

# Stability of some monodromic singularities with two edges in the Newton diagram

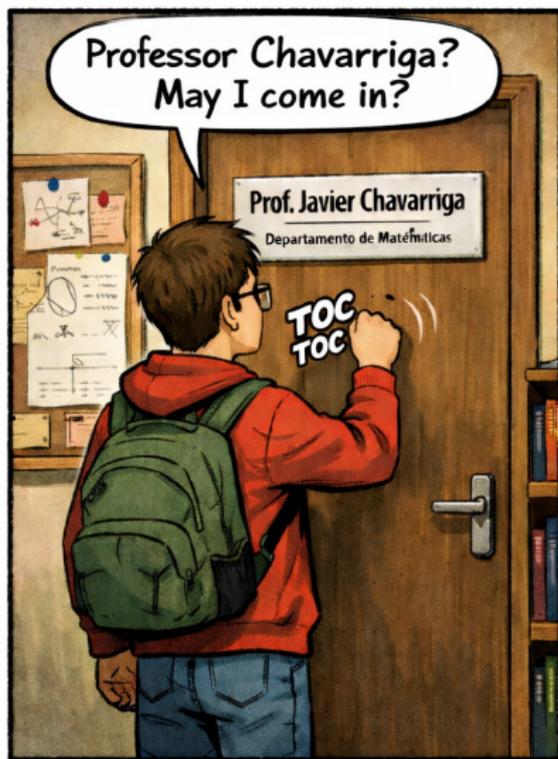
Isaac A. García, Jaume Giné and Víctor Mañosa

SYMPOSIUM ON PLANAR VECTOR FIELDS, LLEIDA,  
JANUARY 12-16, 2026.

This work is motivated by...



# The first time I met Javier



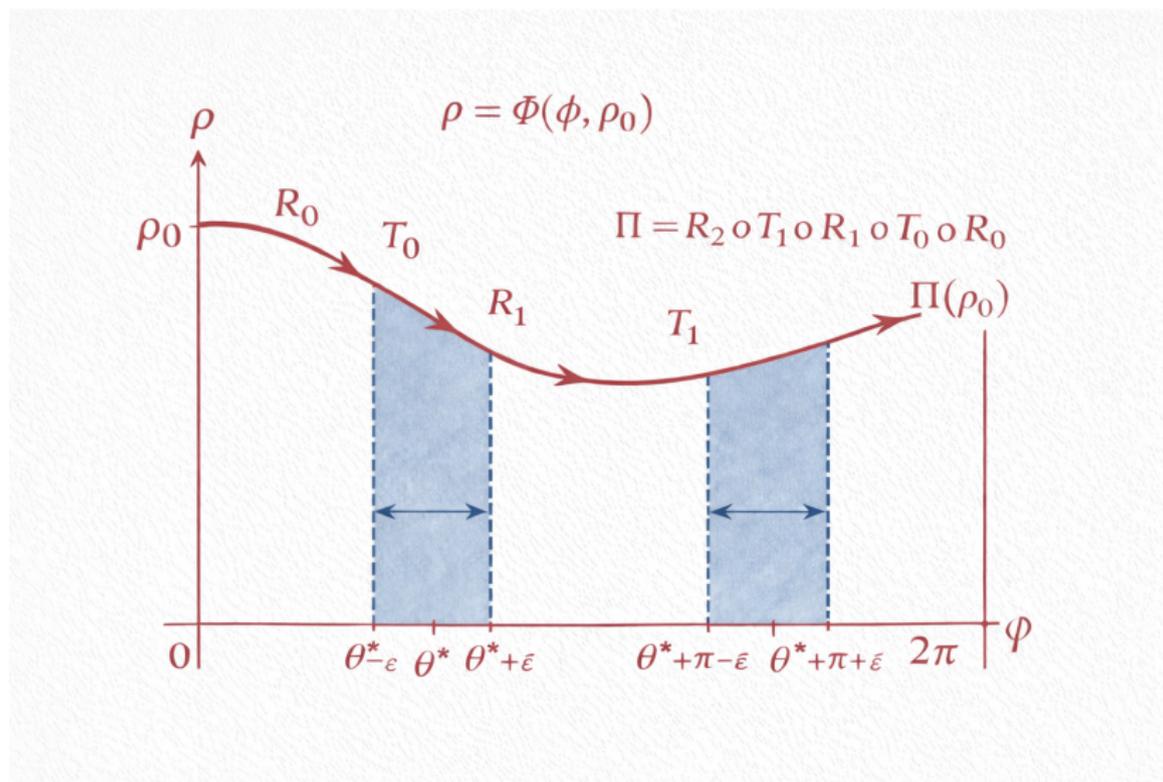
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- A. GASULL, J. LLIBRE, V. MAÑOSA, F. MAÑOSAS, *The focus-centre problem for a type of degenerate system*, Nonlinearity **13** (2000), 699–729.
- A. GASULL, V. MAÑOSA, F. MAÑOSAS, *Monodromy and stability of a class of degenerate planar critical points*, J. Differential Equations **217** (2005), 363–376.
- V. MAÑOSA, *On the center problem for degenerate singular points of planar vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **12** (2002), no. 4, 687–707.

