Reversible periodic linear systems: the planar case

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Abstract

The existence of a time-reversal symmetry in periodic linear systems imposes some structure in the monodromy matrix.

For planar systems, we take advantage of such a structure to study the local geometry of their stability boundaries around some critical points im the space of parameters.

A general result for reversible Hill's equation will be stated. We also revisit the elliptic-elliptic case of Meissner's equation.

Summary

— An example: the inverted pendulum

— Reversibility notions

— Reversibilities: the planar case

— Bifurcating from critical values

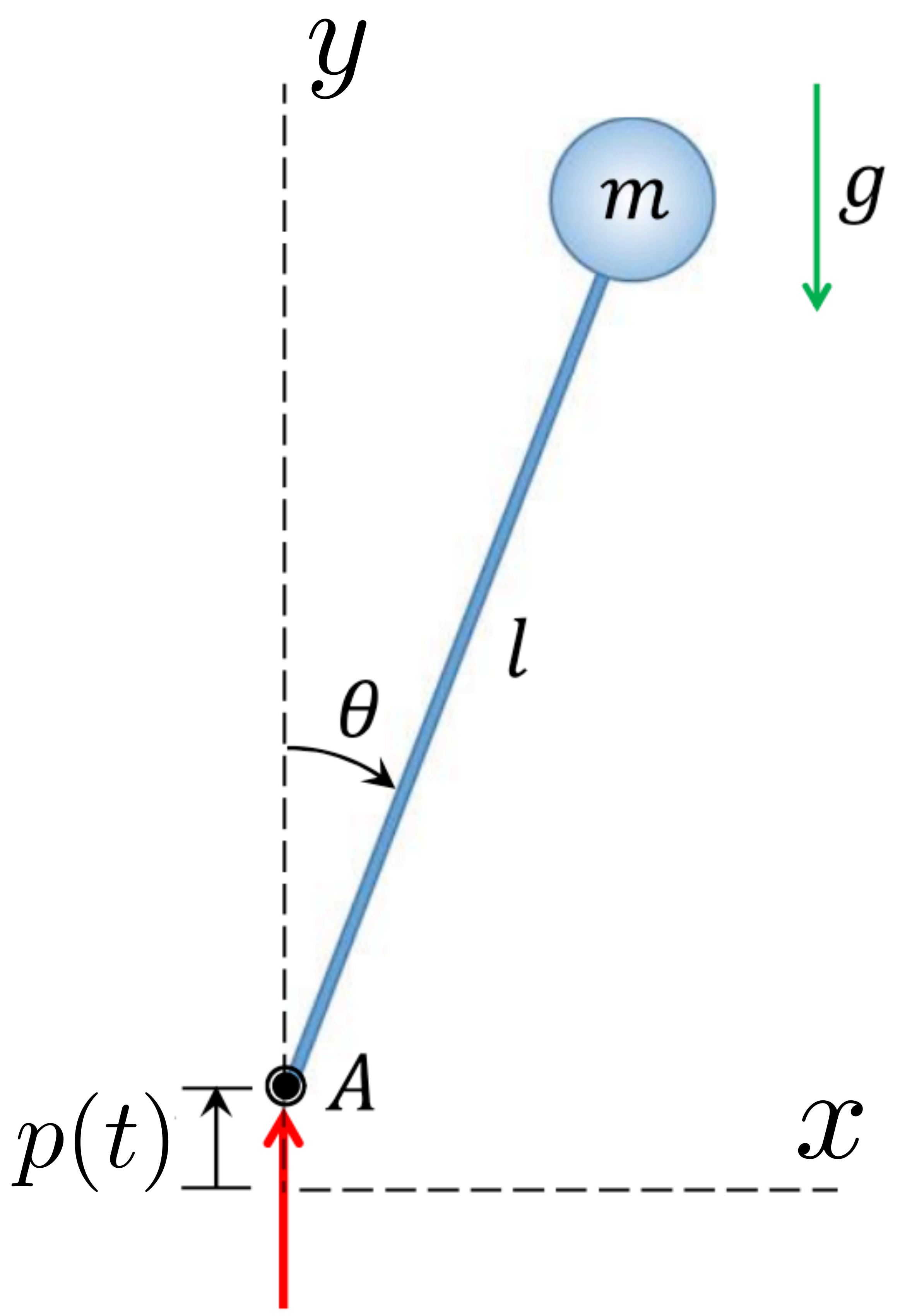
— Practical bifurcation analysis

— Main result for reversible Hill's equation

— Application to Meissner's equation

— Conclusion

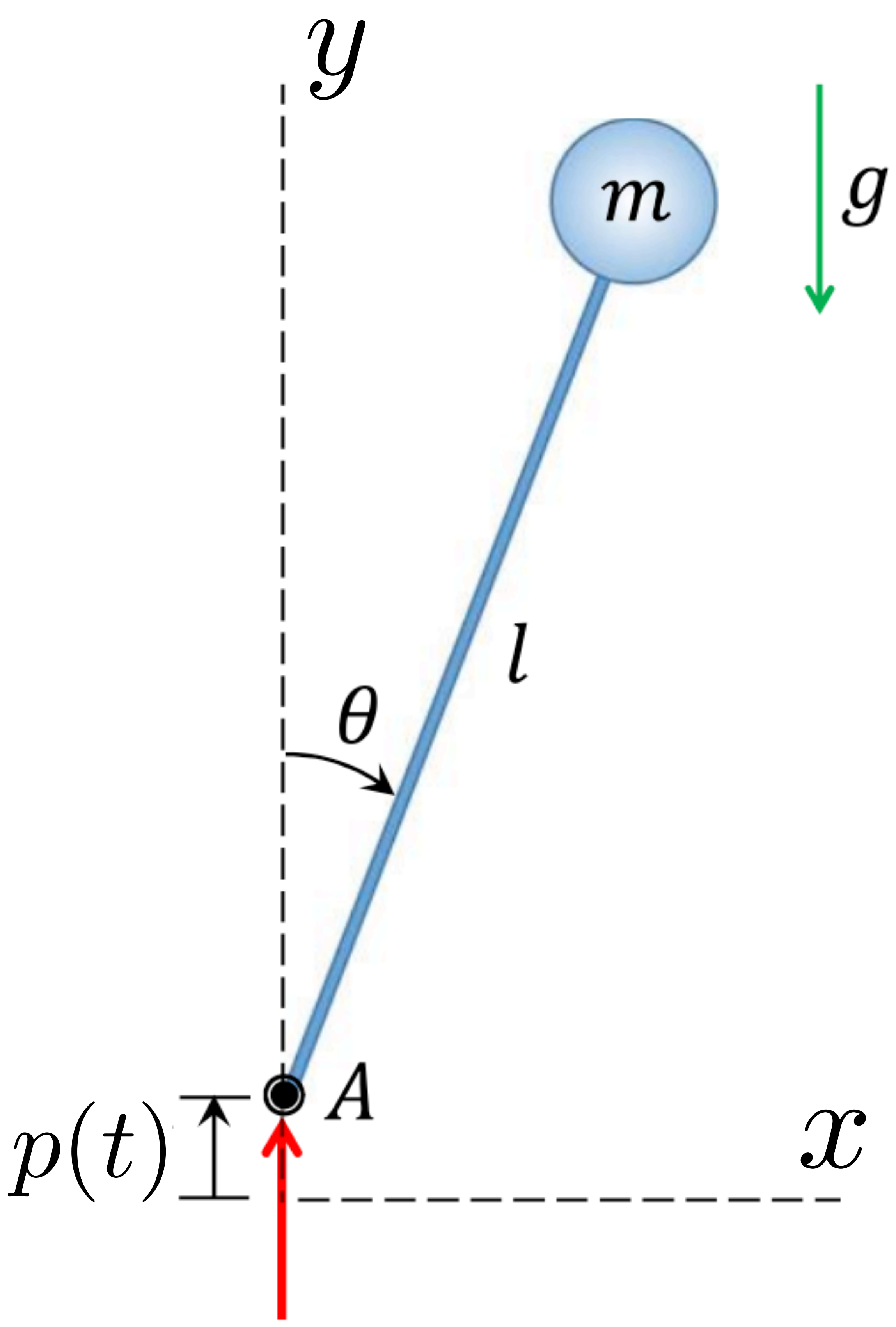
A motivating example: the inverted pendulum



The stabilizing effect of vibrations has been known as early as 1908 through Stephenson's experimental demonstration of stability of an inverted pendulum with the vertically oscillating suspension.

Somewhat later, in 1928, van der Pol and Strutt have drawn the stability diagram for the Mathieu's equation. This diagram implies, among other facts, Stephenson's result.

A motivating example: the inverted pendulum

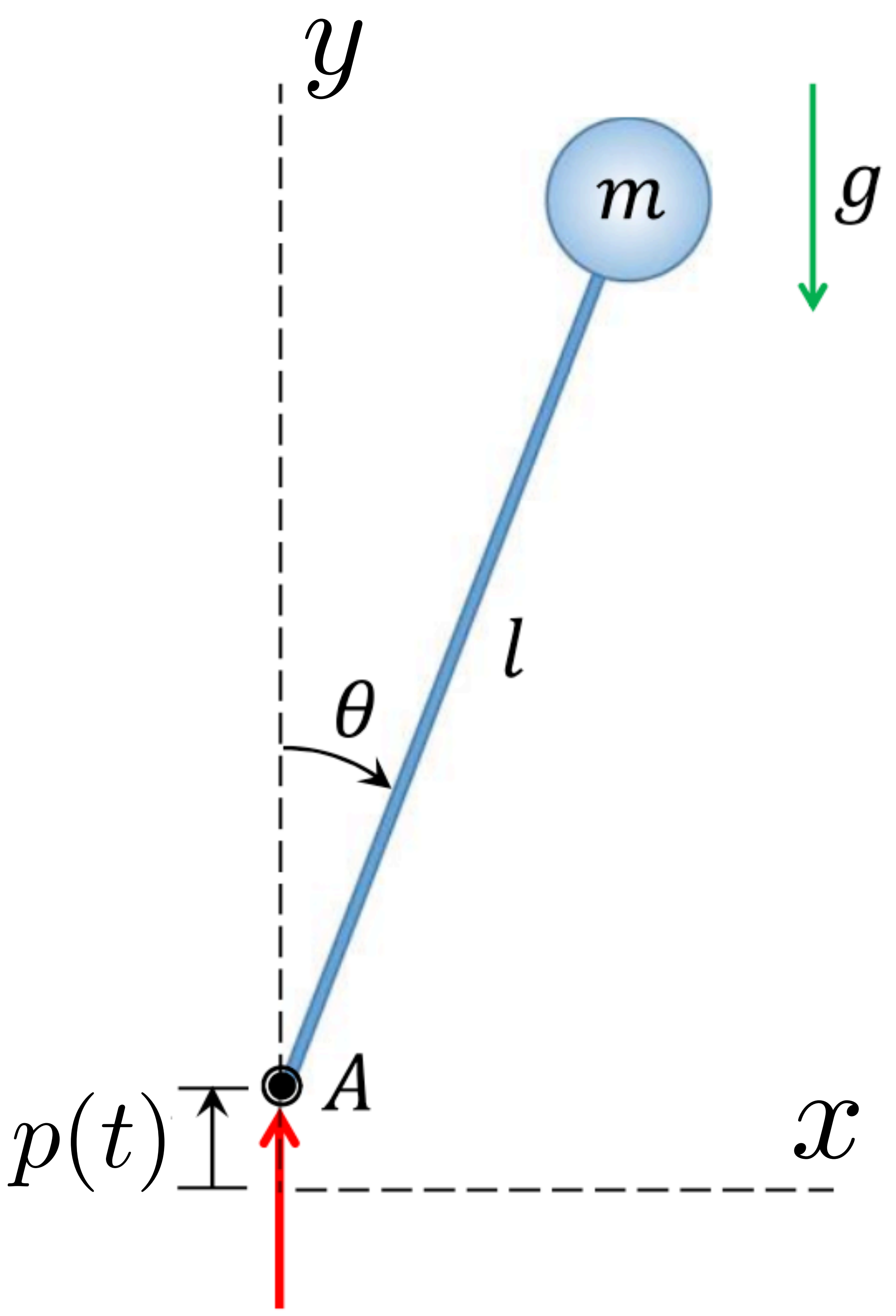


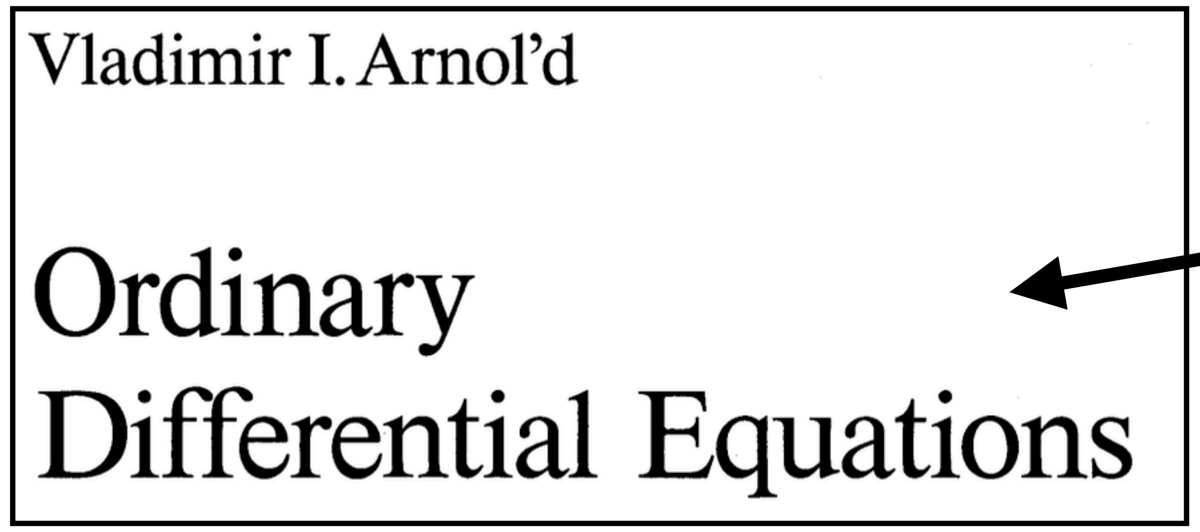
In 1951, Kapitsa studied stability of the inverted pendulum through effective potentials and had also suggested to apply vibrational stabilization to mechanical objects other than pendula, such as large molecules.

The related idea of levitating charged particles via an oscillating electric field (the 'Paul trap') goes back to 1958; for this work Paul was awarded the Nobel Prize in 1989.

Quotation after [**L**]Levi, M. (1999). Geometry and physics of averaging with applications, *Physica D*, **132**: 150–164.

