V SYMPOSIUM ON PLANAR VECTOR FIELDS

New lower bounds for limit tori in 3D polynomial vector fields

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Introduction

Motivated by Hilbert's 16th Problem, the quest to determine the finiteness of the maximum number $H(m)$ of limit cycles that planar polynomial vector fields of degree $m \geqslant 2$ can exhibit has driven research for over a century.

Most advances concerning $H(m)$ have centered on establishing lower bounds. In recent decades, substantial progress particularly in the study of the cyclicity problem, both local and global has led to further improvements in the known lower bounds for $H(m)$ at low degrees. The best-known global estimatesinclude $H(2) \geq 4$ [1], $H(3) \geq 13$ [2], and, for higher degrees, $H(4) \geq 28$, $H(5) \geq 37$, $H(6) \geq 53$, $H(7) \geq 74$, $H(8) \geq 96$, $H(9) \geq 120$, and $H(10) \geq 142$ [3].

In higher dimensions, isolated invariant tori play a role analogous to that of limit cycles in the plane, providing a natural framework for extending classical questions related to existence, stability, number, and distribution. From this perspective, the authors of [4] proposed an analogue of Hilbert's 16th Problem in three-dimensional space: to investigate the maximal number $N(m)$ of isolated invariant tori in polynomial vector fields of degree m.

Among the compact invariant manifolds of vector fields, *normally hyperbolic* ones are of particular interest due to their robust dynamical properties, such as persistence under small perturbations. This motivates the following definition:

$$N_h(m) := \sup \{\tau_h(P, Q, R) : \deg(P), \deg(Q), \deg(R) \leqslant m\},$$

where $\tau_h(P, Q, R)$ denotes the number of normally hyperbolic limit tori of the vector field $X = P\partial_x + Q\partial_y + R\partial_z$. It is evident that $N_h(m) \leqslant N(m)$. Analogously to the Hilbert number $H(m)$, it is an open question whether $N(m)$ or $N_h(m)$ are finite.

Motivated by developments in the planar case to establish and refine lower bounds for $H(m)$, the goal of this work is to improve the known lower bounds for $N_h(m)$ for all $m \geq 2$. Our approach is based on recent advances in the averaging method, which we apply to detect limit tori in three-dimensional vector fields that are close to ones exhibiting monodromic behavior around the z-axis.

Preliminary results

The main tool used to investigate the tori bifurcation is the Averaging theory. For a detailed exposition on this method, we refer the reader to [5].

We consider systems of non-autonomous T-periodic differential equations, given in the standard form

$$\dot{\mathbf{x}} = \sum_{i=1}^{N} \varepsilon^i F_i(t, \mathbf{x}; \mu) + \varepsilon^{N+1} \tilde{F}(t, \mathbf{x}; \mu, \varepsilon), \tag{1}$$

where $\tilde{F} : \mathbb{R} \times D \times \Lambda \times [0, \varepsilon_0] \to \mathbb{R}^2$ and each $F_i : \mathbb{R} \times D \times \Lambda \to \mathbb{R}^2$, $i \in \{1, \ldots, N\}$ is a C^r function, $r > 1$ and T-periodic in the variable t. Here D is a bounded subset of $\mathbb{R}^2$, Λ is an open subset of $\mathbb{R}^n$, $\varepsilon_0 > 0$.

Since F_i and $\tilde{F}$ are periodic, we can consider (1) as a family of autonomous systems in the extended phase space $\mathbb{S}^1 \times D$, where $\mathbb{S}^1 \equiv \mathbb{R}/T\mathbb{Z}$:

$$\begin{aligned} \dot{\mathbf{x}} &= \textstyle\sum_{i=1}^{N} \varepsilon^i F_i(t, \mathbf{x}; \mu) + \varepsilon^{N+1} \tilde{F}(t, \mathbf{x}; \mu, \varepsilon), \\ \dot{t} &= 1. \end{aligned} \tag{2}$$

The Poincaré map $\Pi(\mathbf{x}; \mu, \varepsilon)$ associated to the equation (2) defined on the section $\{t = 0\}$ is given by

$$\Pi(\mathbf{x}; \mu, \varepsilon) = \mathbf{x} + \sum_{i=m}^{N} \varepsilon^i \mathbf{f}_i(\mathbf{x}; \mu) + O(\varepsilon^{N+1}). \tag{3}$$

