Slow-fast solutions of abel equations

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The object of the talk will be the equation

$$\varepsilon \frac{dx}{d\theta} = A_\varepsilon(\theta)x^3 + B_\varepsilon(\theta)x^2, \qquad x, \theta \in \mathbb{R}$$

with 2π-periodic coefficients A_ε and B_ε. The parameter ε is kept small and positive, and like is usual in the study of Abel equations we restrict to the half space $\{x \geq 0\}$, although there is really no need to do so.

Solutions $x(\theta)$ are typically not periodic, and some of the solutions may blow up in finite time, eg. have a vertical asymptote somewhere. Isolated periodic solutions, hence defined over the full $[0, 2\pi]$ are limit cycles of the vector field

$$\begin{cases} \dot{\theta} &= \varepsilon \\ \dot{x} &= A_\varepsilon(\theta)x^3 + B_\varepsilon(\theta)x^2. \end{cases}$$

Goal: characterize limit cycles near the slow-fast limit.

The link between Abel equations and quadratic polynomial equation relates

$$\begin{cases} \dot{x} &= -y + \lambda x + yP_2(x,y) \\ \dot{y} &= \lambda x + y + xQ_2(x,y) \end{cases}$$

to

$$\frac{dX}{d\theta} = a_\lambda(\theta)X^3 + b_\lambda(\theta)X^2 + \lambda X$$

(Cherkas)
The coefficients a_λ resp. b_λ are trigonometric polynomials of degree 6 resp 3.

It is possible to compactify the (r, θ)-space by adding infinity. Also the coefficient space can be compactified, so that after scaling we have

$$\varepsilon \frac{dX}{d\theta} = A_\Lambda(\theta)X^3 + B_\Lambda(\theta)X^2 + \Lambda X$$

Known result: When $\Lambda = 0$ and A and B are linear: ≤ 3 limit cycles (Yu, Huang, Liu, 2024). It solves problem 6 in the Gasull list of problems.

The main equation that we consider in the first slide chooses

$$\Lambda = 0.$$

It is, we believe, the more interesting case. We will consider a framework where we can obtain several results, but it is not the most general one, and it even would not allow us to treat the linear case in full generality when $\Lambda = 0$. It is a consideration between completeness and readability

Basic slow-fast analysis of:

$$X_\varepsilon : \begin{cases} \dot\theta &= \varepsilon \\ \dot x &= A_\varepsilon(\theta)x^3 + B_\varepsilon(\theta)x^2. \end{cases}$$

The fast vector field is given by

$$X_0 : \begin{cases} \dot\theta &= 0 \\ \dot x &= A_0(\theta)x^3 + B_0(\theta)x^2. \end{cases}$$

The critical set is given by $\{x = 0\}$ union

$$\{A_0(\theta)x + B_0(\theta) = 0\}$$

Some peculiarities:

- The "ground solution" $x = 0$ is completely non-hyperbolic
- the critical set may contain vertical lines at common roots of A_0 and B_0. Also these branches will be completely non-hyperbolic
- There will be intersection points of different branches. At these points the linearization of the vector field is completely vanishing. It is more degenerate than the nilpotent case studied in the book, but we will see the study is very similar.

The slow vector field along hyperbolic segments is very elementary:

$$\begin{cases} \theta' &= 1 \\ x' &= \phi'(\theta) \end{cases}$$

In our book we write $\mathcal{X}_\varepsilon = F.\mathcal{Z} + \varepsilon\mathcal{Q} + O(\varepsilon^2)$ where

$$\mathcal{Z}: \begin{cases} \dot{\theta} &= 0 \\ \dot{x} &= 1 \end{cases} \quad \text{and} \quad \mathcal{Q}: \begin{cases} \dot{\theta} &= 1 \\ \dot{x} &= p(\theta)x^3 + \varepsilon q(\theta)x^2 \end{cases}$$

and $F = A_0(\theta)x^3 + B_0(\theta)x^2$. To find the slow vector field starting from $\mathcal{Q}$ one has to add a multiple of the vector field $\mathcal{Z}$ to it until the result becomes tangent to the critical curve. (One projects the vector field $\mathcal{Q}$ along $\mathcal{Z}$ onto the tangent space of $\{F = 0\}$.)

