Integral tori in non-autonomous planar differential equations via averaging theory.

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Abstract

A classic and useful result from the theory of averaging relates the existence of isolated periodic solutions of non-autonomous periodic differential equations with the existence of simple singularities of its so-called *guiding system*, which is an autonomous differential equation given in terms of the first non-vanishing higher order averaged function. We will discuss an extension of this result: that hyperbolic periodic orbits in the phase of the guiding system guarantee the existence of integral tori for the original non-autonomous differential equation. Regularity, convergence, and stability of such tori, as well as the dynamics displayed on them, are also investigated.

Introduction

Consider the family of planar differential systems

$$\dot{x} = \sum_{i=1}^{N} \varepsilon^i F_i(t,x) + \varepsilon^{N+1} F_R(t,x,\varepsilon), \tag{1}$$

defined for $(t,x,\varepsilon) \in \mathbb{R} \times D \times [0,\varepsilon_0]$, with $N \in \mathbb{N}^*$, $D \subset \mathbb{R}^2$ open and bounded, and $\varepsilon_0 > 0$. Assume there are $T > 0$ and $r \geq N$ such that each F_i and F_R are of class C^r, as well as T-periodic in t.

Higher-order averaging guarantees there is a T-periodic near-identity change of variables transforming (1) into

$$\dot{z} = \sum_{i=1}^{N} \varepsilon^i g_i(z) + \varepsilon^{N+1} R(t,x,\varepsilon), \tag{2}$$

effectively pushing time-dependence to $\mathcal{O}(\varepsilon^{N+1})$. Each g_i is named the averaged function of i-th order, and admit recursive formulas for their calculation. For instance,

$$g_1(z) = \frac{1}{T}\int_0^T F_1(t,z)dt. \tag{3}$$

Let $\ell \in \{1,\dots,N\}$ be the first index for which g_i is not identically zero. The *guiding system*

$$\dot{z} = g_\ell(z) \tag{4}$$

acts as an approximation of (2). The motivating question is: what qualitative properties of (1) can be inferred from it?

Equilibria and Periodic Orbits

A classic result concerns periodic solutions of (1).

Theorem. *If z_* is a simple equilibrium of (4), then (1) admits a T-periodic solution for small $\varepsilon > 0$, converging to z_* as $\varepsilon \to 0^+$.*

More geometrically, by adding time as an angular variable τ in (1), the resulting autonomous differential equation

$$\dot{\tau} = 1, \quad \dot{x} = \sum_{i=1}^{N} \varepsilon^i F_i(\tau,x) + \varepsilon^{N+1} F_R(\tau,x,\varepsilon), \tag{5}$$

has a periodic orbit provided its guiding system has a simple equilibrium. Stability and hyperbolicity properties are inherited from this equilibrium as well.

Limit cycles and invariant tori

Bogoliubov and Mitropolsky [1], and later Hale [2], generalized the correspondence of invariant manifolds of (1) with those of its associated **first-order** guiding system to include invariant tori.

Theorem. *If $N = 1$, $F_R = 0$ and (4) admits a hyperbolic limit cycle, then (5) has an invariant torus.*

New Results

The main result of this study removes the constraint of only considering first-order averaging. It also ensures normal hyperbolicity - and thus robustness under perturbations - of the obtained invariant torus.

Theorem A ([4])**.** *If γ is an attracting hyperbolic limit cycle of (4), then (5) has an attracting normally hyperbolic invariant torus for small $\varepsilon > 0$, converging to $\gamma \times \mathbb{S}^1$ as $\varepsilon \to 0^+$.*

Other important conclusions obtained in [3] about the torus M_ε are:

- Each M_ε is of class $C^{r-\ell}$;
- The flow on M_ε defines a $C^{r-\ell}$ first return map p_ε on a transversal section, with continuous rotation number $\rho(\varepsilon)$;
- If $r - \ell \geq 4$, ρ maps zero-measure sets to zero-measure sets, and there is a positive measure set $E \subset [0,\varepsilon_0]$ for which p_ε is $C^{r-\ell-3}$ conjugated to an irrational rotation.

Further work

In [3], we have been able to extend Theorem A to systems in $\mathbb{R}^n$, with $n \in \mathbb{N}$, although without guaranteeing normal hyperbolicity. It was also proved that the number of transversal stable and unstable directions of the torus are the same as those of the limit cycle.

We are presently developing new results that not only guarantee normal hyperbolicity of the resulting torus, but also require only that we find a **normally hyperbolic invariant closed curve** in the phase space of the guiding system.

References

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Acknowledgments

This work has received support from the São Paulo Research Foundation (FAPESP), process number 2020/14232-4.