The functions $\mathbf{f}_i$ are the Melnikov functions, which can be recursively computed [6]. The first two Melnikov functions are given by

$$\begin{aligned} \mathbf{f}_1(\mathbf{z}; \mu) &= \int_0^T F_1(t, \mathbf{z}; \mu) dt, \\ \mathbf{f}_2(\mathbf{z}; \mu) &= \int_0^T \left(F_2(t, \mathbf{z}; \mu) + \partial_x F_1(t, \mathbf{z}; \mu) \int_0^t F_1(s, \mathbf{z}; \mu) ds \right) dt. \end{aligned}$$

Let $\ell \in \{1, \ldots, N\}$ be the index of the first non-vanishing Melnikov function and consider the so-called *guiding system*

$$\dot{\mathbf{z}} = \mathbf{g}_\ell(\mathbf{z}; \mu) = \frac{1}{T} \mathbf{f}_\ell(\mathbf{z}; \mu). \tag{4}$$

Theorem 1 Consider the differential equation (1). Suppose that for some $\ell \in \{1, \ldots, N\}$, $\mathbf{f}_1 = \cdots = \mathbf{f}_{\ell-1} = 0$, $\mathbf{f}_\ell \neq 0$ and that the guiding system (4) has a hyperbolic attracting (resp. repelling) limit cycle γ. Then, for each $\varepsilon > 0$ sufficiently small, the differential equation (1) has a T-periodic solution γ_{int} and a normally hyperbolic attracting (resp. repelling) invariant torus in the extended phase space. In addition, the torus surrounds the periodic solution γ_{int} and converges to $\gamma \times \mathbb{S}^1$ as $\varepsilon \to 0$.

Thus, the problem of finding limit tori in the extended phase space of system (1) reduces to the search for limit cycles in the phase space of the associated guiding system, which, in our case, will be a planar differential equation. From here, the classical tools in the investigation of limit cycle bifurcation can be applied.

The Method

We consider the following 1-parameter family of vector fields

$$X = X_0 + \varepsilon X_1(\mu) + \varepsilon^2 X_2(\mu), \tag{5}$$

where $\mu \in \Lambda \subset \mathbb{R}^M$, $\varepsilon > 0$ is a small parameter, and X_1, X_2 are polynomial vector fields of degree m. The *unperturbed part* is given by

$$X_0 = -y\,\partial_x + x^{2n-1}\,\partial_y,$$

with $n \geqslant 1$, and has the z-axis as an invariant line of singularities, around which nearby orbits exhibit rotational (monodromic) behavior. When $n = 1$, the origin of X_0 is a *Hopf-zero* singularity; for $n > 1$, we refer to it as a *nilpotent-zero* singularity. In this latter case, n is called by *Andreev number* associated with the singularity.

We now search for conditions on the parameters $\mu \in \Lambda$ for invariant tori bifurcations to occur. To fix notation, we set $\mu = (a_{jkl}, b_{jkl}, c_{jkl})$ and

$$X_1(\mu) = \sum_{j+k+l=0}^{m} a_{jkl} x^j y^k z^l \partial_x + \sum_{j+k+l=0}^{m} b_{jkl} x^j y^k z^l \partial_y + \sum_{j+k+l=0}^{m} c_{jkl} x^j y^k z^l \partial_z.$$

We introduce the change of variables

$$x = r\mathrm{Cs}\theta, \quad y = r^n \mathrm{Sn}\theta, \quad z = w,$$

where $\mathrm{Cs}\theta$ and $\mathrm{Sn}\theta$ are the *generalized trigonometric functions*. After this change of variables, in the associated differential system, we have

$$\dot{\theta} = r^{n-1} + \frac{\varepsilon}{r^{n+1}} \left(\sum_{j+k+l=0}^{m} b_{jkl} r^{j+kn+1} \mathrm{Sn}^k\theta \mathrm{Cs}^{j+1}\theta w^l - n a_{jkl} r^{n+j+nk} \mathrm{Sn}^{k+1}\theta \mathrm{Cs}^j\theta w^l \right) + O(\varepsilon^2),$$

and since $\dot{\theta}$ does not vanish at $\varepsilon = 0$ for $r \neq 0$, setting θ as the independent variable, we obtain the differential system

$$\left(\frac{dr}{d\theta}, \frac{dw}{d\theta} \right) = \varepsilon F_1(r, \theta, w; \mu) + \varepsilon^2 \tilde{F}(r, \theta, w; \mu, \varepsilon),$$

which is written in the standard form (1).