Along the vertical branches, there is no reasonable well-defined slow dynamics, along the ground state one would define a "slow dynamics"

$$\begin{cases} \theta' &= 1 \\ x' &= 0 \end{cases}$$

[An intrinsic treatment of the case of double critical curves is initiated in DM & Torregrosa, JDE 2025.]
Because the ground solution is an actual invariant curve of the full vector field, the dynamics around it is characterized by a so-called entry-exit relation

Preliminary study near infinity

Let us study infinity by writing $x = 1/z$. The singular set is then given by

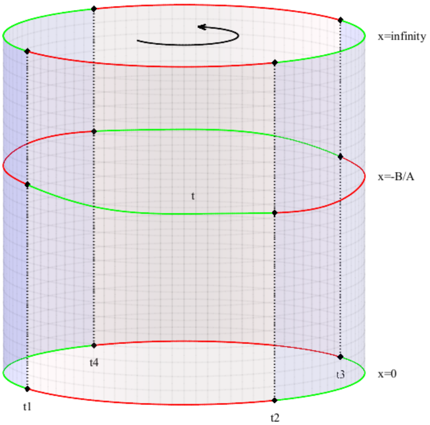
$$A_0 + zB_0 = 0.$$

Then at infinity:

$$\begin{cases} \dot{\theta} &= \varepsilon z \\ \dot{z} &= A_\varepsilon(\theta)) + B_\varepsilon(\theta)z. \end{cases}$$

Branches at infinity $z = 0$ seems to be normally hyperbolic when $B_0 \neq 0$.
NONGENERICITY ASSUMPTION:
Assume all roots of A_0 are distinct and simple.
Under this condition, and when the roots of A_0 are not a root B_0, infinity does not play any role in the singular limit: all orbits in positive or negative time will cross infinity and hence correspond to a solution of the abel equation that blows up along a vertical asymptote. So we will only need to study the following case:
Assume for the rest of the talk that all roots of A_0 are also roots of B_0 AND VICE-VERSA

Study near the zero solution



We distinguish RED and GREEN parts of the cylinder: along RED the ground state is attracting, green is repelling for the ground state. Define

$$I(v,w) = \int_v^w B(\theta)d\theta.$$

Linear Trigonometric example

We take

$$-A_0 = B_0 = \sin\theta - \sin\theta_1, \qquad \theta_1 := \pi - \theta_2.$$

The red region is to the left of θ_1, and the green region to the right of it. Even in this restricted framework we will be able to find the maximum 3 cycles!
(a priori it seems θ_1 is the only free parameter, but we can still choose an ε-perturbation, it has 6 free parameters)

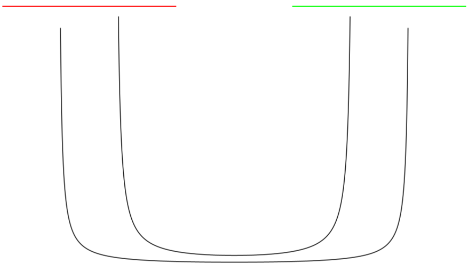
Given

$$\Sigma_M = \{x = x_M\}$$

with

$$0 < x_M < -B_0(\theta)/A_0(\theta) \text{ for all } \theta$$

The entry-exit relation describes how orbits from Σ_M pass near the ground state from and return to Σ_M:



The section Σ_M is divided into green and red points.

Proposition (Schecter, DM)

Let v be red, w be green such that w is the first point for which $I(v,w)=0$ (assuming such a point exists).
Then, there exist a (red) neighbourhood V of v and a neighbourhood of $\epsilon=0$ such that the dynamics is given by a map

$$p_\epsilon : V \to \{\text{green}\},\ \ \epsilon > 0,$$

with $p_\epsilon \to p_0$ smoothly in $(v,\epsilon,\epsilon\log\epsilon)$ and satisfying $p_0(v)=w$.
In fact,

$$\int_{\tilde v}^{p_0(\tilde v)} B(\theta)\, d\theta = 0,\ \ \forall \tilde v \in V$$

If no such w exists, the point (x_M, v) lies in the basin of attraction of the ground orbit.