A motivating example: the inverted pendulum





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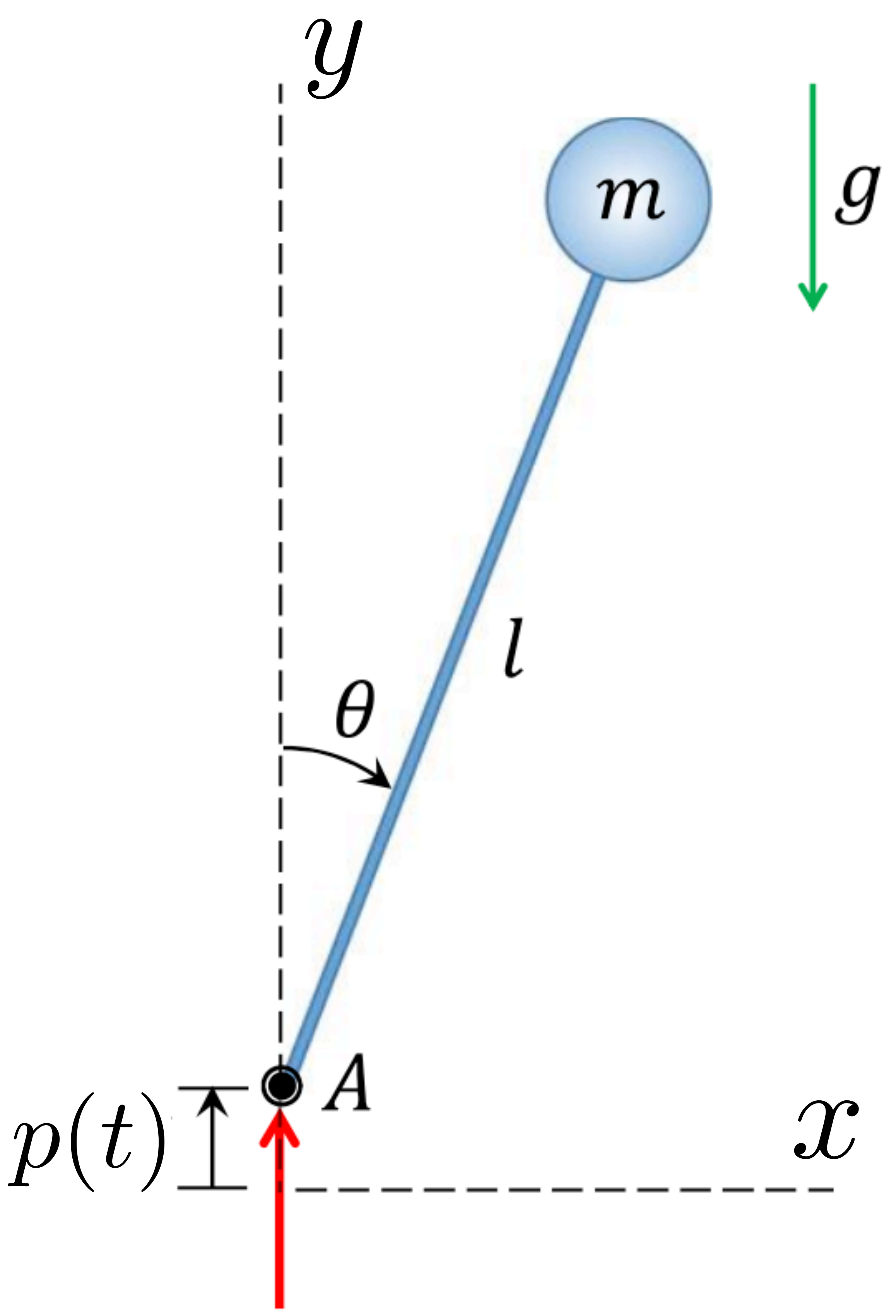
A motivating example: the inverted pendulum



- The suspension point undergoes a periodic oscillation $p(t)$, with period $T = 2\pi/\omega$.
- From $x = l\sin\theta$, and $y = p(t) + l\cos\theta$, we get $\dot{x} = l\dot{\theta}\cos\theta$, and $\dot{y} = \dot{p}(t) - l\dot{\theta}\sin\theta$.

Therefore, $E_{\mathrm{POT}} = mg\,(p(t) + l\cos\theta)$, while

$$E_{\mathrm{KIN}} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(l^2\dot{\theta}^2 - 2\dot{p}(t)l\dot{\theta}\sin\theta + \dot{p}(t)^2\right).$$

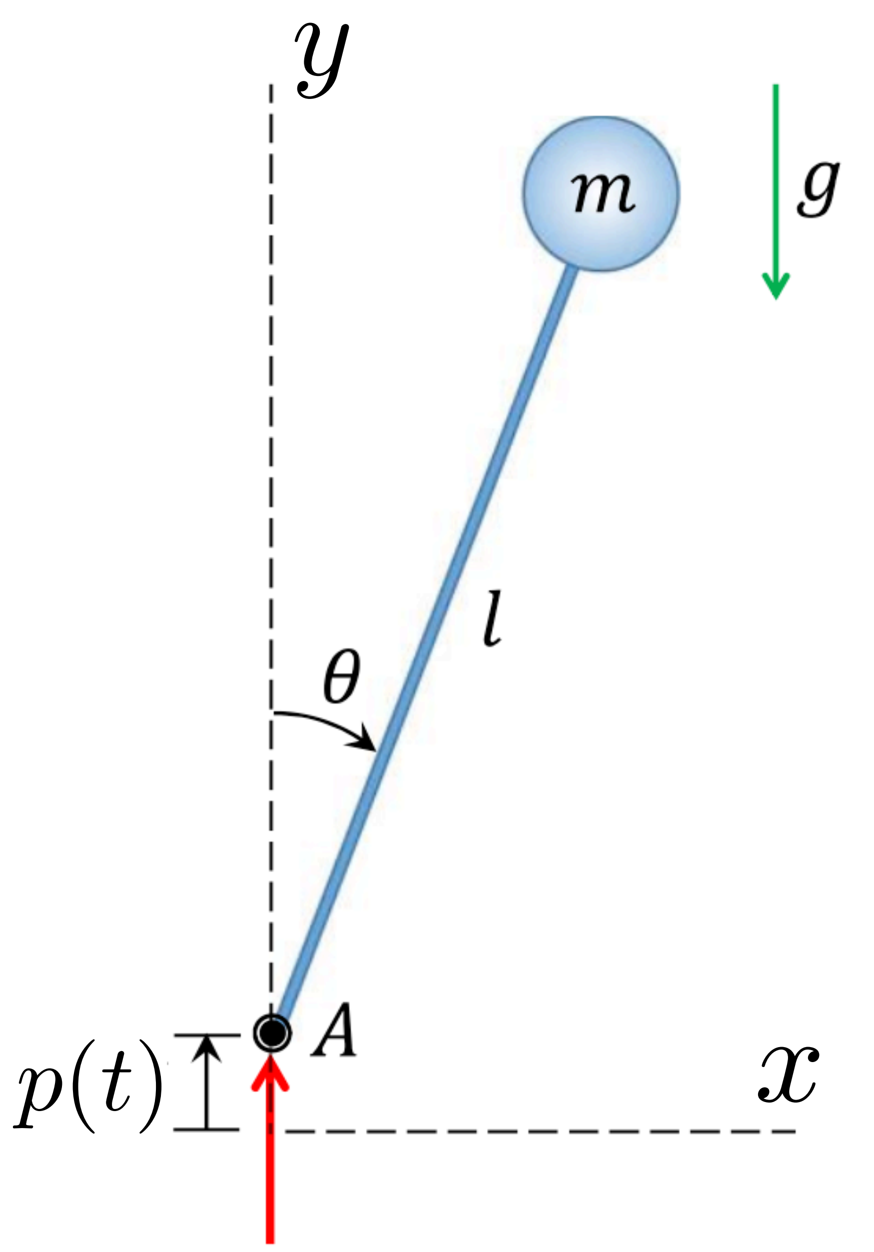
Taking a new time $\tau = \omega t$, so that $(')$ denotes the new derivative, (after dividing by $m\omega^2 l^2$) the normalized Lagrangian becomes

$$L = \frac{1}{2}\left((\theta')^2 - 2\frac{p'(\tau)}{l}\theta'\sin\theta + \frac{p'(\tau)^2}{l^2}\right) - \frac{g}{\omega^2 l}\left(\frac{p(\tau)}{l} + \cos\theta\right)$$

A motivating example: the inverted pendulum

From the Lagrangian $L = \frac{1}{2}\left((\theta')^2 - 2\frac{p'(\tau)}{l}\theta' \sin\theta + \frac{p'(\tau)^2}{l^2}\right) - \Omega^2\left(\frac{p(\tau)}{l} + \cos\theta\right)$,

with $\boxed{\Omega^2 = \frac{g}{\omega^2 l}}$, we get $\boxed{\frac{\partial L}{\partial \theta} = -\frac{p'(\tau)}{l}\theta' \cos\theta + \Omega^2 \sin\theta}$, and $\boxed{\frac{\partial L}{\partial \theta'} = \theta' - \frac{p'(\tau)}{l}\sin\theta.}$



The Euler-Lagrange equation of motion $\frac{d}{d\tau}\left(\frac{\partial L}{\partial \theta'}\right) = \frac{\partial L}{\partial \theta}$ becomes

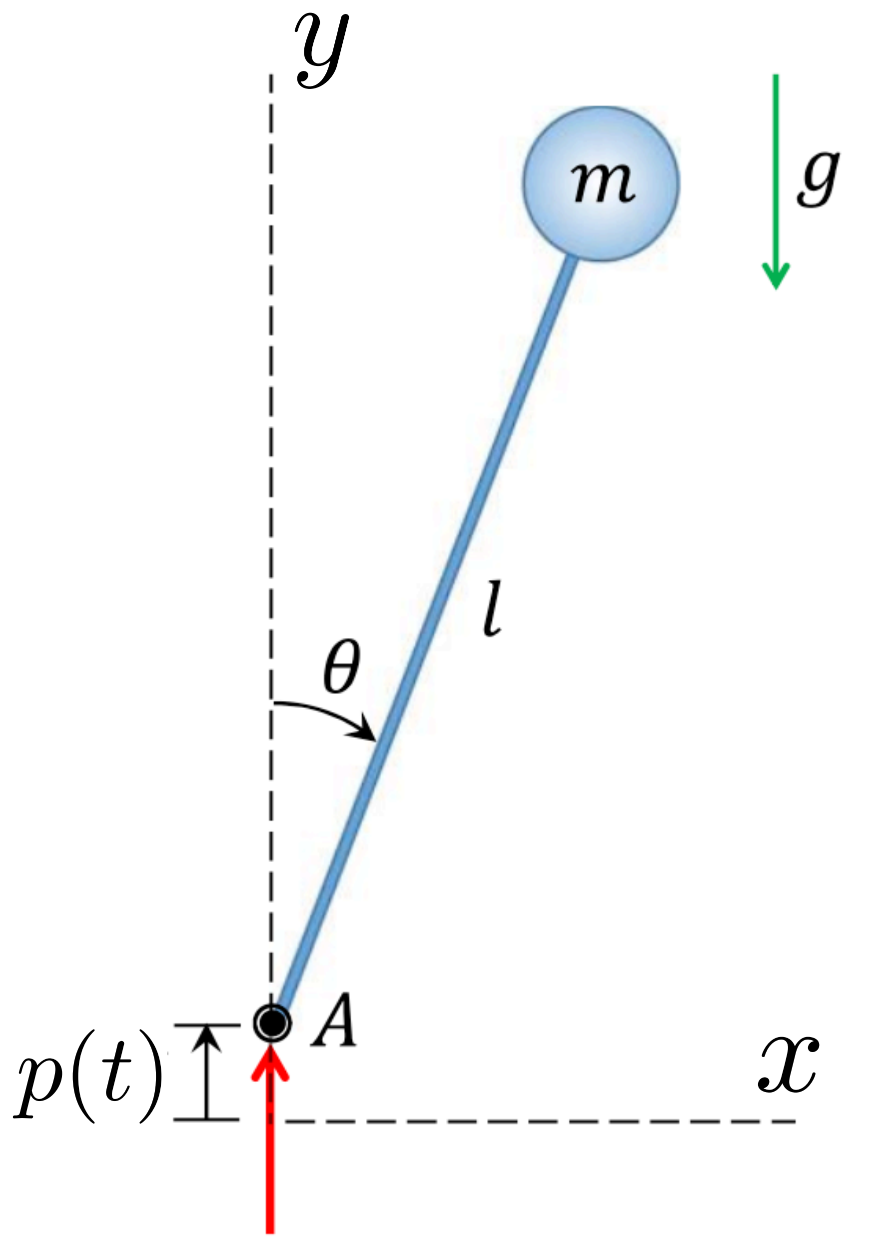
$$\theta'' - \frac{p''(\tau)}{l}\sin\theta - \frac{p'(\tau)}{l}\theta' \cos\theta = -\frac{p'(\tau)}{l}\theta' \cos\theta + \Omega^2 \sin\theta,$$

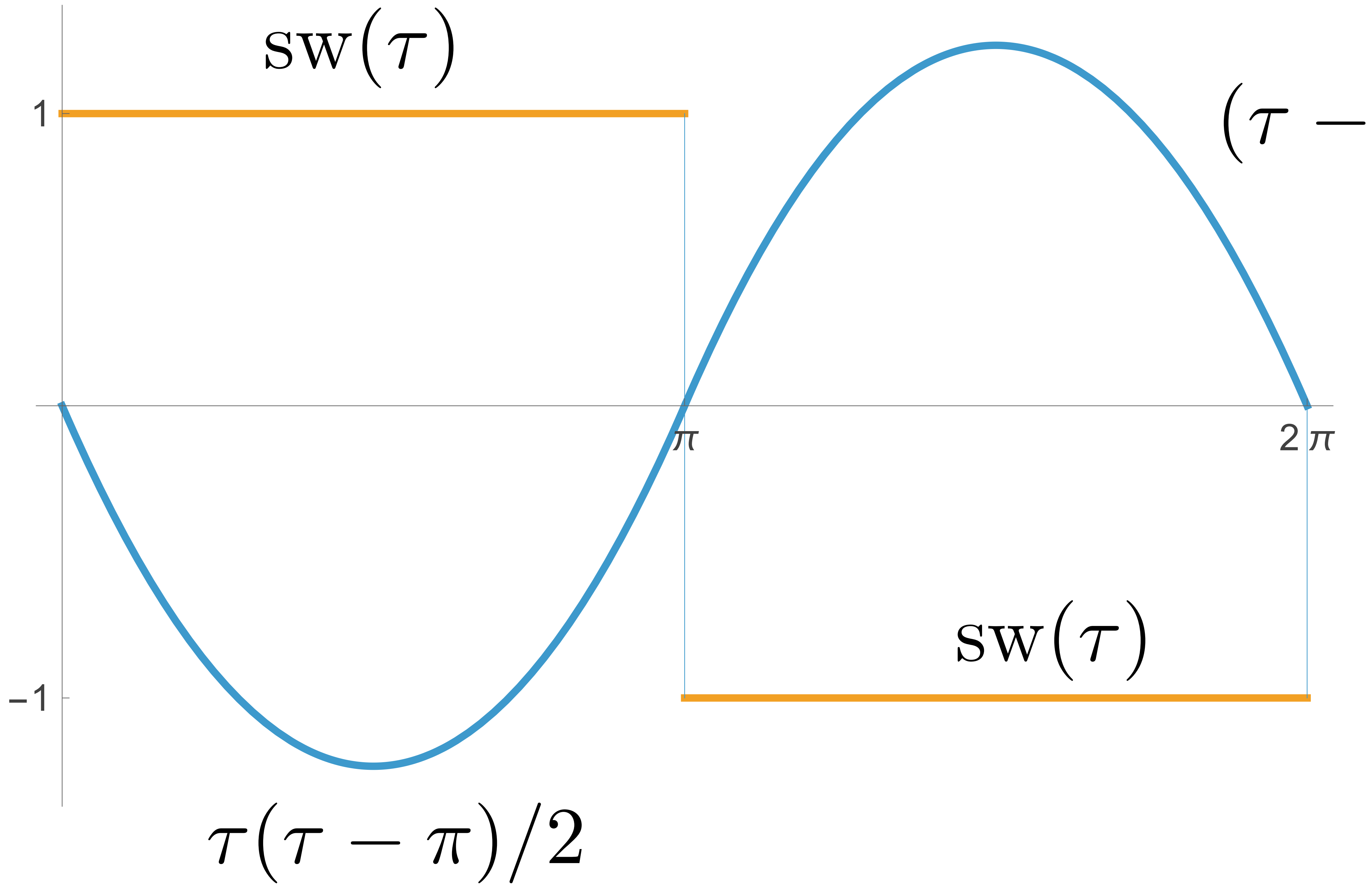
that is

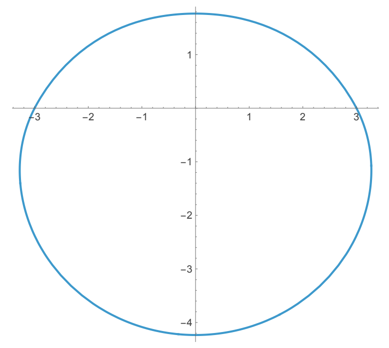
$$\boxed{\theta'' - \left(\Omega^2 + \frac{p''(\tau)}{l}\right)\sin\theta = 0.}$$

A motivating example: the inverted pendulum

We assume small oscillations, so that $\sin\theta \approx \theta$, and for the 2π-periodic function $p(\tau)$ we take the convention $p''(\tau) = A\,\mathrm{sw}(\tau)$, where ‘sw’ stands for a unitary square wave.