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- 
- N.B. MEDVEDEVA, *The problem of distinguishing between a centre and a focus in the space of vector fields with given Newton diagram*, Sb. Math. **211** (2020), 1399–1446.

# Geometric idea in the works of the UAB team:



# The Newton diagram $\mathbf{N}(\mathcal{Z})$ of an analytic planar vector field $\mathcal{Z}$

We will use the following notation. Let

$$\mathcal{Z} = \sum_{(i,j) \in \mathbb{N}^2} a_{ij} x^i y^{j-1} \partial_x + \sum_{(i,j) \in \mathbb{N}^2} b_{ij} x^{i-1} y^j \partial_y \quad (1)$$

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The **Newton diagram**  $\mathbf{N}(\mathcal{Z})$  of  $\mathcal{Z}$  is the polygonal line that appears when considering the boundary (modulus the two open rays) of the convex hull of the set  $\bigcup_{(i,j) \in \text{supp}(\mathcal{Z})} \{(i,j) + \mathbb{R}_+^2\}$ , where

$$\text{supp}(\mathcal{Z}) = \{(i,j) \in \mathbb{N}^2 : (a_{ij}, b_{ij}) \neq (0,0)\}.$$

## Example of Newton diagram for $\mathcal{X}$

Let  $\mathcal{X}$  be

$$\dot{x} = Bx^2y + Ay^5, \quad \dot{y} = Dxy^2 + Cx^5, \quad (2)$$

so that

$$\text{supp}(\mathcal{X}) = \{(2, 2), (0, 6), (6, 0)\}.$$

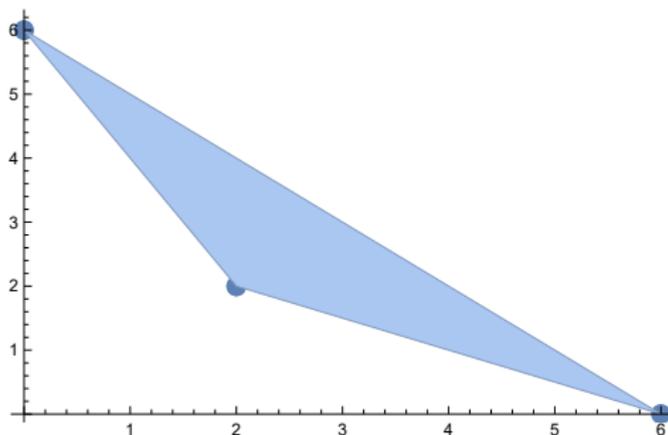
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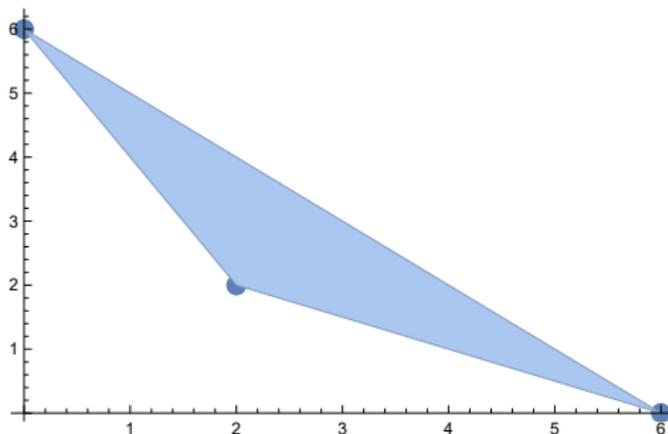
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The **weights** are:  $W(\mathbf{N}(\mathcal{X})) = \{(2, 1), (1, 2)\}$