For a heuristic argument, one may look at a simplified model $\varepsilon \frac{dx}{d\theta} = B_\varepsilon(\theta) x^2$ that can easily be integrated:

$$x(\theta) = \frac{\varepsilon}{\varepsilon / x_M - \int_v^\theta B_\varepsilon(s) ds}$$

from which all assertions easily follow: the entry-exit relation is an implicit solution of the equation

$$\int_v^w B_\varepsilon(s) ds = 0$$

A full proof yielding only continuity would show that the orbit $x(\theta)$ lies between $\underline{x}(\theta)$ and $\overline{x}(\theta)$, both of which are solutions to equations of the simplified kind with versions of B that are close to B in the L_1-norm.

Behaviour near the nonzero critical curve

So we consider

$$\dot{\theta} = \varepsilon, \qquad \dot{x} = A_\varepsilon(\theta)x^3 + B_\varepsilon(\theta)x^2,$$

and write

$$A_\varepsilon = A_0 + \varepsilon p + O(\varepsilon^2), \qquad B_\epsilon = B_0 + \epsilon q + O(\varepsilon^2)$$

Lemma

Given any compact interval K that is fully red or fully green. There exists a smooth graph

$$x = -B_0(\theta)/A_0(\theta) + \varepsilon\Phi_1(\theta) + O(\varepsilon^2)$$

defined for $\theta \in K$ and ε sufficiently small that is locally invariant under the flow of the vector field. One has

$$\Phi_1 := \frac{p(B_0/A_0) - q + (A_0/B_0)'}{A_0}$$

This is just the so-called slow curve or Fenichel curve. Its expansion is typically not valid at contact points, i.e. the transition points from red to green and vice versa.

Definition

Keeping in mind that A_0 and B_0 have a simple zero at θ_k expression for Φ_1 has a simple pole; we define ρ_k to be its residue

$$\rho_k := \lim_{\theta \to \theta_k} (\theta - \theta_k)\Phi_1(\theta)$$

Away from the θ_k, we can also define the slow divergence integral

$$J(v, w) = \int_v^w (-B_0).(-B_0/A_0)d\theta$$

Consider a green section of Σ_M, and let θ_k be right boundary of green.

Proposition

For $\varepsilon > 0$ sufficiently small, the transition map

$$\{green\ section\} \to \{\theta = \theta_*\}$$

with $\theta_* := \theta_k - \delta$ is well defined, following orbits in positive time, and is of the form

$$v \mapsto x = \phi_\varepsilon(\theta_*) - \exp \frac{\tilde{J}(v, \varepsilon)}{\varepsilon}.$$

Furthermore, $\tilde{J}(v, 0) = J(v, \theta_*) < 0$ is the slow divergence integral along $x = -B_0/A_0$ computed on $[v, \theta_*]$:

Here ϕ_ε is a well-chosen slow curve.

Let $[\theta^\ell, \theta_r]$ be a segment of green values and chose any slow curve $x = \phi_\varepsilon(\theta)$ defined on that segment.

Proposition

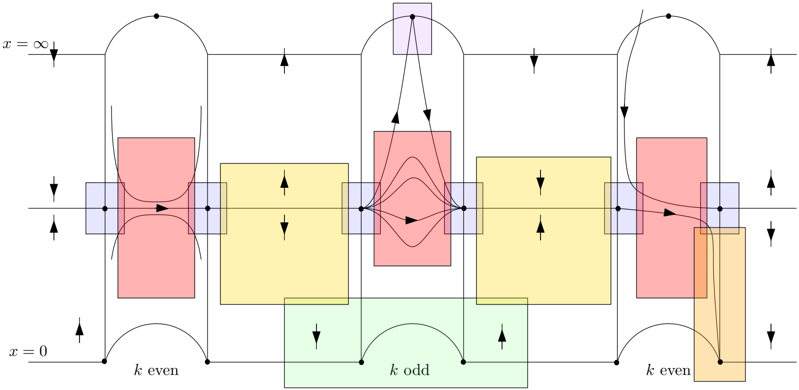
For $\varepsilon > 0$ small enough and x in some compact interval inside $]0, \infty[$, the transition map