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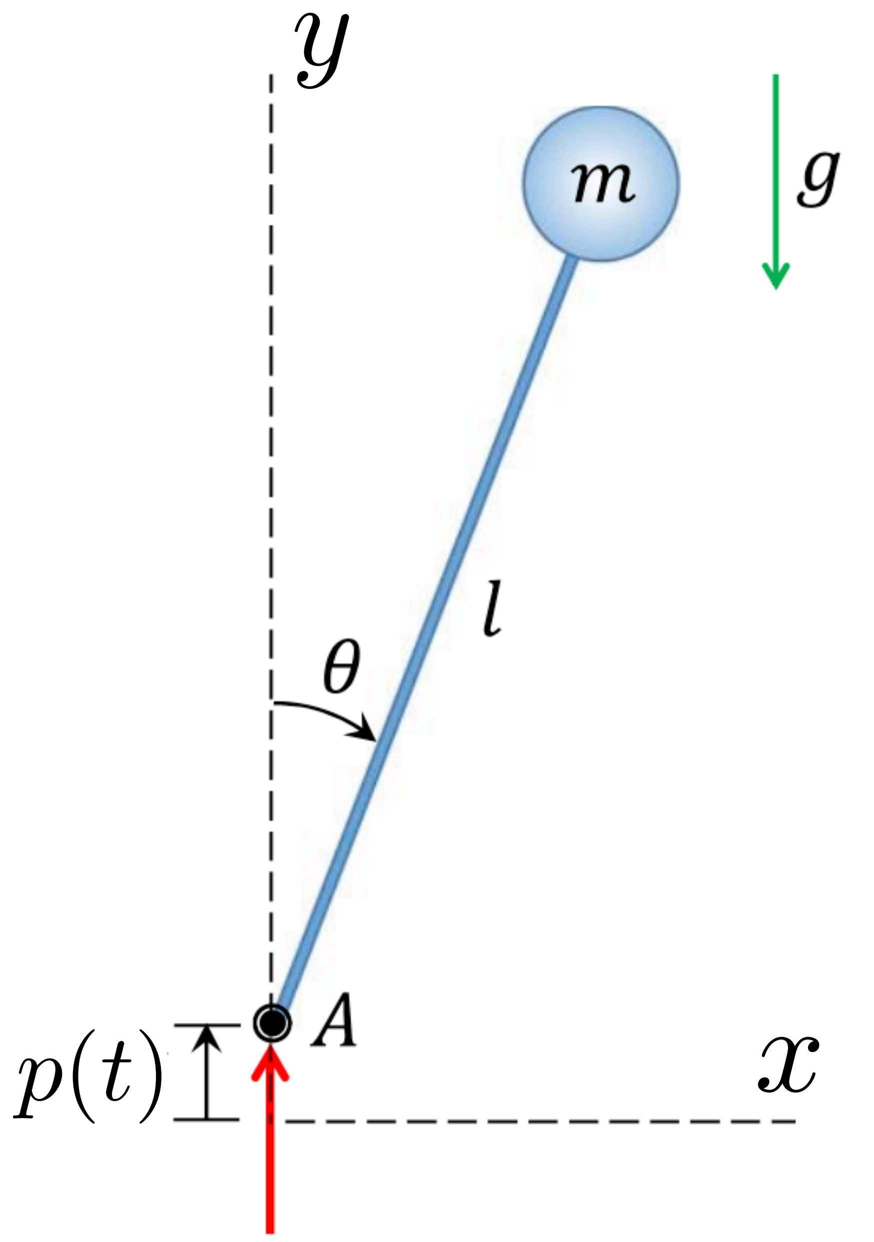

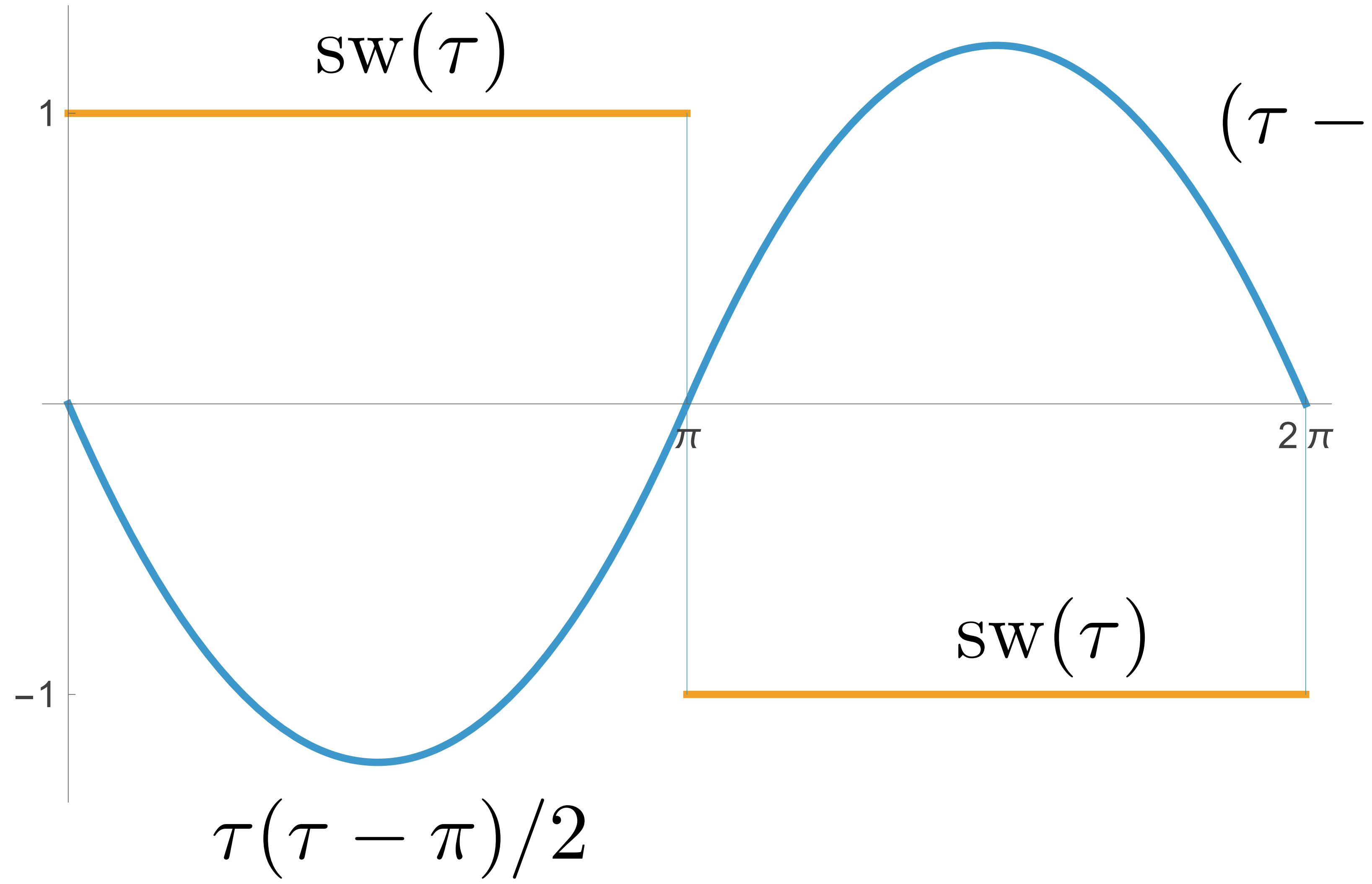

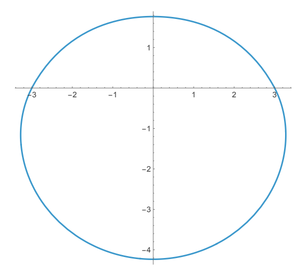

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We get the Hill's equation

$$\theta'' - \left(\Omega^2 + a\,\mathrm{sw}(\tau)\right)\theta = 0, \qquad \text{with } a = \frac{A}{l}.$$

A motivating example: the inverted pendulum

We take advantage of the piecewise constant character of the restoring force term to solve the equation. Assuming $a \geqslant \Omega^2$, and defining $\delta^2 = a + \Omega^2$, and $\epsilon^2 = a - \Omega^2$, we have

$$\theta'' - \delta^2\theta = 0, \text{ for } 0 \leqslant \tau \leqslant \pi,$$

and

$$\theta'' + \epsilon^2\theta = 0, \text{ for } \pi \leqslant \tau \leqslant 2\pi.$$

Remark Note that, by definition, we have $0 \leqslant \epsilon \leqslant \delta$, and $\delta^2 + \epsilon^2 = 2a$, while $\delta^2 - \epsilon^2 = 2\Omega^2$.

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$$\frac{d}{d\tau}\begin{pmatrix}\theta \\ \theta'\end{pmatrix} = \begin{pmatrix}0 & 1 \\ \delta^2 & 0\end{pmatrix}\begin{pmatrix}\theta \\ \theta'\end{pmatrix}, \quad \text{for } 0 \leqslant \tau \leqslant \pi,$$

and

$$\frac{d}{d\tau}\begin{pmatrix}\theta \\ \theta'\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -\epsilon^2 & 0\end{pmatrix}\begin{pmatrix}\theta \\ \theta'\end{pmatrix}, \quad \text{for } \pi \leqslant \tau \leqslant 2\pi.$$

(to be periodically extended for $\tau > 2\pi$)

A motivating example: the inverted pendulum

Reminder. For periodic linear systems in the form

$$\dot{x} = A(t)x, \text{ where } x(t) \in \mathbb{R}^n, \quad A(t+T) = A(t),$$

for all t and some $T > 0$, if $X(t)$ is a fundamental matrix of solutions with $X(0) = I$, being I the identity matrix, then $X(t+T)$ is also a fundamental matrix and there exists a non-singular matrix C such that $X(t+T) = X(t)C$ for all t. Therefore,

$$X(t+T) = X(t)X(T) \text{ and } X(nT) = X(T)^n, \text{ for any } n \in \mathbf{N}.$$

A motivating example: the inverted pendulum

Reminder. Thus, the *monodromy matrix* $M = X(T)$ contains all the information needed to characterize the whole dynamics. In particular, the eigenvalues of M (*characteristic multipliers* or *Floquet multipliers*) characterize the stability of solutions. Thus, if $\mu \in \mathbb{R}$ and $v \in \mathbb{R}^n$ verify $Mv = \mu v$, then for the solution $x(t) = X(t)v$, with $x(0) = v$,

$$x(t+T) = X(t+T)v = X(t)X(T)v = X(t)Mv = X(t)\mu v = \mu x(t).$$

• The existence of T-*periodic* solutions is associated to the existence of a multiplier $\mu = 1$.
• If $\mu = -1$ is an eigenvalue of M, then we have $2T$-periodic solutions.
• Periodic solutions are not isolated (linearity + homogeneity).

A motivating example: the inverted pendulum

Here, exceptionally, we can compute the *principal fundamental matrix of solutions* $X(\tau)$, and the corresponding monodromy matrix $M = X(2\pi)$, by writing

$$M = \exp\left[\begin{pmatrix} 0 & 1 \\ -\epsilon^2 & 0 \end{pmatrix}\pi\right] \cdot \exp\left[\begin{pmatrix} 0 & 1 \\ \delta^2 & 0 \end{pmatrix}\pi\right]$$

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$$M = \begin{pmatrix} \cos\pi\epsilon & \epsilon^{-1}\sin\pi\epsilon \\ -\epsilon\sin\pi\epsilon & \cos\pi\epsilon \end{pmatrix} \cdot \begin{pmatrix} \cosh\pi\delta & \delta^{-1}\sinh\pi\delta \\ \delta\sinh\pi\delta & \cosh\pi\delta \end{pmatrix}$$

A motivating example: the inverted pendulum

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Clearly, $\det M = 1$. For the trace of M, we get

$$\mathrm{tr}(M) = 2 \cos \pi\epsilon \cosh \pi\delta + \left(\frac{\delta}{\epsilon} - \frac{\epsilon}{\delta} \right) \sin \pi\epsilon \sinh \pi\delta.$$

A motivating example: the inverted pendulum

The case $\epsilon = 0$ with $\delta > 0$ gives

$$M = \exp\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\pi\right] \cdot \exp\left[\begin{pmatrix} 0 & 1 \\ \delta^2 & 0 \end{pmatrix}\pi\right] = \begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cosh \pi\delta & \delta^{-1} \sinh \pi\delta \\ \delta \sinh \pi\delta & \cosh \pi\delta \end{pmatrix},$$

where

$$\mathrm{tr}(M) = 2\cosh \pi\delta + \pi\delta \sinh \pi\delta > 2,$$

while for the singular point $\epsilon = \delta = 0$ we have $\mathrm{tr}(M) = 2$.