# Example of Newton diagram for $\mathcal{X}^{[1]}$

Let the first extension  $\mathcal{X}^{[1]}$  of  $\mathcal{X}$  be:

$$\dot{x} = Bx^2y + Ay^5 + a_1xy^3 + a_2x^4, \quad \dot{y} = Dxy^2 + Cx^5 + b_1y^4 + b_2x^3y. \quad (3)$$

so that

$$\text{supp}(\mathcal{X}^{[1]}) = \text{supp}(\mathcal{X}) \cup \{(1, 4), (4, 1)\}.$$

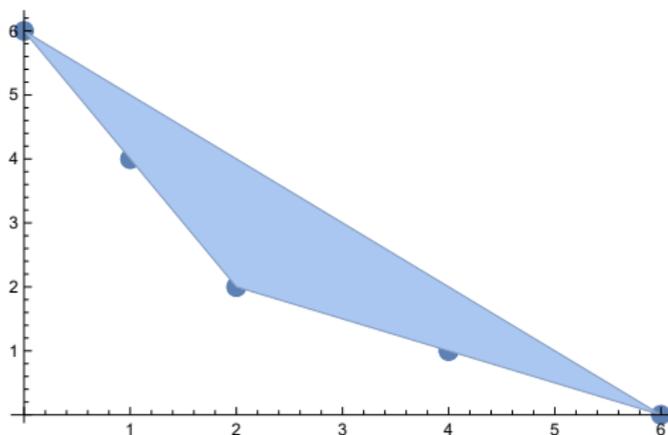
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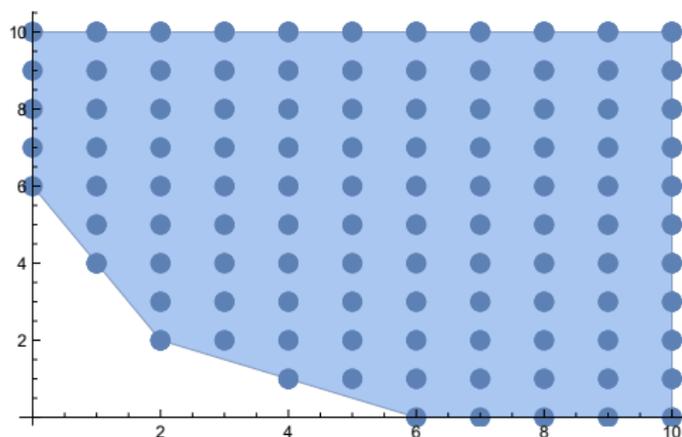
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# The Brunella-Miari principal part $\mathcal{Z}_\Delta$ of an analytic vector field $\mathcal{Z}$

Given  $\mathcal{Z}$ , we define  $\mathcal{Z}_\Delta$  as the biggest part of  $\mathcal{Z}$  such that

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NOTATION:  $\mathcal{Z}_\Delta = \bigoplus_{i=1}^k \mathcal{Z}_{r_i}$  where  $\mathcal{Z} = \mathcal{Z}_{r_1} + \dots$  is the  $(p_i, q_i)$ -quasihomogeneous expansion with  $(p_i, q_i) \in W(\mathbf{N}(\mathcal{Z}))$ .