$$\{\theta = \theta^\ell\} \to \{\theta = \theta^r\}$$

is of the form

$$x^\ell \mapsto x^r = \phi_\varepsilon(\theta^r) + (x^\ell - \phi_\varepsilon(\theta^\ell)) \exp \frac{\tilde{J}(x^\ell, \varepsilon)}{\varepsilon},$$

where $\tilde{J}(x^\ell, 0)$ is the slow divergence integral $J(v, w) < 0$ for $(v, w) = (\theta^\ell, \theta^r)$. It does not depend on x^ℓ. So essentially the transition map behaves like an affine map, exponentially strongly contracting.



$x = \infty$
$x = 0$
k even
k odd
k even

Consider a θ_k, i.e. a common zero of A_0 and B_0.

Lemma

Near $\theta = \theta_k$, there exists a local change of coordinates, reducing the vector field to an equivalent system of the same shape:

$$\begin{cases} \dot{\theta} & = & \varepsilon \\ \dot{x} & = & \pm(\theta_k - \theta)\left[-x^3 + x^2\tilde{B}(\theta)\right] + \varepsilon P_\varepsilon(\theta)x^3. \end{cases}$$

with $\tilde{B} > 0$.

- θ_k on the transition red to green: $\pm = -$.
- θ_k on the transition green to red: $\pm = +$.

Cylindrical blowup

Let us now blow up the vector field

$$\begin{cases} \dot{\theta} & = & \varepsilon \\ \dot{x} & = & \pm(\theta_k - \theta)\left[-x^3 + x^2\tilde{B}(\theta)\right] + \varepsilon P_\varepsilon(\theta)x^3 \\ \dot{\varepsilon} & = & 0 \end{cases}$$

along the line $\theta = \theta_k$, $\varepsilon = 0$.
The line is being replaced by a half cylinder.

$$\theta = \theta_k + rT, \qquad \varepsilon = r^2 E$$

with $(T, E) \in S^1$.

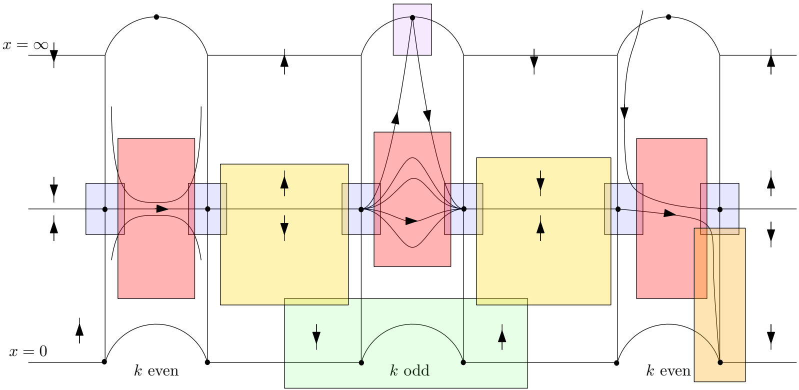
The family chart

Here we write

$$(\theta, \varepsilon) = (\theta_k + rT, r^2)$$

$$\begin{aligned} x' &= (-1)^k T x^2 (x - \tilde{B}(\theta_k + rT)) + r x^3 P(\theta_k + rT), \\ T' &= 1, \\ r' &= 0. \end{aligned}$$

This vector field does not have critical points.



The red parts are studied in the family chart, the purple parts in the matching charts. One could study the orange part in detail also in the matching chart (with yet another blow-up necessary), but we prefer a crude result there

The left matching chart

Here we write

$$(\theta, \varepsilon) = (\theta_k - r, r^2 E),$$

$$\begin{aligned} x' &= -(-1)^k x^2 (x - \tilde{B}(\theta_k - r)) + rEx^3 P(\theta_k - r), \\ r' &= -rE, \\ E' &= 2E^2. \end{aligned}$$

For x in compact intervals that do not contain $\tilde{B}(\theta_k)$ or 0, the dynamics is quasi-vertical, since it is a regular perturbation of

$$\begin{aligned} x' &= -(-1)^k x^2 (x - \tilde{B}(\theta_k)) \\ r' &= 0, \\ E' &= 0. \end{aligned}$$

Let θ_k be the left boundary point of a red section.