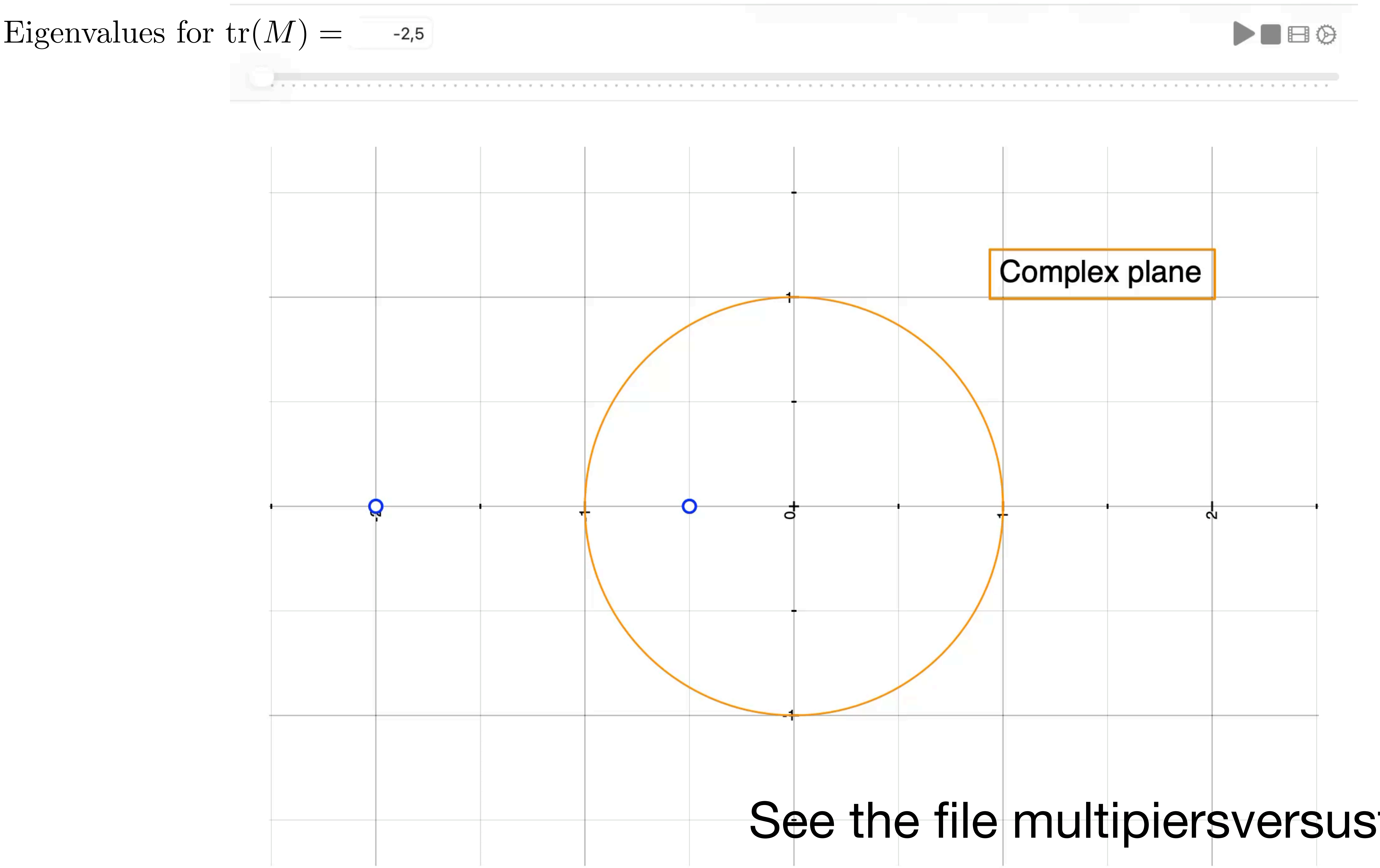
A motivating example: the inverted pendulum

The stability of the upper position $\theta = 0$ can be deduced from the Floquet multipliers, that is, from the eigenvalues μ_1 and μ_2 of the monodromy matrix M. Since these multipliers are the roots of the quadratic

$$\mu^2 - \mathrm{tr}(M)\mu + 1 = 0,$$

we need the condition $|\,\mathrm{tr}(M)| \leqslant 2$, and we emphasize that if $|\,\mathrm{tr}(M)| = 2$ the matrix should be diagonalizable.

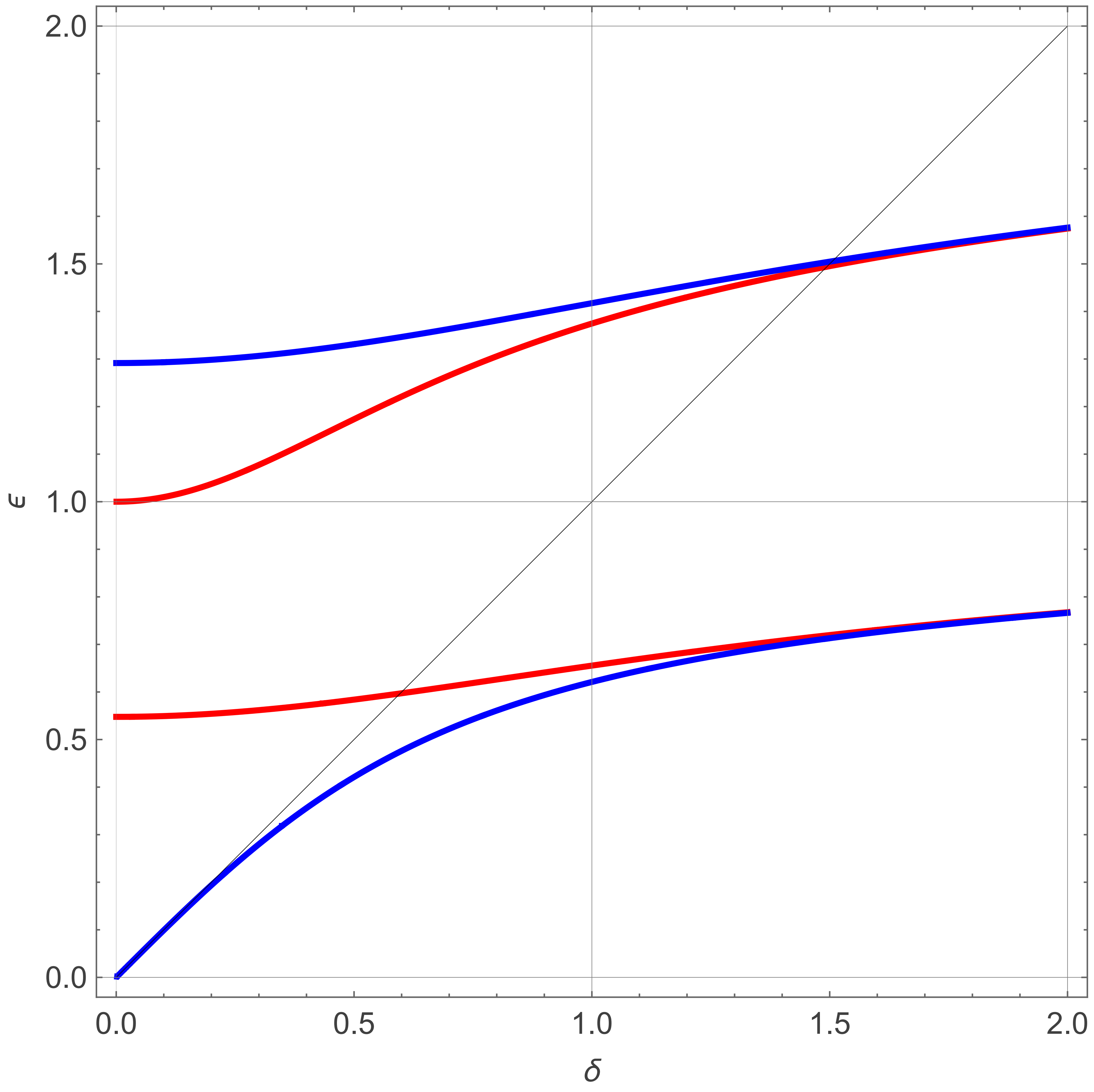
A motivating example: the inverted pendulum



See the file multipiersversustrace.mp4

A motivating example: the inverted pendulum

In looking for parametric regions of stability, we must draw in the positive quadrant of plane (δ, ϵ) the curves corresponding to $\mathrm{tr}(M) = 2$ (in blue), and to $\mathrm{tr}(M) = -2$ (in red), which act as bifurcation curves.



A motivating example: the inverted pendulum

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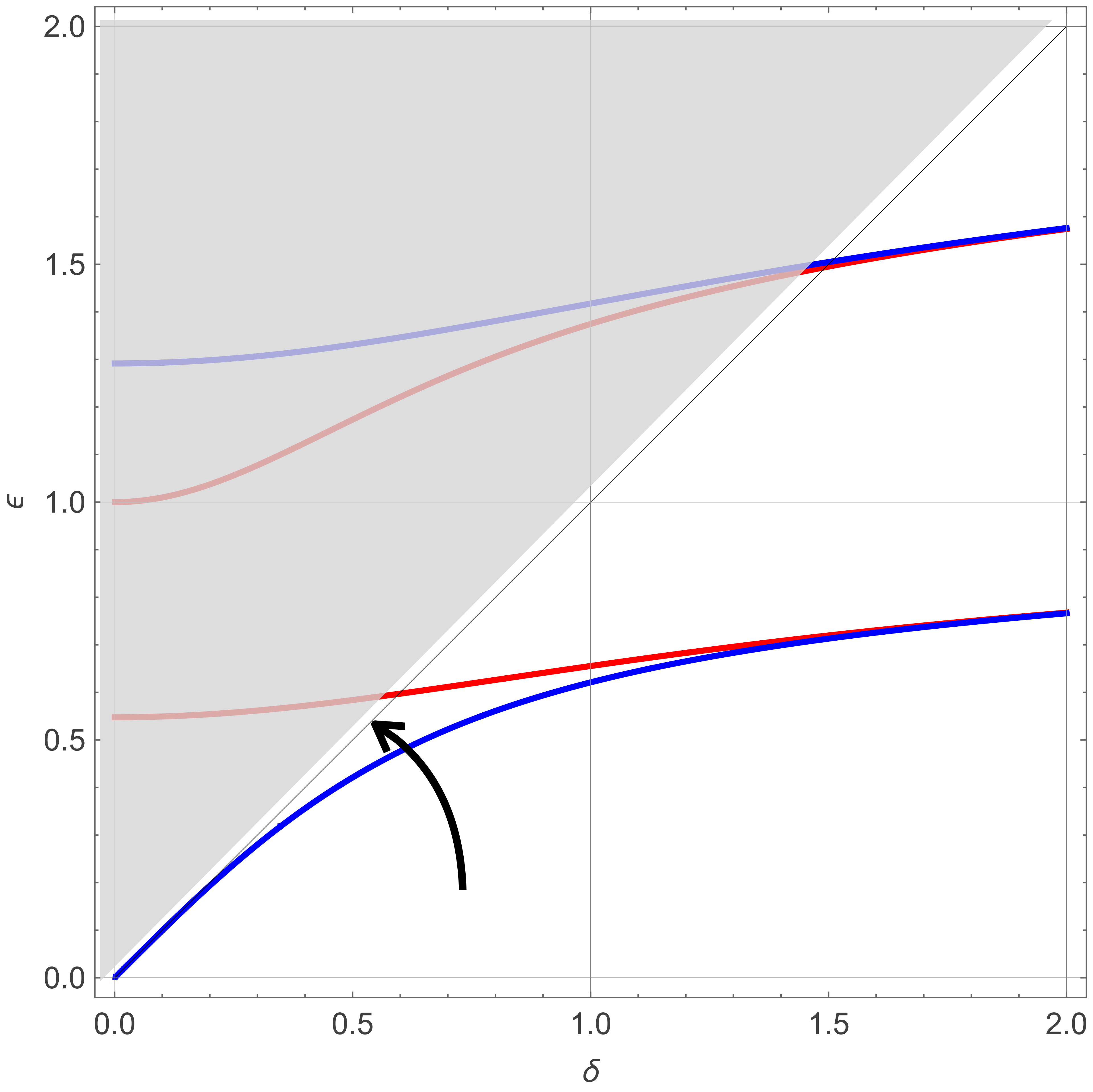
Recall:

$$a = \frac{A}{l}, \quad \Omega^2 = \frac{g}{\omega^2 l},$$
$$\epsilon^2 = a - \Omega^2, \quad \delta^2 = a + \Omega^2.$$

Our physical setting requires $0 \leqslant \epsilon \leqslant \delta$. Keeping constant the amplitude A and increasing ω, we follow the arc $\delta^2 + \epsilon^2 = 2a$.



A motivating example: the inverted pendulum

We can do a local study near the origin in the plane (δ, ϵ) for the curve $\mathrm{tr}(M) = 2$. From the expression

$$\mathrm{tr}(M) = 2\cos\pi\epsilon\cosh\pi\delta + \left(\frac{\delta}{\epsilon} - \frac{\epsilon}{\delta}\right)\sin\pi\epsilon\sinh\pi\delta,$$

we see that $\mathrm{tr}(M) = 2 \iff (\delta^2 - \epsilon^2)\sin\pi\epsilon\sinh\pi\delta = 2\delta\epsilon(1 - \cos\pi\epsilon\cosh\pi\delta)$.

After some standard simplifications, and using the implicit function theorem, we get

$$\delta^2 - \epsilon^2 = \frac{\pi^2}{6}\epsilon^4 + \cdots \iff \Omega^2 = \frac{\pi^2}{12}a^2 + \cdots \iff \frac{g}{\omega^2 l} = \frac{\pi^2}{12}\frac{A^2}{l^2} + \cdots$$

leading to the practical condition $\boxed{\omega A > \frac{2}{\pi}\sqrt{3gl}.}$

A motivating example: the inverted pendulum

See the file video_vibrated_pendulum.mp4

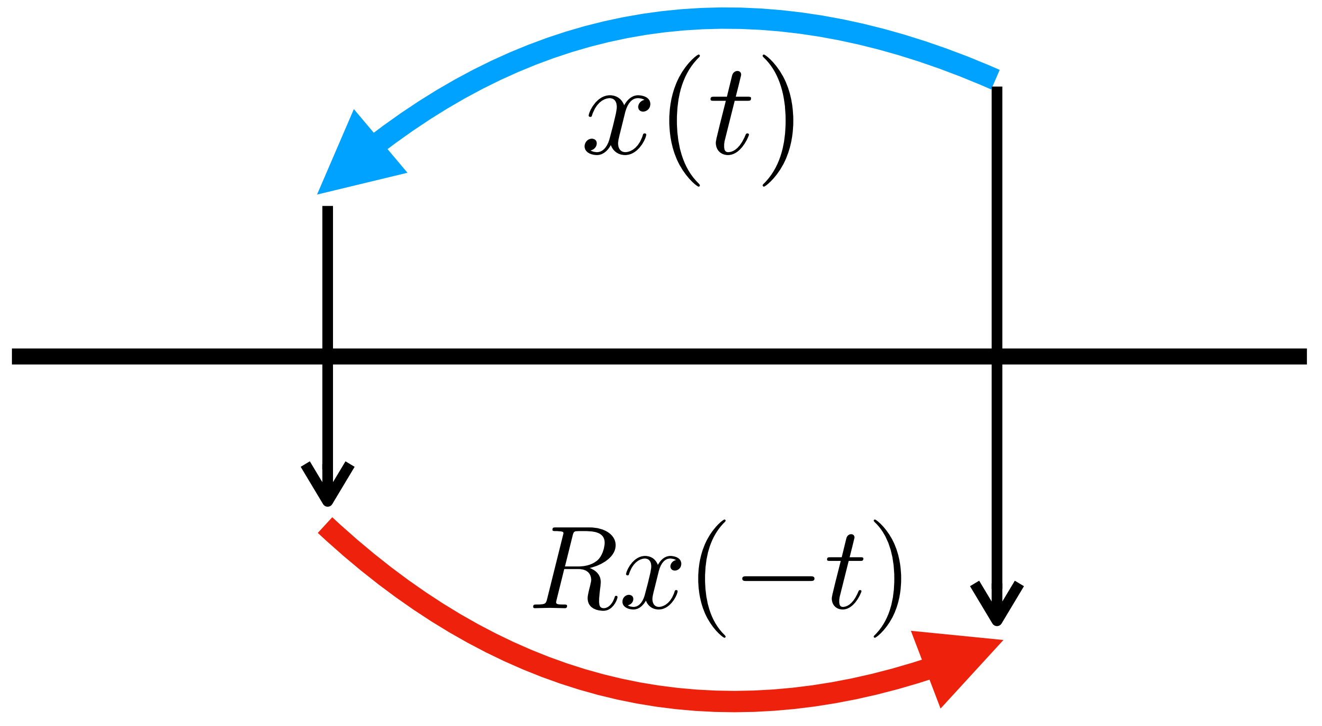
Courtesy of my colleague Daniel J. Pagano,
Autamação e Sistemas, UFSC Florianópolis.