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As example:  $\mathcal{Z} = \mathcal{Z}_\Delta + \dots$  where  $W(\mathbf{N}(\mathcal{Z})) = \{(1, 1), (1, 3)\}$ :

$$\mathcal{Z}_\Delta = (Bx^2y + Ay^3 + a_1xy^2 + a_2x^5)\partial_x + (Dxy^2 + Cx^7 + b_1y^3 + b_2x^4y)\partial_y$$

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(1, 3)-quasihomogeneous expansion:  $\mathcal{Z} = \mathcal{Z}_4 + \dots$  with

$$\mathcal{Z}_4 = (Bx^2y + a_2x^5)\partial_x + (Dxy^2 + Cx^7 + b_2x^4y)\partial_y.$$

# Monodromy and Brunella-Miari principal part

## Non-degenerate $\mathcal{Z}_\Delta$

We say that  $\mathcal{Z}_\Delta$  is non-degenerate when no quasihomogeneous component  $\mathcal{Z}_{r_i}$  of  $\mathcal{Z}_\Delta$  have singularities outside the coordinate axes.

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## Brunella-Miari monodromic result

The origin of  $\mathcal{Z}$  is monodromic  $\Leftrightarrow$  The origin of  $\mathcal{Z}_\Delta$  is monodromic (provided  $\mathcal{Z}_\Delta$  is non-degenerate).

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## Open problem:

Let the origin be a monodromic singularity of both  $\mathcal{Z}$  and  $\mathcal{Z}_\Delta$  and  $\Pi(x) = \eta x + o(x)$  and  $\Pi_\Delta(x) = \eta_\Delta x + o(x)$  its Poincaré maps, respectively. Then, is it true that  $\eta = \eta_\Delta$ ?

# Characteristic directions of a monodromic singularity

We define the  $(p, q)$ -weighted polar coordinates  $(\varphi, \rho)$  as:

$$x = \rho^p \cos \varphi, \quad y = \rho^q \sin \varphi,$$

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## Definition: characteristic directions

We transform a monodromic analytic vector field to  $(p, q)$ -weighted polar coordinates

$$\dot{\varphi} = G_0(\varphi) + O(\rho), \quad \dot{\rho} = R_1(\varphi)\rho + O(\rho^2).$$

The set of characteristic directions of the origin is

$$\Omega_{pq} = G_0^{-1}(0) = \{\theta_* \in \mathbb{S}^1 : G_0(\theta_*) = 0\}.$$

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When  $\Omega_{pq} \neq \emptyset$  then  $\rho = 0$  corresponds to a **monodromic polycycle**.

## Simple example: the nilpotent monodromic case

We consider a vector field  $\mathcal{Z}$  with a nilpotent singularity at the origin

$$\dot{x} = -y, \quad \dot{y} = x^5 + ax^2y, \quad (5)$$

with  $\Lambda = \{a \in \mathbb{R} : 12 - a^2 > 0\}$  and  $W(\mathbf{N}(\mathcal{Z})) = \{(1, 3)\}$ .

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$\eta$  computed using that  $\Omega_{13} = \emptyset$

In (1, 3)-weighted polar coordinates

$$\dot{\varphi} = G_0(\varphi) + O(\rho), \quad \dot{\rho} = R_1(\varphi)\rho + O(\rho^2), \quad G_0(\varphi) > 0.$$

Then,  $d\rho/d\varphi = \mathcal{F}(\varphi)\rho + o(\rho)$  with  $\mathcal{F} = R_1/G_0$  and

$$\eta = \exp\left(\int_0^{2\pi} \mathcal{F}(\varphi)d\varphi\right) = \exp\left(-\frac{1}{4} \int_0^{2\pi} \frac{\cos \varphi \sin^2 \varphi (-4a \cos \varphi + 5 \sin \varphi + \sin(3\varphi))}{(\cos^6 \varphi + a \cos^3 \varphi \sin \varphi + 3 \sin^2 \varphi)} d\varphi\right)$$

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$\eta$  computed using blow-ups  $W(\mathbf{N}(\mathcal{Z})) = \{(1, 3)\}$

The blow up  $z = y/x^3$  and time rescaling  $dt = x^2 d\tau$ :

$$x' = -xz, \quad z' = P(z) = 1 + az + 3z^2.$$

with  $P(z) > 0$  on  $\Lambda$ .