Proposition

Let w the first point for which $I(\theta_k, w) = 0$. Assume such a point exists and it is a green point.
Then, for $\varepsilon > 0$ and $\delta > 0$ sufficiently small, the dynamics defines a smooth transition map

$$p_\epsilon : [\theta_k + \sqrt{\varepsilon}, \theta_k + \delta) \subset \overline{\{red\}} \to \{green\}$$

with a continuous extension towards p_0 as $\varepsilon \to 0$ and satisfying $I(v, p_0(v)) = 0$.

Proof using a similar technique with $\underline{x}$ and $\overline{x}$.
We did not want to study the smoothness of p_ε, it would have required additional blowups.

Study of the contact points

Proposition

Let k be even. Then, for $\delta > 0$ sufficiently small, there is a transition map

$$\{\theta = \theta_*\} \to \{\theta = \theta_k\}$$

with $\theta_* = \theta_k - \delta$ (resp. $\theta_* = \theta_k + \delta$) that is well defined for sufficiently small $\varepsilon > 0$ and for x in a compact interval inside $]0, \infty[$, by following the orbits in positive time (resp. negative time), and is given by

$$x \mapsto V_\pm(\varepsilon) + (x - \phi_\varepsilon(\theta_*)) \exp \frac{J(x, \varepsilon)}{\varepsilon}.$$

The function V_+ (resp. V_-) is smooth w.r.t. $\sqrt{\varepsilon}$ and $J(x, \varepsilon)$ is smooth w.r.t. x and $(\sqrt{\varepsilon}, \varepsilon \ln \varepsilon)$ (up to $\varepsilon = 0$). Furthermore, $J(x, 0)$ is the slow divergence integral $\int_{\theta_*}^{\theta_k} \frac{B_0(t)^2}{A_0(t)} dt$ (it does not depend on x).

Elements of the proof

- choose local center manifold
- choose normal form. It defines a vector field for which the orbits can be manually integrated.

$$
\begin{aligned}
z' &= h(r, E)z, \\
r' &= -rE, \\
E' &= 2E^2.
\end{aligned}
$$

- Prove using the normal form that the family of slow curves extend smoothly in the matching chart. The ε-family defines another local center manifold
- Repeat the construction of normal form, now with the slow curve flattened to $z = 0$.
- Characterize the transition map through explicit integration

Proposition

Regarding the functions $V_\pm$ (in the case k even) in Proposition 5, we have

$$V_- - V_+ = \sqrt{\varepsilon}\Delta + O(\varepsilon)$$

with $\Delta = -\sqrt{2\pi}\rho_k \tilde{B}(\theta_k)$.

Conclusion:

when $\rho_k > 0$, the orbits jump down when $\rho_k < 0$, the orbits jump up

If the system has additional parameter λ that controls ρ_k, meaning that

$$\frac{\partial \rho_k}{\partial \lambda} \neq 0$$

then λ is a canard breaking parameter.

Proof.

Let us recall the expression of the vector field in the family chart, which for even k is

$$x' = Tx^2(x - \tilde{B}(\theta_k + rT)) + rx^3 P(\theta_k + rT),$$
$$T' = 1.$$

We observe that at $r = 0$, the curve $x = \tilde{B}(\theta_k)$ is an invariant curve. Hence we may look for orbits of the form $x = \tilde{B}(\theta_k) + x_1(T)r + O(r^2)$. It is elementary to set up an equation for $x_1(T)$:

$$\frac{dx_1}{dT} = T\tilde{B}(\theta_k)^2(x_1 - \tilde{B}'(\theta_k)T) + \tilde{B}(\theta_k)^3 P(\theta_k),$$

which can be solved using the variation of constants method (and taking into account the expression for ρ_k):

$$x_1(T) = T\tilde{B}'(\theta_k) - \rho_k \tilde{B}(\theta_k)^2 \int_C^T \exp\left(\tilde{B}(\theta_k)^2 \frac{T^2 - s^2}{2}\right) ds.$$

...