Reversibility notions

Restricting our attention to linear *reversing symmetries*, we say that the autonomous dynamical system $\dot{x} = F(x)$, where $F : \mathbb{R}^n \to \mathbb{R}^n$, is R-reversible, with R a real non-singular constant matrix, if

$$F(Rx) = -RF(x), \text{ for all } x \in \mathbb{R}^n.$$

It is easy to deduce that whenever $x(t)$ is a solution of the system, then $Rx(-t)$ is also a solution: 'the future is determined by the past of an alternative present'.



Reversibility notions

Typically, R is an involution, that is, $R^2 = I$ with $R \neq \pm I$.

Example. Consider the differential system

$$\begin{array}{rcl} \dot{x} & = & -y, \\ \dot{y} & = & x - y^2, \end{array} \quad \text{and } R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Reversibility notions

The above notions can be extended to non-automous systems $\dot{x} = F(x, t)$, where $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$, by accepting some freedom in the time origin for reversal.

Definition. Introducing, for a certain $t_0 \in \mathbb{R}$ and a given matrix R, the map $R_{t_0} : (x, t) \to (Rx, t_0 - t)$, we say that the above non-autonomous system is R_{t_0}-*reversible* if it is invariant under the transformation R_{t_0}, that is

$$RF(x, t) = -F(Rx, t_0 - t), \text{ for all } (x, t).$$

Reversibility notions

As suggested in [LR], taking a new time $\tau = t - t_0/2$, for the translated vector field $\widetilde{F}(x,\tau) = F(x, \tau + t_0/2)$, the extended (autonomous) dynamical system

$$\frac{d(x,\tau)}{d\tau} = \left(\widetilde{F}(x,\tau), 1\right) = \left(F\left(x, \tau + \frac{t_0}{2}\right), 1\right),$$

becomes R_0-reversible, where $R_0 : (x,\tau) \to (Rx, -\tau)$. Effectively, for $R\widetilde{F}(x,\tau)$ we can write

$$RF\left(x, \tau + \frac{t_0}{2}\right) = -F\left(Rx, t_0 - \tau - \frac{t_0}{2}\right) = -F\left(Rx, \frac{t_0}{2} - \tau\right) = -\widetilde{F}(Rx, -\tau).$$

[LR] Lamb J.S.W., Roberts J.A.G. (1998), Time-reversal symmetry in dinamical systems: a survey, *Physica D*, **112**, 1–39.

Reversibility notions

In the context of T-periodic linear systems, assume that for the system

$$\dot{x} = A(t)x, \quad A(t+T) = A(t) \text{ for all } t,$$

where $x \in \mathbb{R}^n$, we know that, for a given involution R and a certain value t_0, for all t the property

$$RA(t) = -A(t_0 - t)R = -A(-(t - t_0))R$$

holds. If we define the new time $\tau = t - t_0/2$ then $t = \tau + t_0/2$, and we see that

$$RA(\tau + t_0/2) = -A(t_0 - \tau - t_0/2)R = -A(-\tau + t_0/2))R.$$

In short, by defining $\tilde{A}(\tau) = A(\tau + t_0/2)$ the reversibility property simplifies to

$$R\tilde{A}(\tau) = -\tilde{A}(-\tau)R.$$

Reversibility notions

Suppose that $X(t)$ with $X(0) = I$ is a fundamental matrix of solutions for the original system, that is $\dot{X}(t) = A(t)X(t)$. Then we conclude that $X(\tau + t_0/2)$ is a solution matrix of the translated system

$$\dot{\tilde{x}} = \tilde{A}(\tau)\tilde{x}.$$

For the fundamental matrix of solutions $\widetilde{X}(\tau)$, with $\widetilde{X}(0) = I$ and $\dot{\widetilde{X}}(\tau) = \tilde{A}(\tau)\widetilde{X}(\tau)$, we can write $X(\tau + t_0/2) = \widetilde{X}(\tau)X(t_0/2)$, so that, putting $\tau = T$, we get

$$X(T + t_0/2) = \boxed{X(t_0/2)X(T) = \widetilde{X}(T)X(t_0/2)},$$

that is, the two monodromy matrices $M = X(T)$ and $\widetilde{M} = \widetilde{X}(T)$ are similar.

The matrix $\widetilde{M} = \widetilde{X}(T)$ can be more suitable than $M = X(T)$, as shown later.

Reversibility notions

In the sequel, we assume that, after a translation in time if needed, we have a reversible, T-periodic linear system

$$\dot{x} = A(t)x, \quad A(t+T) = A(t), \quad RA(t) = -A(-t)R, \text{ for all } t, \text{ with } R^2 = I.$$

Proposition The following statements hold.

(a) Whenever $x(t)$ is a solution of the system, then $Rx(-t)$ is also a solution.

(b) If $M = X(T)$ is the monodromy matrix, then $R = MRM$, both products RM and MR are involutions, and so $\det(RM) = \det(MR) = \pm 1$.

(c) The monodromy matrix M is unimodal, that is, $\det M = 1$.

(d) The matrix M is ruled by $X(T/2)$, that is, $M = RX(T/2)^{-1}RX(T/2)$.

Reversibility notions

Proof (a) First, we note that by the change of time variable $t \to -t$ we also have $RA(-t) = -A(t)R$. Take now $y(t) = Rx(-t)$, where $x(t)$ is a given solution, that is, $\dot{x}(t) = A(t)x(t)$. By using again the change $t \to -t$, we also have $\dot{x}(-t) = A(-t)x(-t)$. Taking time derivative on $y(t)$, we get

$$\dot{y}(t) = -R\dot{x}(-t) = -RA(-t)x(-t) = A(t)Rx(-t) = A(t)y(t),$$

and so $y(t)$ is also a solution. The statement follows.

Reversibility notions

(b) Since the matrix $RX(-t)$ has solutions in its columns by the reversibility, there exists a nonsingular matrix C such that $RX(-t) = X(t)C$. Using that $X(0) = I$, we get $C = R$, and so $RX(-t) = X(t)R$ for all t, and in particular

$$RX(-T) = X(T)R.$$

By putting $t = -T$ in the equality $X(t+T) = X(t)X(T)$, we see that $I = X(-T)X(T)$. Therefore, by right-multiplying by $X(T)$ we get

$$R = X(T)RX(T).$$

Left-multiplying by R the last equality, we see that $I = RX(T)RX(T)$ and the statement follows. Note that $I = X(T)RX(T)R$ is also true.

Reversibility notions

(c) We know that both $\det(RX(T)) = \pm 1$ and $\det R = \pm 1$, being involutions. However,

$$\det(X(T)) = \exp\left(\int_0^T \operatorname{trace} A(t)\, dt\right) > 0,$$

and the statement follows.

(d) By putting $t = -T/2$ in the equality $X(t+T) = X(t)X(T)$, we get $X(T/2) = X(-T/2)X(T)$, and so $RX(T/2) = RX(-T/2)X(T)$. But from $RX(-t) = X(t)R$ we also have $RX(-T/2) = X(T/2)R$, and so

$$RX(T/2) = X(T/2)RX(T).$$

The conclusion follows by multiplying by $RX(T/2)^{-1}$, and the proof is complete.

Reversibility notions

Remark. Given a involution R, for invertible maps $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ it is usual to say that it is an R-reversible map whenever $\Pi = R\,\Pi^{-1}R$. In such a case, we see that the equalities

$$R\,\Pi = \Pi^{-1}R, \quad \Pi R = R\,\Pi^{-1}, \quad \Pi R\,\Pi = R, \quad R\,\Pi R\,\Pi = I, \quad \text{and } \Pi R\,\Pi R = I,$$

are equivalent. In this sense, statement (b) above says that the monodromy matrix $M = X(T)$ is an R-reversible map.

Reversibility notions

Lemma (Reversibility and changes of variables) Given a system that admits a time-reversal symmetry R, after any invertible linear change of variables of the form $y = Px$, the new system has a reversibility given by the matrix $S = P^{-1} R\, P$.

Proof. From the assumption, we have $RA(t) = -A(-t)R$. After the change of variables, the system becomes $\dot{y} = B(t)y$ with $B(t) = P^{-1}A(t)P$. Now, we see that

$$SB(t) = \left(P^{-1}R\, P\right)\left(P^{-1}A(t)P\right) = -P^{-1}A(-t)RP = -P^{-1}A(-t)PP^{-1}RP = -B(-t)S,$$

as required. The conclusion follows.

Reversibilities: the planar case

Lemma (Canonical reversibility)
Any proper involution in $\mathbb{R}^2$ is similar to $R = \operatorname{diag}(1, -1)$.

Proof. A direct computation shows that if $R^2 = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}^2 = \begin{pmatrix} r_{11}^2 + r_{12}r_{21} & r_{12}(r_{11} + r_{22}) \\ r_{21}(r_{11} + r_{22}) & r_{22}^2 + r_{12}r_{21} \end{pmatrix}$

must be equal to the identity matrix, we need $r_{11}^2 + r_{12}r_{21} = r_{22}^2 + r_{12}r_{21} = 1$, so that $r_{11}^2 = r_{22}^2$, and $r_{12}(r_{11} + r_{22}) = r_{21}(r_{11} + r_{22}) = 0$. Two cases appear.
If $r_{11} + r_{22} \neq 0$ then $r_{12} = r_{21} = 0$, and so $r_{11}^2 = r_{22}^2 = 1$, concluding that $r_{11} = r_{22} = \pm 1$; then the original matrix is plus or minus the identity matrix, not a proper involution.
When $r_{11} + r_{22} = 0$, we can write that

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} \\ (1 - r_{11}^2)/r_{12} & -r_{11} \end{pmatrix}, \text{ when } r_{12} \neq 0; \text{ otherwise, } \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ r_{21} & \mp 1 \end{pmatrix}.$$

Since in these two cases the matrix has null trace and determinant equal to -1, it is similar to the matrix $R = \operatorname{diag}(1, -1)$, as stated.

Reversibilities: the planar case

Proposition (Reversibility induced structure in the monodromy matrix).

Given a planar T-periodic linear system $\dot{x} = A(t)x$ and the matrix $R = \mathrm{diag}(1, -1)$, the following statements hold.

(a) The system is R-reversible if and only if for the functions in the entries of matrix $A(t)$, given by

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix},$$

both $a_{11}(t)$ and $a_{22}(t)$ are odd, while $a_{12}(t)$ and $a_{21}(t)$ are even.

Reversibilities: the planar case

Proposition (Reversibility induced structure in the monodromy matrix).

Given a planar T-periodic linear system $\dot{x} = A(t)x$ and the matrix $R = \mathrm{diag}(1, -1)$, the following statements hold.

(b) If the system is R-reversible, then the monodromy matrix $M = X(T)$ has the structure

$$M = \begin{pmatrix} a_M & b_M \\ c_M & a_M \end{pmatrix}, \text{ with } \det(M) = a_M^2 - b_M c_M = 1.$$

Reversibilities: the planar case

Proof. **(a)** The equality $RA(t) = -A(-t)R$ for all $t \in \mathbb{R}$ is equivalent to

$$\begin{pmatrix} a_{11}(t) & a_{12}(t) \\ -a_{21}(t) & -a_{22}(t) \end{pmatrix} = \begin{pmatrix} -a_{11}(-t) & a_{12}(-t) \\ -a_{21}(-t) & a_{22}(-t) \end{pmatrix}$$

for all t, so that statement (a) follows.
(b) It suffices to impose for a generic $M = X(T)$ the equality $R = MRM$, namely

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a_M & b_M \\ c_M & d_M \end{pmatrix} \begin{pmatrix} a_M & b_M \\ -c_M & -d_M \end{pmatrix} = \begin{pmatrix} a_M^2 - b_M c_M & b_M(a_M - d_M) \\ c_M(a_M - d_M) & b_M c_M - d_M^2 \end{pmatrix}.$$

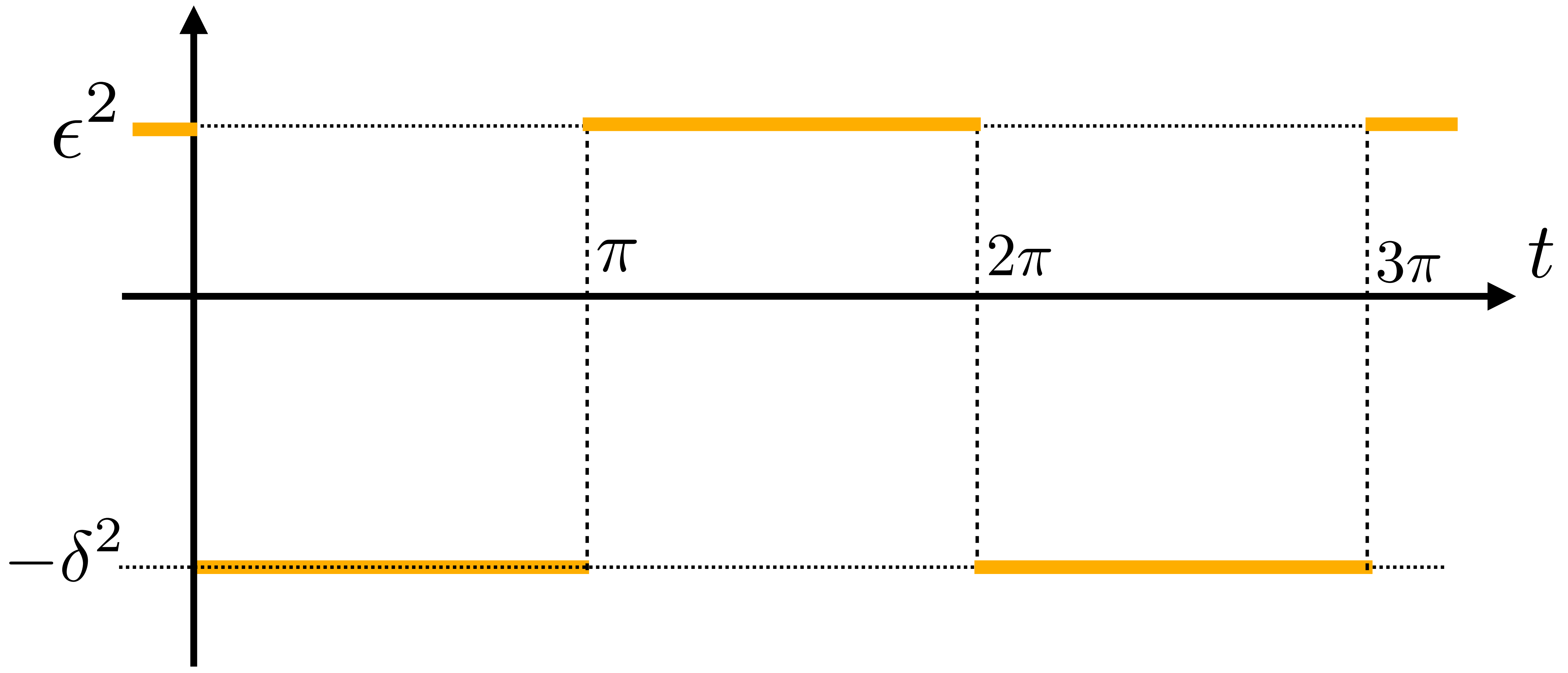
If we suppose $a_M \neq d_M$, then we have $b_M = c_M = 0$, and so $a_M^2 = d_M^2 = 1$, leading to $a_M = -d_M = \pm 1$. Hence, $a_M d_M = -1$, but then $\det M \neq 1$. Thus, $a_M = d_M$.

