

# Limit Cycles in Piecewise ODE

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## Introduction

### The State of the Art

Piecewise linear systems are currently being studied due to the accurate description that these systems provide of some phenomena in nature. Indeed, piecewise linear systems seem to present almost the same dynamical behavior of general nonlinear systems.

### The Aim of the Communication

Hilbert 16th Problem studies the upper bound  $\mathcal{H}(n)$  of the number of limit cycles of a polynomial ODE in terms of the degree  $n$  of the field. We provide a characterization of global centers and give an upper bound of the number of limit cycles.

## Setting out the Problem

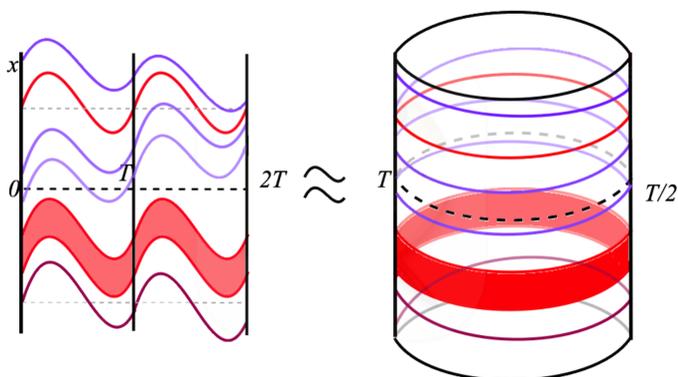
### Definition

We consider the scalar linear piecewise ODE

$$x' = a(t)|x| + b(t)$$

where  $a(t)$  and  $b(t)$  are trigonometric polynomials.

A  $T$ -periodic solution ( $u(t, x) = u(t + T, x) \quad \forall t \in \mathbb{R}$ ) is called a *limit cycle* if it is isolated in the set of periodic solutions. We say that a periodic solution is a *center* if it is not a limit cycle, and that a center is *global* if every solution is periodic.



A limit cycle and a center on the cylinder.

We study the following two problems:

*Is it possible to give explicit conditions on  $a(t)$  and  $b(t)$  such that all the solutions are periodic?*

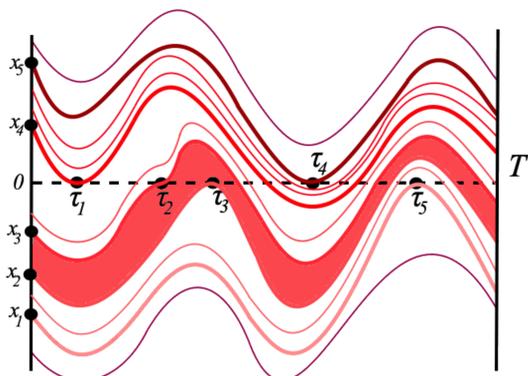
*Is it possible to give an upper bound for the number of limit cycles?*

## Piecewise Linear ODEs

### Properties

- All solutions are bounded, i.e.,  $\forall x \in \mathbb{R} \quad |\max_{t \in [0, 2\pi]} u(t, x)| < +\infty$ .
- There exists, at most, two periodic solutions with definite sign.
- There exists a correspondence between tangent solutions to the line  $x = 0$  and the zeroes of  $b(t)$ .

### The Structure of the Solutions Set



There exists bands of solutions which have the same number of zeroes. These bands have a finite number of limit cycles, otherwise they are a center [1].

## Global Centers

### Limit Cycles with two Zeroes

Let  $x < 0$ ,  $u(t, x)$  a periodic solution and  $t_1, t_2$  its **two zeroes**. Then, the following equations hold:

$$\int_{t_1}^{t_2} b(t) \exp\left(\int_t^{t_2} a(s) ds\right) dt = 0.$$

$$\int_{t_1}^{t_2} a(s) ds = 0, \quad \int_{t_1}^{t_2} b(t) dt = 0.$$

We prove that there is a correspondence between trigonometric polynomials of degree  $n$  and **Laurent** polynomials of degree  $n$  with **complex-conjugate** coefficients. This correspondence, together with the previous equations, allows us to apply known results, leading to the following factorization of  $a(t)$  and  $b(t)$ :

**Theorem** (J.L. Bravo & R. Trinidad-Forte).  $x' = a(t)|x| + b(t)$  has a global center if and only if there exist  $P, Q \in \mathbb{R}[x]$  and  $h \in \mathbb{R}[\sin(t), \cos(t)]$ , such that

$$a(t) = P(h(t))h'(t), \quad b(t) = Q(h(t))h'(t).$$

## Upper Bound of the number of Limit Cycles

### The Zeroes of the Limit Cycles

The Zeroes of the limit cycles are solutions of the following implicit system of equations:

$$\int_{t_1}^{t_2} b(t) \exp\left(\int_t^{t_2} a(s) ds\right) dt = 0,$$

$$\int_{t_2}^{t_3} b(t) \exp\left(\int_t^{t_3} a(s) ds\right) dt = 0,$$

$$\vdots$$

$$\int_{t_{k-1}}^{t_k} b(t) \exp\left(\int_t^{t_k} a(s) ds\right) dt = 0,$$

$$\int_{t_k}^{t_1+2\pi} b(t) \exp\left(\int_t^{t_1+2\pi} a(s) ds\right) dt = 0.$$

### Pfaffian Functions

Pfaffian functions are analytic functions whose derivatives satisfy a triangular system of differential equations with polynomial coefficients.

**Theorem** (Khovanskii). [2] Consider a system of equations  $f_1 = \dots = f_n = 0$ , where  $f_i$ ,  $1 \leq i \leq n$  are Pfaffian functions in a domain  $G \subset \mathbb{R}^n$ , having a common Pfaffian chain of order  $r$  and degrees  $(\alpha, \beta_i)$  respectively. Then the number of non-degenerate solutions of this system does not exceed

$$\mathcal{M}(n, r, \alpha, \beta_1, \dots, \beta_n) := 2^{\frac{r(r-1)}{2}} \beta_1 \dots \beta_n (\min\{n, r\} \alpha + \beta_1 + \dots + \beta_n - n + 1)^r.$$

**Theorem** (J.L. Bravo & R. Trinidad-Forte). Let  $m = \deg(b)$  y  $N = \max\{\deg(a^+), \deg(a^-), \deg(b)\}$ . The number of limit cycles is bounded by

$$2^{(m+2)(2m+3)} (6Nm + 2m + 1)^{2m+4} + 2$$

## References

- Bravo, J.L., Fernández, M. & Tineo, A.: *Periodic solutions of a periodic scalar piecewise ODE*, Commun. Pure Appl. Anal. **6**, 213-228 (2007).
- Khovanskii, A.G.: *Fewnomials*, Transl. Math. Monogr. **88**, Amer. Math. Soc., Providence, RI, (1991).

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