It can be seen that the center manifold in the left matching chart corresponds to those orbits for which $x_1(T)$ has non-exponential growth towards $T=-\infty$ (it is found by choosing $C=-\infty$). Similarly, $V_+(r)$ corresponds to the orbit with $C=+\infty$. To conclude, we find

$$V_\pm(r)=\tilde{B}(\theta_k)+r\left(-\rho_k\tilde{B}(\theta_k)^2\int_{\pm\infty}^{0}\exp\left(-\tilde{B}(\theta_k)^2\frac{s^2}{2}\right)ds\right)+O(r^2),$$

□

Proposition (Jump Behaviour)

When k is even and $\rho_k \neq 0$, the transition map

$$\{\theta = \theta_k, |x - V_-| \leq K\varepsilon\} \to \{x = x_*\}$$

with $x_* \in \left]0, \tilde{B}(\theta_k)\right[$ in case $\rho_k > 0$ and $x_* \in \left]\tilde{B}(\theta_k), \infty\right[$ in case $\rho_k < 0$, is well-defined and continuous for sufficiently small $\varepsilon > 0$ (and K arbitrary). During the transition map, all orbits tend, as $\varepsilon \to 0$ uniformly and in Hausdorff sense to the limit set defined by a part of the critical curve, between θ_* and θ_k, together with a vertical segment along $\theta = \theta_k$, between $x = \tilde{B}(\theta_k)$ and $x = x_*$. In particular, the orbits reach the end section at a point with

$$\theta = \theta_k + (\varepsilon \ln(1/\varepsilon))^{1/2}(t + o(1)), \tag{1}$$

for some $t \in [t_0, t_1]$ (with t_0 and t_1 some strictly positive constants).

First result

Assume that $\rho_k > 0$ for all even k. Assume also that $\int_{\theta_k}^{\theta_\ell} B(\theta)d\theta \neq 0$. Then the system has at most two cycles. It has exactly two when additionally

$$\int_0^{2\pi} B(\theta)d\theta > 0.$$

(elementary proof)

Second result

Assume that there are two θ_k, and that there is a parameter λ controlling ρ_2. Then there exist canard type orbits, toggling between the ground state and the critical curve. The stability is predicted by $J(\theta_{exit}, \theta_{entry})$, and zeros of J show presence of extra cycles.
Proof: using IFT to glue two orbits together at θ_2.
The result is extended to cases where additional jump points are considered. In that case, we do not use the IFT but the intermediate value theorem.
When there are multiple parameters, we can also obtain multilayer canards of the $(TP)^N$ kind. The same method: IFT, or the poincare-miranda theorem.

Third result

When there is a $\int_{\theta_k}^{\theta_\ell} B(\theta)d\theta = 0$ for some even k and ℓ: the transition from θ_k to θ_ℓ is seen as something similar to a jump breaking connection, if there is an extra parameter λ controlling the integral. All other statements remain the same.

Back to the linear case

Using linear ε-perturbation of the vector field we can control ρ_1 and ρ_2, but we cannot control the stability of canard orbits independently from the stability of the ground state. We can only ensure existence of two cycles.
We need something more.

Behaviour near infinity

Lemma

In the left and the right matching charts, the orbit $r = E = 0$ *crosses* $z = 0$*, as well as all nearby orbits.*

Proof.

In the left matching chart at infinity, the vector field is given (after multiplication by z) by

$$\begin{aligned} \dot{z} &= (-1)^k - rEP(\theta_k - r) + O(z) \\ \dot{r} &= -rEz \\ \dot{E} &= 2E^2 z \end{aligned}$$

So along the orbit $r = E = 0$, $\dot{z}$ is nonzero. Similar for the system in the right matching chart. □

So it remains to study the family chart at infinity . Recall that $z = \frac{1}{x}$:

Lemma

In the family chart, the blown-up system has a unique singularity on $z = 0$*, located at*

$$T = (-1)^{k+1} P(\theta_k) r + O(r^2).$$

When k *is even, it is a center or focus and no periodic orbit can approach the singularity at infinity. When* k *is odd, it is a saddle with eigenvalues* $\lambda_\pm := \pm 1 - \frac{1}{2} P(\theta_k)\tilde{B}(\theta_k) r + O(r^2)$ *with an incoming separatrix in the quarter plane* $\{z > 0, T < 0\}$ *and an outgoing separatrix in the quarter plane* $\{z > 0, T > 0\}$.