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The blow up  $v = x^3/y$  and time rescaling  $dt = (yv)^{2/3} d\tau$ :

$$y' = y(a + v), \quad v' = Q(v) = -3 - av - v^2,$$

with  $Q(v) < 0$  on  $\Lambda$ .



# Type $\mathcal{A}$ characteristic direction

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We say that  $\theta_* = 0$  is of **type  $\mathcal{A}$**  with weights  $(p, q) \in W(\mathbf{N}(\mathcal{Z}))$  and  $(p, q) \neq (p_1, q_1)$  if it can be desingularized using the following two blow-ups:

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- **FIRST BLOW-UP:**  $(x, y) \mapsto (z, w)$  with  $z = x^q/y^p$  and  $w = y/x$  which gives a hyperbolic saddle at the origin of

$$\dot{z} = Z(z, w) = -az + \dots, \quad \dot{w} = W(z, w) = bw + \dots; \quad (\text{A})$$

with  $a, b \neq 0$  and  $\lambda = a/b > 0$  being its **hyperbolicity ratio**.

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- **SECOND BLOW-UP:**  $(x, y) \mapsto (x, v)$  with  $v = y^p/x^q$  giving a regular flow in a neighborhood of the **regular solution  $x = 0$**  of

$$\dot{x} = X(x, v), \quad \dot{v} = V(x, v), \quad (\text{B1})$$

that is with  $X(0, v) = 0$  and  $V(0, v) \neq 0$ ,

# The desingularization scheme for the type $\mathcal{A}$ characteristic direction

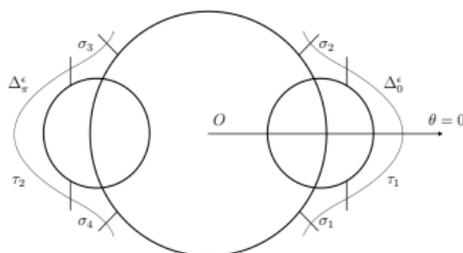


FIGURE 1. Transition maps scheme for  $\mathcal{A}$ -type characteristic directions.

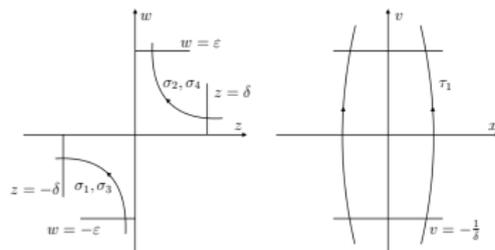


FIGURE 2. Transition maps in local coordinates for  $\mathcal{A}$ -type characteristic directions in the case  $p$  and  $q$  odd.

Type  $\mathcal{A}$  characteristic direction when  $\#\{\Omega_{pq} \cap [0, \pi)\} \geq 2$

Let  $R_{-\theta^*}$  be a planar rotation of angle  $-\theta^*$ .

Type  $\mathcal{A}$  characteristic direction  $\theta_* \neq 0$

We say that  $0 \neq \theta^* \in \Omega_{pq}$  is of type  $\mathcal{A}$  for the vector field  $\mathcal{Z}$  if 0 is a type  $\mathcal{A}$  characteristic of the rotated vector field  $R_{-\theta^*}^* \mathcal{Z}$ .

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Remark: Complementary characteristic directions

Let  $\theta^*$  be a type  $\mathcal{A}$  characteristic direction with respect to some weights  $(p, q) \in W(\mathbf{N}(\mathcal{Z}))$ . Then  $\theta^* + \pi$  is too.