Reversibilities: the planar case

Coming back to our inverted pendulum example, using t for the time and taking $x_1 = \theta$, $x_2 = \theta'$, we were dealing with the 2π-periodic Hill's equation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -p(t) & 0 \end{pmatrix} x, \quad p(t) = \begin{cases} -\delta^2, & 0 \leq t \leq \pi, \\ \epsilon^2, & \pi \leq t \leq 2\pi, \end{cases}$$

which, is R_π-reversible, as $p(t) = p(\pi - t)$.



Reversibilities: the planar case

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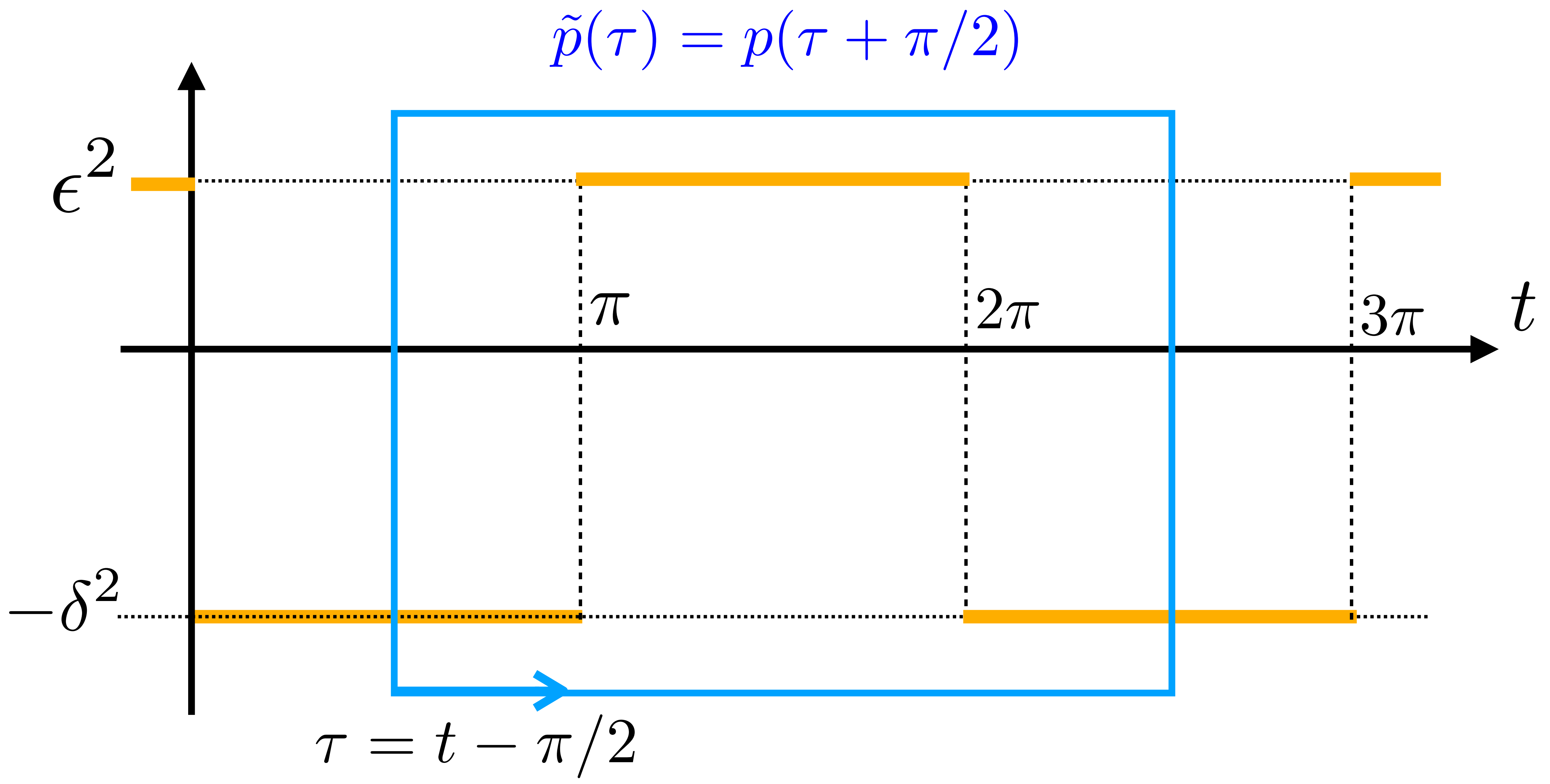
Defining $\tilde{p}(\tau) = p(\tau + \pi/2)$, we get

$$\boxed{\tilde{p}(\tau) = \tilde{p}(-\tau)}$$



Reversibilities: the planar case

From $M = \begin{pmatrix} \cos \pi\epsilon & \epsilon^{-1} \sin \pi\epsilon \\ -\epsilon \sin \pi\epsilon & \cos \pi\epsilon \end{pmatrix} \cdot \begin{pmatrix} \cosh \pi\delta & \delta^{-1} \sinh \pi\delta \\ \delta \sinh \pi\delta & \cosh \pi\delta \end{pmatrix}$

Reversibilities: the planar case

From $$M = \begin{pmatrix} \cos\pi\epsilon\cosh\pi\delta + \dfrac{\delta}{\epsilon}\sin\pi\epsilon\sinh\pi\delta & \delta^{-1}\cos\pi\epsilon\sinh\pi\delta + \epsilon^{-1}\sin\pi\epsilon\cosh\pi\delta \\ \delta\cos\pi\epsilon\sinh\pi\delta - \epsilon\sin\pi\epsilon\cosh\pi\delta & \cos\pi\epsilon\cosh\pi\delta - \dfrac{\epsilon}{\delta}\sin\pi\epsilon\sinh\pi\delta \end{pmatrix}$$

Reversibilities: the planar case

From $M = \begin{pmatrix} \cos\pi\epsilon\cosh\pi\delta + \dfrac{\delta}{\epsilon}\sin\pi\epsilon\sinh\pi\delta & \delta^{-1}\cos\pi\epsilon\sinh\pi\delta + \epsilon^{-1}\sin\pi\epsilon\cosh\pi\delta \\ \delta\cos\pi\epsilon\sinh\pi\delta - \epsilon\sin\pi\epsilon\cosh\pi\delta & \cos\pi\epsilon\cosh\pi\delta - \dfrac{\epsilon}{\delta}\sin\pi\epsilon\sinh\pi\delta \end{pmatrix}$

we can pass, for the translated in time, R-reversible system, to

$$\widetilde{M} = \exp\left[\begin{pmatrix} 0 & 1 \\ \delta^2 & 0 \end{pmatrix}\frac{\pi}{2}\right]\cdot\exp\left[\begin{pmatrix} 0 & 1 \\ -\epsilon^2 & 0 \end{pmatrix}\pi\right]\cdot\exp\left[\begin{pmatrix} 0 & 1 \\ \delta^2 & 0 \end{pmatrix}\frac{\pi}{2}\right]$$

Reversibilities: the planar case

From $$M=\begin{pmatrix} \cos\pi\epsilon\cosh\pi\delta+\dfrac{\delta}{\epsilon}\sin\pi\epsilon\sinh\pi\delta & \delta^{-1}\cos\pi\epsilon\sinh\pi\delta+\epsilon^{-1}\sin\pi\epsilon\cosh\pi\delta \\ \delta\cos\pi\epsilon\sinh\pi\delta-\epsilon\sin\pi\epsilon\cosh\pi\delta & \cos\pi\epsilon\cosh\pi\delta-\dfrac{\epsilon}{\delta}\sin\pi\epsilon\sinh\pi\delta \end{pmatrix}$$

we can pass, for the translated in time, R-reversible system, to

$$\widetilde{M}=\begin{pmatrix} \cos\pi\epsilon\cosh\pi\delta+\dfrac{\delta^2-\epsilon^2}{2\delta\epsilon}\sin\pi\epsilon\sinh\pi\delta & \dfrac{\cos\pi\epsilon\sinh\pi\delta}{\delta}+\dfrac{\delta^2+\epsilon^2+(\delta^2-\epsilon^2)\cosh\pi\delta}{2\delta^2\epsilon}\sin\pi\epsilon \\ \delta\cos\pi\epsilon\sinh\pi\delta-\dfrac{\delta^2+\epsilon^2+(\delta^2-\epsilon^2)\cosh\pi\delta}{2\epsilon}\sin\pi\epsilon & \cos\pi\epsilon\cosh\pi\delta+\dfrac{\delta^2-\epsilon^2}{2\delta\epsilon}\sin\pi\epsilon\sinh\pi\delta \end{pmatrix}$$

showing the structure predicted by the above proposition.

Most of the classical examples are R-reversible

Another example of reversible periodic system comes from the Ince-Hill equation, namely

$$(1 + a\cos\omega t)\ddot{x} + (b\sin\omega t)\dot{x} + (c + d\cos\omega t)x = 0.$$

Effectively, after introducing the variable $y = \dot{x}$, we can rewrite the equation as the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{c + d\cos\omega t}{1 + a\cos\omega t} & -\dfrac{b\sin\omega t}{1 + a\cos\omega t} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where we assume $|a| < 1$. We note that we have got a R-reversible $(2\pi/\omega)$-periodic system, according to statement (a) of the above proposition.

Bifurcating from critical values

Our goal is to get, for reversible planar systems, a deeper insight about the parametric stability boundaries given by the conditions $\text{trace}(M) = \pm 2$.
From statement (b) of above proposition, since the diagonal entries of M are equal, the condition $\text{trace}(M) = 2$ corresponds to $a_M = 1$, and there appear the three cases

$$M = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & b_M \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ c_M & 1 \end{pmatrix},$$

where $b_M \neq 0$ and $c_M \neq 0$. Similarly, if $\text{trace}(M) = -2$, then we have the cases

$$M = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} -1 & b_M \\ 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} -1 & 0 \\ c_M & -1 \end{pmatrix}.$$

Stable

Unstable

Bifurcating from critical values

We want to characterize, in terms of the parameters, the geometric structure of the bifurcation sets (i.e. the stability boundaries) in a neighborhood of their stable situations ($M = \pm I$).

Since the three parameters defining M evolve on the surface $a_M^2 - b_M c_M = 1$, it seems natural to study the possible expansions of the monodromy matrix $M(\varepsilon)$ assuming for its entries (a_M, b_M, c_M) some dependence on a bidimensional parameter $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$, so that $M(0) = \pm I$.

We will denote the first derivatives of $M(\varepsilon)$ by

$$M_i(\varepsilon) = \frac{\partial M(\varepsilon)}{\partial \varepsilon_i}, \quad M_{ij}(\varepsilon) = \frac{\partial^2 M(\varepsilon)}{\partial \varepsilon_i \partial \varepsilon_j}, \quad i, j = 1, 2.$$

Bifurcating from critical values

Lemma (Parametric derivatives of the monodromy matrix entries) Let us consider the monodromy matrix $M(\varepsilon)$ of a T-periodic linear system that admits the time-reversal symmetry $R = \mathrm{diag}(1, -1)$. Assuming for the entries of M a dependence $(a_M(\varepsilon), b_M(\varepsilon), c_M(\varepsilon))$ on $\varepsilon = (\varepsilon_1, \varepsilon_2)$ with $M(0) = \pm I$, that is $(a_M(0), b_M(0), c_M(0)) = (\pm 1, 0, 0)$, we have

$$M_i(0) = \frac{\partial M}{\partial \varepsilon_i}(0) = \begin{pmatrix} 0 & b_i \\ c_i & 0 \end{pmatrix},$$

where $a_i = \frac{\partial a_M}{\partial \varepsilon_i}(0) = 0$, $\quad b_i = \frac{\partial b_M}{\partial \varepsilon_i}(0)$, $\quad c_i = \frac{\partial c_M}{\partial \varepsilon_i}(0)$, and furthermore

$$a_{ii} = \frac{\partial^2 a_M}{\partial \varepsilon_i^2}(0) = \pm b_i c_i, \qquad a_{ij} = \frac{\partial^2 a_M}{\partial \varepsilon_i \partial \varepsilon_j}(0) = \pm \frac{1}{2}(b_i c_j + b_j c_i),$$

so that $\mathrm{trace}\, M_{ii}(0) = \pm 2 b_i c_i$, $\quad \mathrm{trace}\, M_{ij}(0) = \pm(b_i c_j + b_j c_i)$, for $i, j = 1, 2$.

Bifurcating from critical values

Starting from a parameter comfiguration for $M(\varepsilon)$ such that $M(0) = \pm I$, we can study the expansion

$$M(\varepsilon) = \pm I + \varepsilon_1 M_1(0) + \varepsilon_2 M_2(0) + \frac{1}{2}\left(\varepsilon_1^2 M_{11}(0) + 2\varepsilon_1\varepsilon_2 M_{12}(0) + \varepsilon_2^2 M_{22}(0)\right) + \cdots$$

and, in particular, we are specially interested in the function

$$\boxed{\Phi(\varepsilon) = \operatorname{trace} M(\varepsilon) \mp 2,}$$

so that we can identify the local stability boundaries by solving the equation $\Phi(\varepsilon) = 0$. From the above proposition, we know that the main diagonal entries both for $M_1(0)$ and $M_2(0)$ vanish, and so the most significative terms for the expansion of the function Φ are quadratic.

Bifurcating from critical values

Substituting the second order derivatives a_{ij} previously obtained, we get

$$\operatorname{trace}(M(\varepsilon)) = \pm 2 + a_{11}\varepsilon_1^2 + 2a_{12}\varepsilon_1\varepsilon_2 + a_{22}\varepsilon_2^2 + \cdots$$

and therefore

$$\Phi(\varepsilon) = \varepsilon^\top \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \varepsilon + \cdots = \pm\varepsilon^\top \begin{pmatrix} b_1c_1 & \frac{b_1c_2+b_2c_1}{2} \\ \frac{b_1c_2+b_2c_1}{2} & b_2c_2 \end{pmatrix} \varepsilon + \cdots$$

We are so impelled to study the character of the quadratic form defined by the symmetric matrix

$$Q = \begin{pmatrix} b_1c_1 & \frac{b_1c_2+c_1b_2}{2} \\ \frac{b_1c_2+c_1b_2}{2} & b_2c_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ b_1 & b_2 \end{pmatrix}.$$

Bifurcating from critical values

Proposition (Relevance of anti-diagonal entries) Whenever the gradient vectors $\nabla b_M(0) = (b_1, b_2)^\top$ and $\nabla c_M(0) = (c_1, c_2)^\top$ are linearly independent the matrix Q is indefinite, so that in the parameter plane $(\varepsilon_1, \varepsilon_2)$ the point $\varepsilon = 0$ is for $\Phi(\varepsilon)$ an isolated non-degenerate critical point of saddle type.
Otherwise, when $b_1 c_2 - b_2 c_1 = 0$, it is degenerate.

Proof. The conclusion follows easily from the equality

$$\det(Q) = -\frac{1}{4}(b_1 c_2 - b_2 c_1)^2 \leq 0.$$

Bifurcating from critical values

Proposition (Relevance of anti-diagonal entries) Whenever the gradient vectors $\nabla b_M(0) = (b_1, b_2)^\top$ and $\nabla c_M(0) = (c_1, c_2)^\top$ are linearly independent the matrix Q is indefinite, so that in the parameter plane $(\varepsilon_1, \varepsilon_2)$ the point $\varepsilon = 0$ is for $\Phi(\varepsilon)$ an isolated non-degenerate critical point of saddle type. Otherwise, when $b_1 c_2 - b_2 c_1 = 0$, it is degenerate.