The next step is to look at the transition map near the saddle, the so-called Dulac map. We will only require limited information in this paper.

For $T_0 > 0$ small enough, the saddle separatrices intersect $\{T = \pm T_0\}$ at z-coordinates $z_\pm(r)$. So we look at the transition map

$$\{T = -T_0, z > z_-(r)\} \to \{T = T_0, z > z_+(r)\}$$

Proposition

For $r \geq 0$ sufficiently small, the Dulac map can be decomposed as

$$H_+^{-1} \circ D \circ H_-$$

where $H_\pm$ are some C^1 diffeomorphisms taking $z = z_\pm(r)$ to $Z = 0$ and with

$$D(Z) = Z^{-\lambda_-/\lambda_+}.$$

Canard-spike

Given a system with two vertical lines. Let λ be a breaking parameter and let μ control the ratio of eigenvalues. Then the equation to solve is

$$Z^{1+\sqrt{\varepsilon}\mu} \exp \frac{J}{\varepsilon} - Z \exp \frac{K}{\varepsilon} = \lambda + O(\varepsilon).$$

It can be solved using IFT w.r.t. λ for each value of Z. Note that for any fixed $\varepsilon > 0$, the divergence integral goes to infinity, with the sign controlled by μ as $Z \to 0$. More in particular, there is a saddle-node of limit cycles (when $J \neq K$) near the saddle homoclinic. We use this to find the three limit cycles in the linear trigonometric.

Of course there is no reason to restrict to one spike

- Each saddle separatrix initiates a slow curve above a green segment or a red segment in inverse time.
- We construct a sequence of transition maps from $\{\theta = \theta_k\}$ with k odd to $\{\theta = \theta_{k\pm1}\}$, each domain section parameterized by a coordinate z_k near infinity.
- So the system of difference maps is given by

$$\begin{aligned} z_1^{\lambda_1^+} e^{J_{12}/\varepsilon} - z_3^{\lambda_3^-} e^{J_{43}/\varepsilon} &= \lambda_2 \\ z_3^{\lambda_3^+} e^{J_{34}/\varepsilon} - z_5^{\lambda_5^-} e^{J_{45}/\varepsilon} &= \lambda_4 \\ \dots \\ z_n^{\lambda_n^+} e^{J_{n,n+1}/\varepsilon} - z_1^{\lambda_1^-} e^{J_{n1}/\varepsilon} &= \lambda_{n+1} \end{aligned}$$

Assuming that there are some additional free parameters controlling all integrals and after blowing up around $z_k = 0$, we can reduce this system of equations to the well known

$$(a_n + (\dots(a_1 + Z^{r_1})^{r_2})\dots)^{r_n} = Z\Phi(Z)$$

Unfortunately all r_i are $1 + o_\varepsilon(1)$...

Outlook: [Adapted Gasull's problem 6] Prove that there are at most 4? limit cycles of

$$\varepsilon \frac{dx}{d\theta} = A(\theta)x^3 + B(\theta)x^2 + C(\theta)x$$

near the singular limit, in case A, B and C are linear trigonometric, possibly restricting to $C(\theta) = constant$.
What do we still need to do so?

- case of double roots of B and or A
- case of C tending to 0 together with epsilon. It may lead to unexpected entry–exit behaviour [DM, Zhang, ongoing work]
- passage through jump points
- canard behaviour at the zero solution, i.e. when $\int_{\theta_k}^{\theta_\ell} B(\theta)d\theta = 0$.
- only 1 spike is possible. It is a degenerate saddle-loop study.

Thank you for your attention!