Now we are set to apply the averaging method to study the bifurcation of invariant tori in the original system (5). Computing explicitly the first averaged function $\mathbf{g}_1(r, w; \mu)$, after a change of parameters, we obtain the equivalent system

$$\begin{aligned} r' &= \sum_{j+k+l=1}^{m} a_{jkl} r^{j+nk} w^l + r^{1-n} \sum_{j+k+l=1}^{m} b_{jkl} r^{j+nk} w^l, \\ w' &= \sum_{j+k+l=0}^{m} c_{jkl} r^{j+nk} w^l, \end{aligned} \tag{6}$$

for the following admissible parameters

$$a_{jkl} : j + 1 \equiv k \equiv 0 \mod 2, \quad b_{jkl} : j \equiv k + 1 \equiv 0 \mod 2, \quad c_{jkl} : j \equiv k \equiv 0 \mod 2.$$

Once this process is completed, one may search for equilibrium points (ρ, w_0) of (6) in the region $r > 0$ whose associated eigenvalues are conjugated complex with non zero imaginary part, and study whether a *degenerate Hopf bifurcation* occurs or not. In practice, it is more effective to assume conditions on the parameters $(a_{jkl}, b_{jkl}, c_{jkl})$ such an equilibrium point exists.

Remark 1 The guiding system (6) is polynomial with degree $d(m, n)$ given by

$$d(m, n) = \begin{cases} n(m-1) + 1, & \text{for } m \text{ odd}, \\ nm, & \text{for } m \text{ even} \end{cases}$$

Thus, the number of limit tori one can detect via the first ordered averaged system is bounded by $H(d(m, n))$, which suggests that in the search for better lower bounds on $N(m)$, the nilpotent-zero singular points have more potential. For instance, for quintic systems (5), the first order averaged function $\mathbf{g}_1$ has also degree 5, but for $n = 2, 3$, the corresponding degrees are 9 and 13.

New lower bounds

Performing this investigation, we were able to obtain for $n = 1, 2, 3$, $m = 2, 3, 4, 5$, examples of 3D vector fields with the following number of limit tori:

m \ n	1	2	3
2	3	-	-
3	3	5	-
4	6	7	-
5	9	11	13

As a consequence, we obtain our main result:

Theorem 2 $N_h(2) \geqslant 3$, $N_h(3) \geqslant 5$, $N_h(4) \geqslant 7$, and $N_h(5) \geqslant 13$.

The above lower bounds for N_h are the best known in the literature.

Notably, and in contrast to the planar case, the best lower bounds for $N_h(m)$ were obtained from nilpotent-zero singularities. Moreover, it was shown in [7] that the function $N_h(m)$ is strictly increasing. Another important property of $N_h(m)$ is that if there exists a three-dimensional vector field of degree m_0 with τ_0 limit tori, then, as shown in [4] via a Christopher–Lloyd-type transformation, one obtains

$$N\left(2^k(m_0 + 2) - 2\right) \geq 8^k \tau_0. \tag{7}$$

This inequality also holds for N_h, provided that the τ_0 limit tori are normally hyperbolic. Consequently, lower bounds established for low degrees can be systematically extended to higher degrees.

In view of these results, the lower bounds for $N(m)$ established in the above table can be extended to higher degrees, as presented below. These improved lower bounds surpass those previously obtained in the literature.

m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
old lower bounds	1	2	2	2	8	8	16	16	28	28	37	37	64	64	64	$\cdots$
new lower bounds	3	5	7	13	24	25	40	41	56	57	104	105	192	193	200	$\cdots$

References

[1] L. Chen, M. Wang, *The relative position, and the number, of limit cycles of a quadratic differential system*, Acta Math. Sinica, 22, (1979).

[2] J. Li, Y. Liu, *New results on the study of $\mathbb{Z}_q$-equivariant planar polynomial vector fields*, Qual. Theory Dyn. Syst., 9 (2010).

[3] R. Prohens, J. Torregrosa, *New lower bounds for the Hilbert numbers using reversible centers*, Nonlinearity, 32 (2019).

[4] D. Novaes, P. Pereira, *A version of Hilbert's 16th problem for 3D polynomial vector fields: Counting isolated invariant tori*, Math. Nachr., 298 (2025).

[5] M. Cândido, D. Novaes, P. Pereira, *A mechanism for detecting normally hyperbolic invariant tori in differential equations*, Journal de Mathématiques Pures et Appliquées, 177 (2023).

[6] D. Novaes, *Higher order stroboscopic averaged functions: a general relationship with Melnikov functions*, Electron. J. Qual. Theory Differ. Equ. (2021).

[7] L. Arakaki, D. Novaes, P. Pereira, *Variation in the number of normally hyperbolic limit tori in 3D polynomial vector fields via time reparametrization and Hopf-zero bifurcation*, preprint (2025).

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