# Stability result for type $\mathcal{A}$ characteristic directions

We assume:

- The origin is a monodromic point of an analytic  $\mathcal{Z}$ ;
- The set  $\Omega_{pq} \cap [0, \pi) = \{\theta_1^*, \dots, \theta_n^*\}$  is of type  $\mathcal{A}$ ;

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Then we consider:

- The differential equation  $d\rho/d\varphi = \mathcal{F}(\varphi)\rho + o(\rho)$  of the orbits of  $\mathcal{Z}$  in  $(p, q)$ -weighted polar coordinates  $(\varphi, \rho)$ .

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- Let  $\lambda_i$  be the hyperbolicity ratio of the saddle at the origin of the differential system  $\dot{z} = Z_i(z, w)$  and  $\dot{w} = W_i(z, w)$  associated to the first blow-up of each characteristic direction  $\theta_i^*$ ;

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Then we consider:

- The differential equation  $d\rho/d\varphi = \mathcal{F}(\varphi)\rho + o(\rho)$  of the orbits of  $\mathcal{Z}$  in  $(p, q)$ -weighted polar coordinates  $(\varphi, \rho)$ .
- Let  $\lambda_i$  be the hyperbolicity ratio of the saddle at the origin of the differential system  $\dot{z} = Z_i(z, w)$  and  $\dot{w} = W_i(z, w)$  associated to the first blow-up of each characteristic direction  $\theta_i^*$ ;
- Let  $\dot{x} = X_i(x, v)$  and  $\dot{v} = V_i(x, v)$  be the differential system with regular orbit  $x = 0$  associated to the second blow-up of each characteristic direction  $\theta_i^*$ ;

# Stability result for type $\mathcal{A}$ characteristic directions

## Theorem

*We assume:*

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We define  $\mathcal{A}^* \subseteq \{1, \dots, n\}$  such that, for any  $i \in \mathcal{A}^*$ ,  $\theta_i^*$  is of type  $\mathcal{A}$  with *odd* weights  $(p_i, q_i)$ .

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Then, the Poincaré map for  $x \gtrsim 0$  is  $\Pi(x) = \eta x + o(x)$  with

$$\eta = \exp \left\{ \text{PV} \int_0^{2\pi} \mathcal{F}(\varphi) d\varphi + 2 \sum_{i \in \mathcal{A}^*} \frac{1}{\lambda_i p_i} \text{PV} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left( \frac{X_i(x, v)}{V_i(x, v)} \right) \Big|_{x=0} dv \right\}$$

where  $\mathcal{F} : \mathbb{S}^1 \setminus \Omega_{pq} \rightarrow \mathbb{R}$  and PV stands for the principal value.

The minimal model with  $\#W(\mathbf{N}(\mathcal{X})) = 2$  and  $\mathcal{X} = \mathcal{X}_\Delta$

The **minimal model** (minimum number of monomials) is given by the 4-parameter family of polynomial planar vector fields  $\mathcal{X}$

$$\dot{x} = Ay^{2s-1} + Bx^{2\alpha}y^{2\beta-1}, \quad \dot{y} = Cx^{2r-1} + Dx^{2\alpha-1}y^{2\beta}, \quad (7)$$

with arbitrary degree such that  $\mathbf{N}(\mathcal{X})$  satisfies the necessary monodromic conditions

- All the vertices of  $\mathbf{N}(\mathcal{X})$  have even coordinates;
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- The weights are:

$$W(\mathbf{N}(\mathcal{X})) = \{(p_1, q_1), (p_2, q_2)\} = \{(s - \beta, \alpha), (\beta, r - \alpha)\}.$$

# The monodromy of the minimal model $\mathcal{X}$

## The monodromic set $\Lambda$

The origin is a monodromic singular point of the minimal model  $\mathcal{X}$  if and only if its exponents belong to the set

$$\mathcal{E} = \{r, s, \alpha, \beta \in \mathbb{N} \setminus \{0\} : s - \beta \geq 1, r - \alpha \geq 1, (r - \alpha)(s - \beta) > \alpha\beta\},$$

and its coefficients lie in the monodromic parameter space

$$\Lambda = \{(A, B, C, D) \in \mathbb{R}^4 : AC < 0, A(D(s - \beta) - B\alpha) < 0, C(B(\alpha - r) + D\beta) > 0\}.$$