Proof. The conclusion follows easily from the equality

$$\det(Q) = -\frac{1}{4}(b_1 c_2 - b_2 c_1)^2 \leq 0.$$

Moreover, the quadratic part factorizes as follows,

$$\varepsilon^\top Q\, \varepsilon = \frac{1}{2}\left(\nabla b_M(0)^\top \varepsilon, \nabla c_M(0)^\top \varepsilon\right) \begin{pmatrix} \nabla c_M(0)^\top \varepsilon \\ \nabla b_M(0)^\top \varepsilon \end{pmatrix} = \left(\nabla b_M(0)^\top \varepsilon\right) \cdot \left(\nabla c_M(0)^\top \varepsilon\right).$$

Bifurcating from critical values

Proposition (Relevance of anti-diagonal entries) Whenever the gradient vectors $\nabla b_M(0) = (b_1, b_2)^\top$ and $\nabla c_M(0) = (c_1, c_2)^\top$ are linearly independent the matrix Q is indefinite, so that in the parameter plane $(\varepsilon_1, \varepsilon_2)$ the point $\varepsilon = 0$ is for $\Phi(\varepsilon)$ an isolated non-degenerate critical point of saddle type. Otherwise, when $b_1 c_2 - b_2 c_1 = 0$, it is degenerate.

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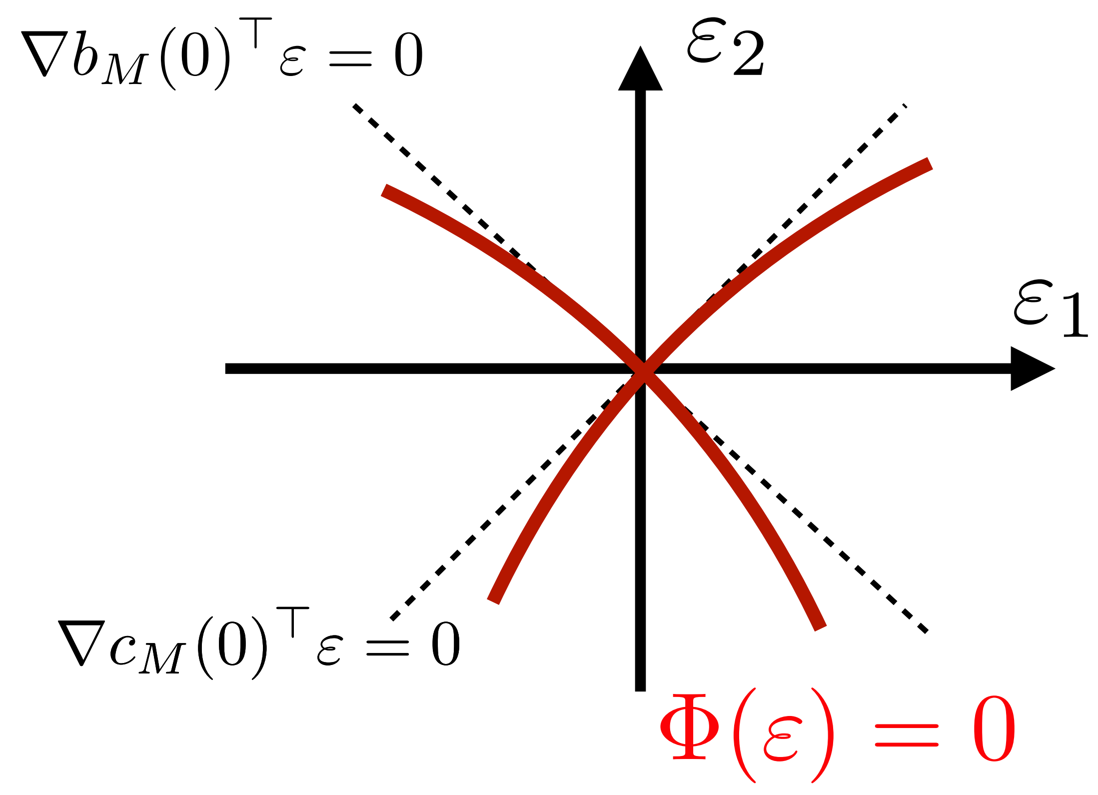
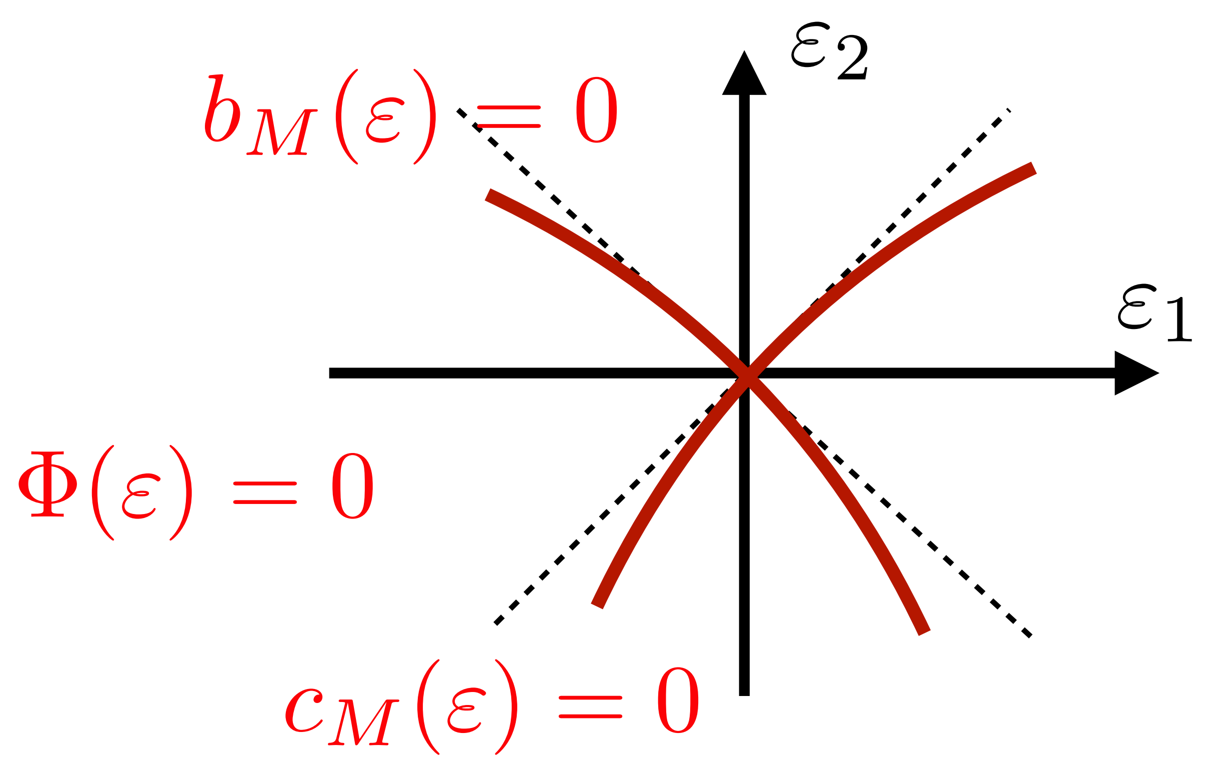


Moreover, the quadratic part factorizes as follows,

$$\varepsilon^\top Q\, \varepsilon = \frac{1}{2}\left(\nabla b_M(0)^\top \varepsilon, \nabla c_M(0)^\top \varepsilon\right) \begin{pmatrix} \nabla c_M(0)^\top \varepsilon \\ \nabla b_M(0)^\top \varepsilon \end{pmatrix} = \left(\nabla b_M(0)^\top \varepsilon\right) \cdot \left(\nabla c_M(0)^\top \varepsilon\right).$$

Bifurcating from critical values

Lemma (Anti-diagonal entries rule the curves where $\Phi(\varepsilon) = 0$)
The stability boundaries around any point where $M(0) = \pm I$ are determined by the curves corresponding to $b_M(\varepsilon) = 0$ and $c_M(\varepsilon) = 0$.



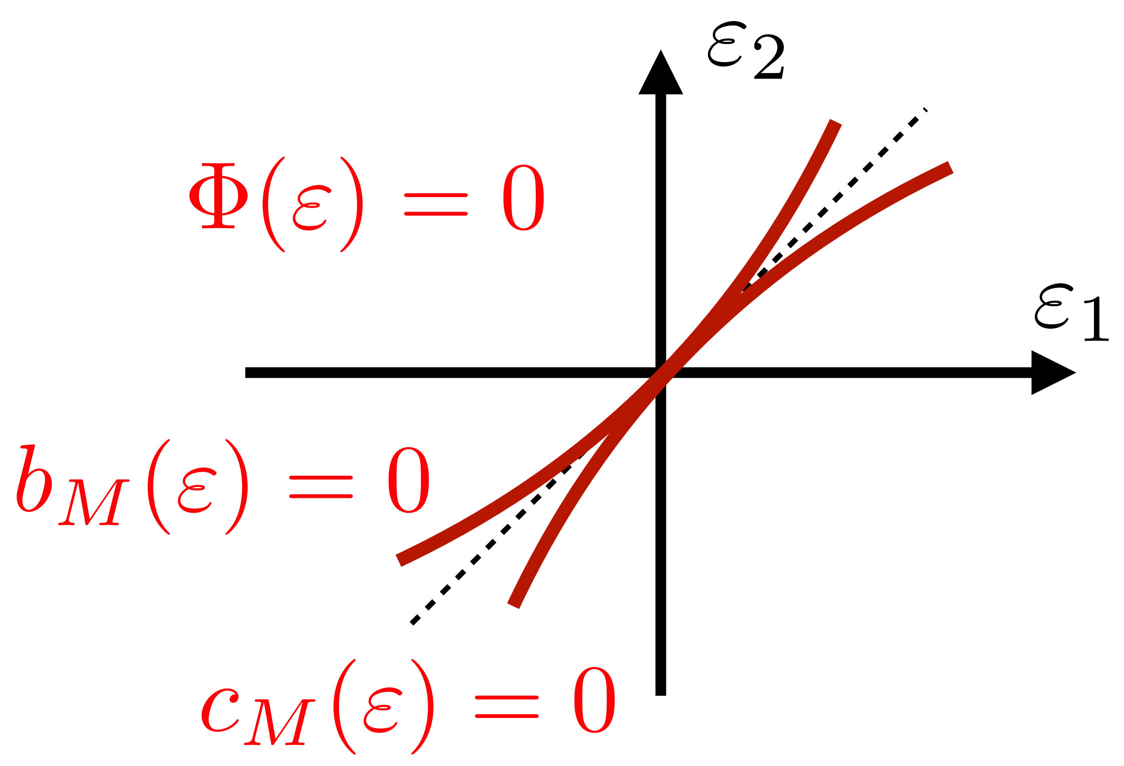
Proof. We have

$$0 = \Phi(\varepsilon) = \operatorname{trace} M(\varepsilon) \mp 2 = 2\,(a_M(\varepsilon) \mp 1) = 2\frac{a_M(\varepsilon)^2 - 1}{a_M(\varepsilon) \pm 1} = 2\frac{b_M(\varepsilon) \cdot c_M(\varepsilon)}{a_M(\varepsilon) \pm 1},$$

because we know that $a_M(\varepsilon)^2 - 1 = b_M(\varepsilon) \cdot c_M(\varepsilon)$, and $a_M(\varepsilon) \pm 1 \neq 0$.

Bifurcating from critical values

Proposition (The degenerate case, but not so much)
If the gradient vectors $\nabla b_M(0) = (b_1, b_2)^\top$ and $\nabla c_M(0) = (c_1, c_2)^\top$ are linearly dependent, but with $\nabla b_M(0) \neq (0,0)^\top$ and $\nabla c_M(0) \neq (0,0)^\top$, then in the parameter plane $(\varepsilon_1, \varepsilon_2)$, the point $\varepsilon = 0$ generically becomes for $\Phi(\varepsilon)$ an isolated degenerate critical point of type double cusp.



Bifurcating from critical values

Proof. Since $b_1 c_2 = b_2 c_1$, without loss of generality we assume $b_1 \neq 0$. Then, we get $c_1 \neq 0$, because otherwise we would also have $c_2 = 0$, contrarily to the hypothesis. We have so $\det M_1(0) = -b_1 c_1 \neq 0$. Take $\alpha = -\dfrac{b_2}{b_1} = -\dfrac{c_2}{c_1}$, and use the new parameters $(\varepsilon_1, \varepsilon_2) = (\delta_1 + \alpha\delta_2, \delta_2)$ in the expansion

$$b_M(\varepsilon_1, \varepsilon_2) = b_1\varepsilon_1 + b_2\varepsilon_2 + \frac{1}{2}\left(b_{11}\varepsilon_1^2 + 2b_{12}\varepsilon_1\varepsilon_2 + \varepsilon_2^2\right) + \cdots$$

The new expression becomes $b_M(\delta_1, \delta_2) = b1\delta_1 + \dfrac{1}{2}\left(b_{11}(\delta_1 + \alpha\delta_2)^2 + 2b_{12}(\delta_1 + \alpha\delta_2)\delta_2 + \delta_2^2\right)$, that is, $\dfrac{1}{2}\left(b_{11}\delta_1^2 + 2(b_{11} + b_{12})\alpha\delta_1\delta_2 + (b_{11}\alpha^2 + 2b_{12}\alpha + b_{22})\delta_2^2\right)$,or in a more compact form,

$$b_M(\delta_1, \delta_2) = b_1\delta_1 + \frac{1}{2}\left(b_{11}\delta_1^2 + 2b_{12}^*\delta_1\delta_2 + b_{22}^*\delta_2^2\right).$$

Bifurcating from critical values

Proof (Cont'd). Applying the i.f.t. to the equation $b_M(\delta_1, \delta_2) = 0$ at the point $(\delta_1, \delta_2) = (0, 0)$, we get that there locally exists a unique function $\delta_1 = g_b(\delta_2)$ whose graph corresponds to the points where $b_M(\delta_1, \delta_2) = 0$. In fact, we obtain the locally parabolic function

$$\delta_1 = -\frac{b_{22}^*}{2b_1}\delta_2^2 + \cdots, \text{ with } b_{22}^* = b_{11}\alpha^2 + 2b_{12}\alpha + b_{22}.$$

Similarly, for the points where $c_M(\delta_1, \delta_2) = 0$, we get

$$\delta_1 = -\frac{c_{22}^*}{2c_1}\delta_2^2 + \cdots, \text{ with } c_{22}^* = c_{11}\alpha^2 + 2c_{12}\alpha + c_{22},$$

which is locally another parabola.

The proof finishes by requiring the generic condition $0 \neq \frac{b_{22}^*}{2b_1} \neq \frac{c_{22}^*}{2c_1} \neq 0$, namely

$$0 \neq \frac{1}{b_1^3}(b_2, -b_1)\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}\begin{pmatrix} b_2 \\ -b_1 \end{pmatrix} \neq \frac{1}{c_1^3}(c_2, -c_1)\begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}\begin{pmatrix} c_2 \\ -c_1 \end{pmatrix} \neq 0.$$

Bifurcating from critical values

We can also take advantage of the fact that for R-reversible systems the monodromy matrix $M = X(T)$ is ruled by $X(T/2)$, as follows. Recall also that, under R-reversibility we know that for planar systems, when $\mathrm{tr}(M) = \pm 2$ then $a_M = \pm 1$ and at least one of the conditions $b_M = 0$ or $c_M = 0$ holds.

Lemma (for critical situations at least one entry in $X(T/2)$ vanishes). For a planar R-reversible system, the following statements hold.

