L)}^{\mu_L(D_L)} \frac{-y}{D_R^2 y^2 - T_R^2 y / T_L + T_R^2 / T_L} dy$$

with

$$\mu_L(D_L) = \frac{T_L - \sqrt{T_L^2 - 4D_L}}{2D_L} > 0$$

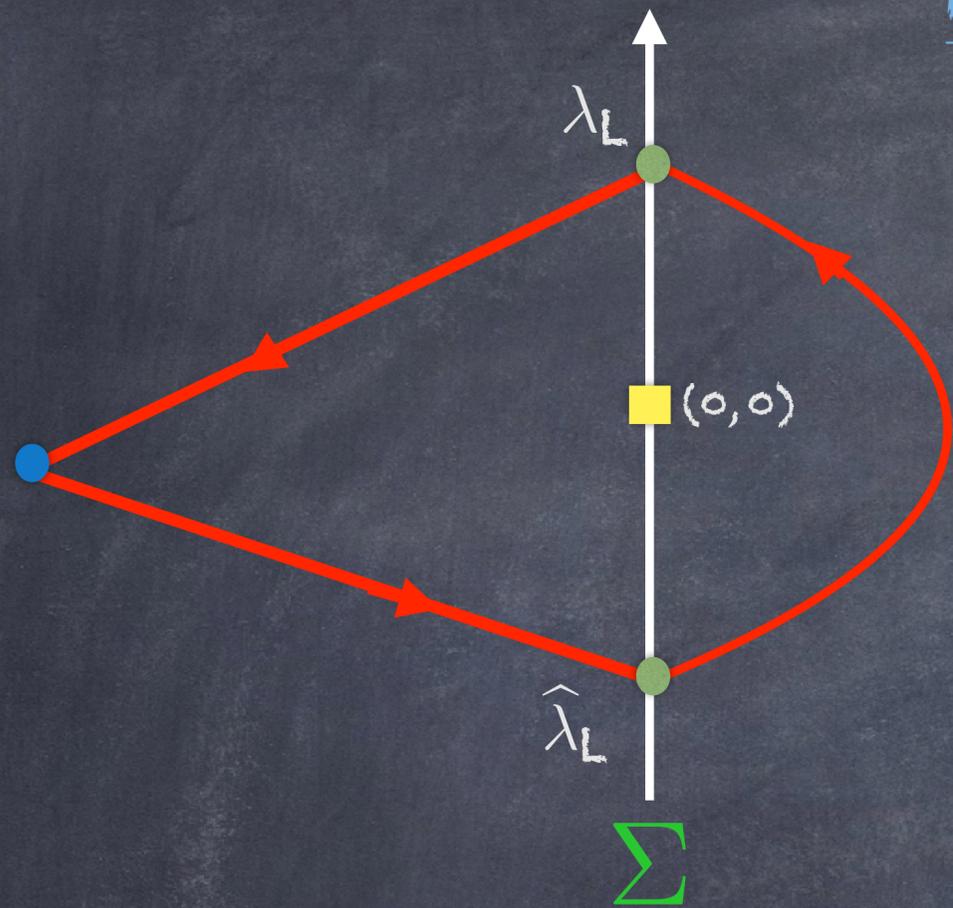
$$\hat{\mu}_L(D_L) = \frac{T_L + \sqrt{T_L^2 - 4D_L}}{2D_L} < 0$$



$$\int_{\hat{\lambda}_L}^{\lambda_L} \frac{-y}{W_R(y)} dy = 0$$

$$W_R(y) = D_R y^2 - a_R T_R y + a_R^2 \geq 0$$

# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions



$$\int_{\hat{\lambda}_L}^{\lambda_L} \frac{-y}{W_R(y)} dy = 0$$

$$\mu_L(D_L) = \frac{T_L - \sqrt{T_L^2 - 4D_L}}{2D_L} > 0$$

$$\hat{\mu}_L(D_L) = \frac{T_L + \sqrt{T_L^2 - 4D_L}}{2D_L} < 0$$

$$H(D_L) = \int_{\hat{\mu}_L(D_L)}^{\mu_L(D_L)} \frac{-y}{D_R^2 y^2 - T_R^2 y / T_L + T_R^2 / T_L^2} dy$$

$$H'(D_L) = \frac{T_L^3}{D_L \xi_\infty \sqrt{T_L^2 - 4D_L}} \neq 0$$

$$\lim_{D_L \rightarrow -\infty} H(D_L) = 0$$

$$\Rightarrow H(D_L) \neq 0 \Rightarrow \int_{\hat{\lambda}_L}^{\lambda_L} \frac{-y}{W_R(y)} dy \neq 0$$

In conclusion, the system has no a saddle homoclinic when  $\xi_0 = 0$

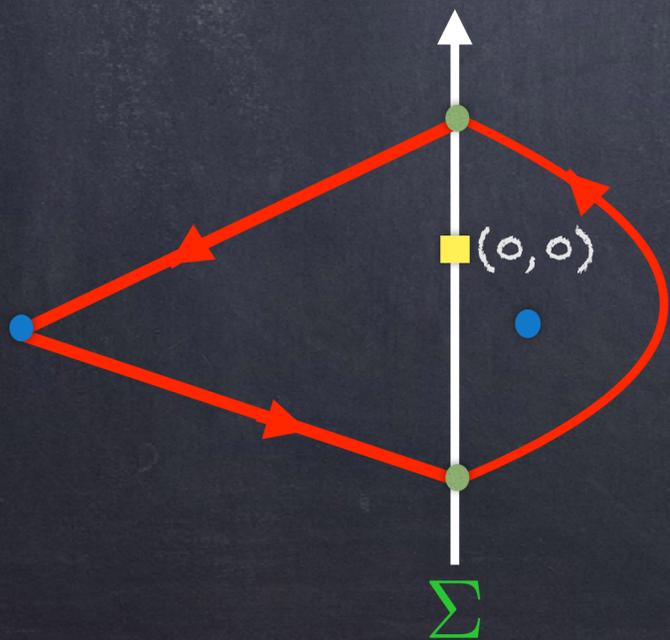
# Planar Piecewise Linear System (PWL) with Two Zones without sliding regions

The attractiveness of the saddle homoclinic is given by  $\xi_0$ .

## Theorem:

Suppose that  $a_L^2 + a_R^2 \neq 0$ ,  $T_L T_R < 0$  and the system admits a unique monodromic singularity. Denote  $c_2 = a_L^2 D_R - a_R^2 D_L$ .

- If  $\xi_0 > 0$  or  $\xi_0 = 0$  and  $T_L c_2 < 0$ , then the singularity is asymptotically stable.
- If  $\xi_0 < 0$  or  $\xi_0 = 0$  and  $T_L c_2 > 0$ , then the singularity is unstable.
- If  $\xi_0 = c_2 = 0$ , then  $\xi_0 = \xi_\infty = 0$  and the system has a crossing period annulus.



One can do the same (with a little more work), when the system has a **fold-fold connection**.

En memoria de nuestro compañero  
Jorge Galán Vioque



August 1966 - September 2025

Jornadas Científicas en memoria de Jorge: 12 de junio de 2026

# V SYMPOSIUM ON PLANAR VECTOR FIELDS !THAN YOU FOR YOUR ATTENTION!

Isolated invariant closed curves in planar piecewise  
linear systems with no sliding regions

Victoriano Carmona, Fernando Fernández-Sánchez

Dpto. Matemática Aplicada II & IMUS  
Universidad de Sevilla



Douglas D. Novaes

Departamento de Matemática & IMECC  
Universidade Estadual de Campinas



Lleida, January 12-16, 2026