# The characteristic directions of the monodromic minimal model $\mathcal{X}$

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- $\Omega_{p_1, q_1} \cap [0, \pi) = \{0\}$ ;
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## A symmetry result for the monodromic minimal model

If  $0 \in \Omega_{p_1, q_1}$  is of type  $\mathcal{A}$  then  $\pi/2 \in \Omega_{p_2, q_2}$  is too.

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The proof is based on the fact that if  $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$  is a minimal model, then:

- $R_{-\pi/2}^* \mathcal{X} = -Q(x, y)\partial_x - P(x, y)\partial_y$ ;
- $W(\mathbf{N}(\mathcal{X})) = \{(p_1, q_1), (p_2, q_2)\} \Rightarrow W(\mathbf{N}(R_{-\pi/2}^* \mathcal{X})) = \{(q_2, p_2), (q_1, p_1)\}$ .

# The monodromic minimal model $\mathcal{X}$ on $\mathcal{E}^* \cap \Lambda^*$

We define the subset of exponents  $\mathcal{E}^* \subset \mathcal{E}$  as

$$\mathcal{E}^* = \{(\alpha, \beta, r, s) \in \mathcal{E} : e_i(\alpha, \beta, r, s) \in \mathbb{N} \cup \{0\}, i = 1, \dots, 8\}, \quad (8)$$

where

$$\begin{aligned} e_1 &= \frac{2(s - \alpha - \beta)}{r - \alpha - \beta}, & e_2 &= \frac{2(-rs + s\alpha + r\beta)}{-r + \alpha + \beta}, & e_3 &= \frac{2(r - \alpha - \beta)}{s - \alpha - \beta}, \\ e_4 &= \frac{2(rs - s\alpha - r\beta)}{s - \alpha - \beta}, & e_5 &= \frac{2(rs - s\alpha - r\beta)}{\alpha}, & e_6 &= \frac{2r - \alpha}{\alpha}, \\ e_7 &= \frac{2(rs - s\alpha - r\beta)}{\beta}, & e_8 &= \frac{2s - \beta}{\beta}. \end{aligned}$$

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We also define the monodromic parameter subset  $\Lambda^* \subset \Lambda$  by

$$\Lambda^* = \Lambda \cap \{(B\alpha - D(s - \beta))(B - D) > 0, (B(r - \alpha) - D\beta)(B - D) > 0\}.$$

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The minimal model  $\mathcal{X}$  on  $\mathcal{E}^* \cap \Lambda^*$  has a type-A desingularization.

# The monodromic minimal model $\mathcal{X}$ on $\mathcal{E}^\dagger \cap \Lambda^\dagger$

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The minimal model  $\mathcal{X}$  on  $\mathcal{E}^\dagger \cap \Lambda^\dagger$  has a type- $\mathcal{A}$  desingularization.

# An example where $\mathcal{X}$ , $\mathcal{X}^{[1]}$ , and $\mathcal{X}^{[2]}$ share type- $\mathcal{A}$ desingularization

We consider the minimal model  $\mathcal{X}$  with  $(\alpha, \beta, r, s) = (1, 1, 3, 2)$

$$\dot{x} = Bx^2y + Ay^3, \quad \dot{y} = Cx^5 + Dxy^2, \quad (9)$$

so that  $W(\mathbf{N}(\mathcal{X})) = \{(1, 1), (1, 2)\}$  and

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type  $\mathcal{A}$  characteristic directions

It follows that  $\Omega_{11} \cap [0, \pi) = \{0\}$  is of type  $\mathcal{A}$ .

# Stability of the origin for the example

The first extension  $\mathcal{X}^{[1]}$  of the minimal model  $\mathcal{X}$  given by (9) is:

$$\dot{x} = Bx^2y + Ay^3 + a_1xy^2 + a_2x^4, \quad \dot{y} = Cx^5 + Dxy^2 + b_1y^3 + b_2x^3y.$$

The origin of  $\mathcal{X}^{[1]}$  is monodromic on  $\hat{\Lambda} = \Lambda \cap \{\delta_1 < 0, \Delta_1 < 0\}$ ,  
 $\delta_1 = (a_1 - b_1)^2 + 4A(D - B)$ ,  $\Delta_1 = (2a_2 - b_2)^2 + 4C(2B - D)$ .

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The second extension  $\mathcal{X}^{[2]}$  is

$$\begin{aligned} \dot{x} &= Bx^2y + Ay^3 + a_1xy^2 + a_2x^4 + \sum_{(i,j) \in S} a_{i,j-1}x^i y^{j-1}, \\ \dot{y} &= Cx^5 + Dxy^2 + b_1y^3 + b_2x^3y + \sum_{(i,j) \in S} b_{i-1,j}x^{i-1} y^j, \end{aligned} \quad (10)$$

where  $S = \{(i, j) \in \mathbb{N}^2 : i + j > 4, i + 2j > 6\}$  and the origin of  $\mathcal{X}^{[2]}$  restricted to  $\hat{\Lambda}$  is monodromic.

# Stability of the origin for the example

## Stability of $\mathcal{X}^{[1]}$ and $\mathcal{X}^{[2]}$

The Poincaré map of  $\mathcal{X}^{[1]}$  and  $\mathcal{X}^{[2]}$  in  $\hat{\Lambda}$  is  $\Pi(x) = \eta x + o(x)$  with

$$\eta = \exp \left\{ 2 \text{PV} \int_0^{2\pi} \frac{(b_1 + \cot \theta (A + D + \cot \theta (a_1 + B \cot \theta))) \sin^2 \theta}{D - A - B + (A - B + D) \cos(2\theta) + (b_1 - a_1) \sin^2 \theta} d\theta \right\}$$

# Minimal models $\mathcal{X}$ without type- $\mathcal{A}$ desingularization

The minimal model  $\mathcal{X}$  with  $(\alpha, \beta, r, s) = (1, 1, 4, 2) \in \mathcal{E}$  is:

$$\dot{x} = Bx^2y + Ay^3, \quad \dot{y} = Dxy^2 + Cx^7, \quad (11)$$

with  $\Lambda = \{AC < 0, AD < AB, CD > 3BC\}$  and  $W(\mathbf{N}(\mathcal{X})) = \{(1, 1), (1, 3)\}$ .

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- $\Omega_{11} \cap [0, \pi) = \{0\}$  is NOT of type- $\mathcal{A}$ .
- We know (**we do not show here**) a different desingularization scheme shared for all  $\mathcal{X}$ ,  $\mathcal{X}^{[1]}$  and  $\mathcal{X}^{[2]}$  and the explicit expression of  $\eta$  for  $\mathcal{X}^{[2]}$ .

# Possible behaviours

- There are minimal models  $\mathcal{X}$  with type- $\mathcal{A}$  desingularization not shared with  $\mathcal{X}^{[1]}$ .

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- There are minimal models with associated non-hyperbolic polycycle.

# Minimal models with associated non-hyperbolic polycycle

Minimal model with  $(\alpha, \beta, r, s) = (1, 1, 3, 2)$  and  $(C, D) = (1, B)$

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- 1 Blow-up:  $(x, y) \mapsto (x, w)$ ,  $w = y/x^2$  and time-rescaling  $x^3$ :

$$\dot{x} = x f(x, w), \quad \dot{w} = 1 - Bw^2 + x g(x, w),$$

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- 2 Blow-up:  $(x, y) \mapsto (u, z)$ ,  $u = y/x$ ,  $z = x^2/y$  and time-rescaling  $z^2u^3$ :

$$\dot{z} = Bz + z(-z^2 + u(b + 2Au)), \quad \dot{u} = u(z^2 - Au^2),$$

that has a **saddle-node** at the origin.