(a) If $\mathrm{tr}(X(T)) = 2$ and for $i, j = 1, 2$, with $i \neq j$, we have $\mathbf{e}_i^\top X(T)\mathbf{e}_j = 0$, then $\mathbf{e}_i^\top X(T/2)\mathbf{e}_j = 0$. The reciprocal is also true.

(b) If $\mathrm{tr}(X(T)) = -2$ and for $i, j = 1, 2$, with $i \neq j$, we have $\mathbf{e}_i^\top X(T)\mathbf{e}_j = 0$, then $\mathbf{e}_j^\top X(T/2)\mathbf{e}_j = 0$. The reciprocal is also true.

For the proof, use $RX(T/2) = X(T/2)RX(T)$ or $X(T) = RX(T/2)^{-1}RX(T/2)$.

Bifurcating from critical values

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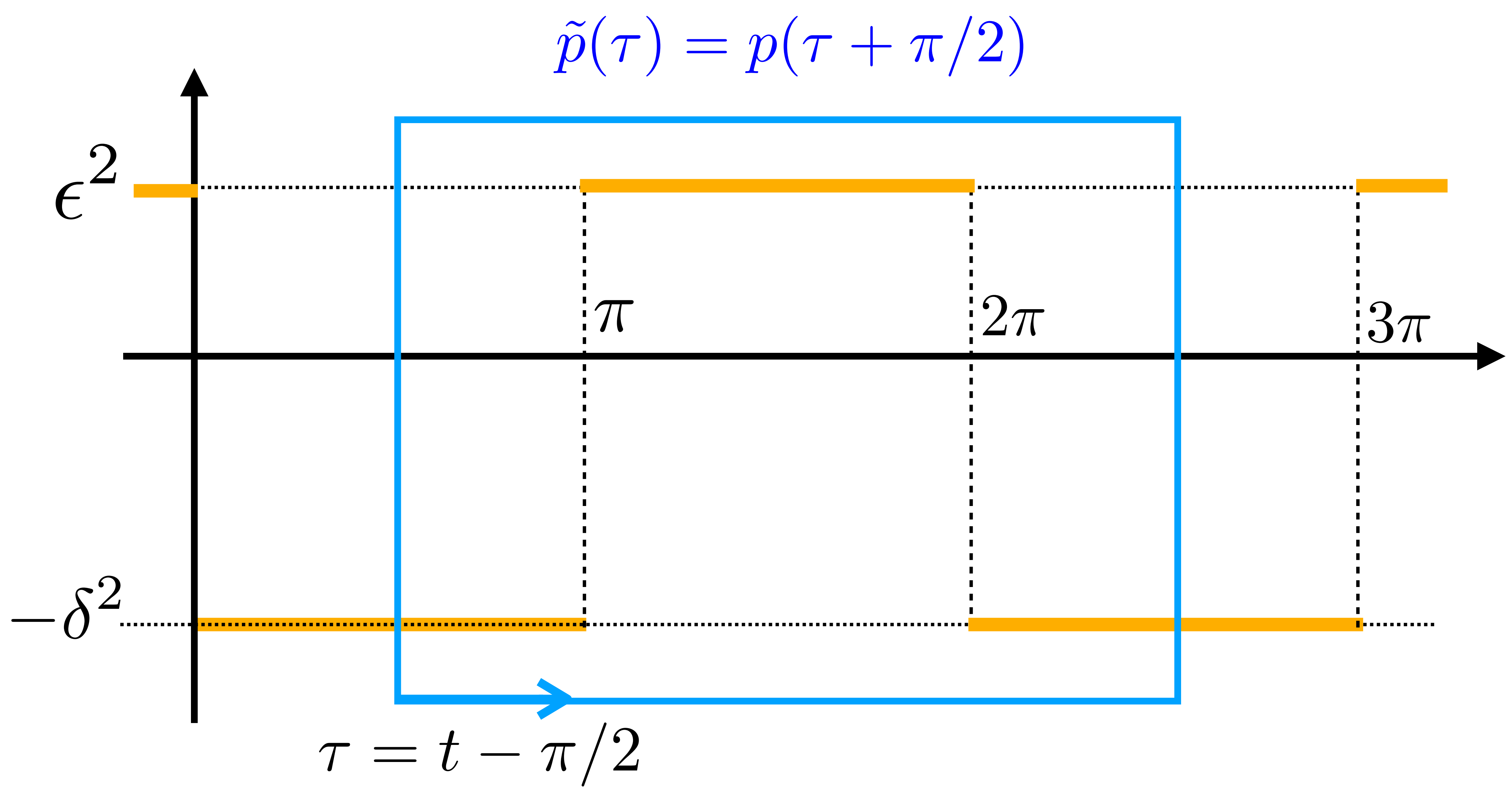
The four entries of $X(T/2)$ determine, when vanishing, stability boundaries!

Bifurcating from critical values

Coming back again to the inverted pendulum problem, for the translated in time, R-reversible version, we can write

$$X(\pi) = \exp\left[\begin{pmatrix} 0 & 1 \\ -\epsilon^2 & 0 \end{pmatrix}\frac{\pi}{2}\right] \cdot \exp\left[\begin{pmatrix} 0 & 1 \\ \delta^2 & 0 \end{pmatrix}\frac{\pi}{2}\right].$$



Bifurcating from critical values

Coming back again to the inverted pendulum problem, for the translated in time, R-reversible version, we can write

$$X(\pi) = \begin{pmatrix} \cos(\pi\epsilon/2) & 2\epsilon^{-1}\sin(\pi\epsilon/2) \\ -(\epsilon/2)\sin(\pi\epsilon/2) & \cos(\pi\epsilon/2) \end{pmatrix} \cdot \begin{pmatrix} \cosh(\pi\delta/2) & 2\delta^{-1}\sinh(\pi\delta/2) \\ (\delta/2)\sinh(\pi\delta/2) & \cosh(\pi\delta/2) \end{pmatrix},$$

so that

$$\mathbf{e}_1^\top X(\pi)\mathbf{e}_2 = 2\delta^{-1}\sinh(\pi\delta/2)\cos(\pi\epsilon/2) + 2\epsilon^{-1}\cosh(\pi\delta/2)\sin(\pi\epsilon/2),$$

$$\boxed{\mathbf{e}_2^\top X(\pi)\mathbf{e}_1 = (\delta/2)\sinh(\pi\delta/2)\cos(\pi\epsilon/2) - (\epsilon/2)\cosh(\pi\delta/2)\sin(\pi\epsilon/2).}$$

Therefore, using statement (a) of the above lemma

$$\boxed{\mathbf{e}_2^\top X(\pi)\mathbf{e}_1 = 0 \Longleftrightarrow c_M(\delta,\epsilon) = 0, \text{ and } \operatorname{tr}(M) = \operatorname{tr}(X(2\pi)) = 2.}$$

Bifurcating from critical values

We conclude that $c_M(\delta,\epsilon) = 0$ and $\mathrm{tr}(M) = 2$ when $\mathbf{e}_2^\top X(\pi)\mathbf{e}_1 = 0$, that is,

$$\boxed{x \tanh x = y \tan y}, \quad \text{with } x = \frac{\pi\delta}{2}, \quad \text{and } y = \frac{\pi\epsilon}{2}.$$

This approach is computationally shorter and more elegant in getting the stability boundary of the inverted pendulum.

Practical bifurcation analysis

Since the monodromy matrix is generally not easy to obtain, in computing its first derivatives with respect to parameters, it can be useful to resort to some initial value problems, as follows. Let $X(t;\varepsilon)$ denote a fundamental matrix of solutions, so that

$$\dot{X}(t;\varepsilon) = A(t;\varepsilon)X(t;\varepsilon), \quad X(0;\varepsilon) = I.$$

By changing the order of derivation, for $i = 1, 2$, we get

$$\frac{d}{dt}\left(\frac{\partial X(t;\varepsilon)}{\partial \varepsilon_i}\right) = \frac{\partial}{\partial \varepsilon_i}\dot{X}(t;\varepsilon) = A(t;\varepsilon)\frac{\partial X(t;\varepsilon)}{\partial \varepsilon_i} + \frac{\partial A(t;\varepsilon)}{\partial \varepsilon_i}X(t;\varepsilon).$$

Practical bifurcation analysis

After substituting for $\varepsilon = 0$, we have

$$\frac{d}{dt}\left(\frac{\partial X(t;0)}{\partial \varepsilon_i}\right) = A(t;0)\frac{\partial X(t;0)}{\partial \varepsilon_i} + \frac{\partial A(t;0)}{\partial \varepsilon_i}X(t;0),$$

getting so the initial value problem

$$\frac{dU_i(t)}{dt} = A(t;0)U_i(t) + \frac{\partial A(t;0)}{\partial \varepsilon_i}X(t;0), \quad U_i(0) = 0,$$

for the matrix $U_i(t) = \frac{\partial}{\partial \varepsilon_i}X(t;0)$. We must solve for $U_i(T)$.

See p. 281 of C. Chicone *Ordinary Differential Equations with Applications*, TAM 34 (2006).

Practical bifurcation analysis

To solve the above equation, a first natural step is to write

$$U_i(t) = \exp\left(\int_0^t A(s;0)\,ds\right) V_i(t),$$

getting so a new simpler initial value problem for $V_i(t)$, namely

$$\frac{dV_i(t)}{dt} = \exp\left(-\int_0^t A(s;0)\,ds\right) \frac{\partial A(t;0)}{\partial \varepsilon_i} X(t;0). \quad V_i(0) = 0.$$

Practical bifurcation analysis

Therefore, for $T = 2\pi$ we conclude that

$$V_i(2\pi) = \int_0^{2\pi} \exp\left(-\int_0^t A(s;0)\,ds\right) \frac{\partial A(t;0)}{\partial \varepsilon_i} X(t;0)\,dt,$$

and

$$M_i(0) = U_i(2\pi) = \exp\left(\int_0^{2\pi} A(t;0)\,dt\right) V_i(2\pi), \text{ for } i = 1,2.$$

Practical bifurcation analysis

Once known $U_i(t)$, this procedure can be extended for second order derivatives, if needed. For instance, we can write

$$\frac{d}{dt}\left(\frac{\partial^2 X(t;\varepsilon)}{\partial\varepsilon_i^2}\right) = A(t;\varepsilon)\frac{\partial^2 X(t;\varepsilon)}{\partial\varepsilon_i^2} + 2\frac{\partial A(t;\varepsilon)}{\partial\varepsilon_i}\frac{\partial X(t;\varepsilon)}{\partial\varepsilon_i} + \frac{\partial^2 A(t;\varepsilon)}{\partial\varepsilon_i^2}X(t;\varepsilon),$$

for $i = 1, 2$. After substituting for $\varepsilon = 0$, we get the initial value problem

$$\frac{dU_{ii}(t)}{dt} = A(t;0)U_{ii}(t) + 2\frac{\partial A(t;0)}{\partial\varepsilon_i}U_i(t) + \frac{\partial^2 A(t;0)}{\partial\varepsilon_i^2}X(t;0), \quad U_{ii}(0) = 0,$$

when our goal is to compute $U_{ii}(t) = \dfrac{\partial^2 X(t;0)}{\partial\varepsilon_i^2}$, to obtain $M_{\varepsilon_i\varepsilon_i}(0) = U_{ii}(2\pi)$.

Practical bifurcation analysis: Hill's equation

Consider the R-reversible Hill'equation, that is,

$$\ddot{x} + (\delta + \epsilon\, p(t))\, x = 0,$$

where $p(t)$ is an **even** 2π-periodic function with **zero mean value**, being δ and ϵ constant parameters. Note also that the simple harmonic oscillator is obtained for $\epsilon = 0$ and $\delta > 0$, so that the natural frequency becomes $\omega_0 = \sqrt{\delta}$.

If we take $x_1 = x$, and $x_2 = \dot{x}$, then we get the planar system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = A(t; \delta, \epsilon) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\delta - \epsilon\, p(t) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Practical bifurcation analysis: Hill's equation

Let $X(t;\delta\,,\epsilon)$, with $X(0;\delta\,,\epsilon) = I$, be a fundamental matrix of solutions for such a system, and $M(\delta,\epsilon\,) = X(2\pi;\delta,\epsilon\,)$ its monodromy matrix. In general, **we cannot compute such monodromy matrix**, except for exceptional cases. For instance, when $\epsilon = 0$ the system matrix becomes a constant matrix A_0, so that we van write (assuming $\delta > 0$)

$$X(t;\delta,0) = \exp\left(A_0 t\right) = \cos(\sqrt{\delta}t)\, I + \frac{\sin(\sqrt{\delta}t)}{\sqrt{\delta}} \begin{pmatrix} 0 & 1 \\ -\delta & 0 \end{pmatrix},$$

where

$$A_0 = A(t;\delta,0) = \begin{pmatrix} 0 & 1 \\ -\delta & 0 \end{pmatrix}.$$

Practical bifurcation analysis: Hill's equation

Clearly, we get $M(\delta,0) = \pm I$ whenever $\cos(2\pi\sqrt{\delta}) = \pm 1, \sin(\sqrt{2\pi\delta}) = 0$, that is, when $\delta = \delta_n = n^2/4$, for $n = 1, 2, \ldots$

For the specific case $\delta = 0$, we get $M(0,0) = \exp\left(A_0 t\right)|_{t=2\pi} = \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix}$.

In order to compute the expansion

$$M(\delta,\epsilon) = M(\delta_n,0) + M_\delta(\delta_n,0)(\delta-\delta_n) + M_\epsilon(\delta_n,0)\epsilon + \cdots$$

around the points $(\delta,\epsilon) = (\delta_n, 0)$ in the parameter plane, we will resort to the initial value problems indicated before,by putting $\varepsilon_1 = \delta - \delta_n$ and $\varepsilon_2 = \epsilon$.

Practical bifurcation analysis: Hill's equation

With some abuse of notation, we write

$$A(t;\varepsilon) = \begin{pmatrix} 0 & 1 \\ -\delta_n - \varepsilon_1 - \varepsilon_2 p(t) & 0 \end{pmatrix},$$

and so

$$A(t;0) = \begin{pmatrix} 0 & 1 \\ -\delta_n & 0 \end{pmatrix}, \quad \frac{\partial A(t;0)}{\partial \varepsilon_1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \frac{\partial A(t;0)}{\partial \varepsilon_2} = \begin{pmatrix} 0 & 0 \\ -p(t) & 0 \end{pmatrix}.$$

Practical bifurcation analysis: Hill's equation

Next, we have

$$\exp\left(-\int_0^t A(s;0)\,ds\right) = \begin{pmatrix} \cos\frac{nt}{2} & -\frac{2}{n}\sin\frac{nt}{2} \\ \frac{n}{2}\sin\frac{nt}{2} & \cos\frac{nt}{2} \end{pmatrix}, \quad X(t;0) = \begin{pmatrix} \cos\frac{nt}{2} & \frac{2}{n}\sin\frac{nt}{2} \\ -\frac{n}{2}\sin\frac{nt}{2} & \cos\frac{nt}{2} \end{pmatrix},$$

and so

$$V_1(2\pi) = \int_0^{2\pi} \begin{pmatrix} \cos\frac{nt}{2} & -\frac{2}{n}\sin\frac{nt}{2} \\ \frac{n}{2}\sin\frac{nt}{2} & \cos\frac{nt}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{nt}{2} & \frac{2}{n}\sin\frac{nt}{2} \\ -\frac{n}{2}\sin\frac{nt}{2} & \cos\frac{nt}{2} \end{pmatrix} dt = \begin{pmatrix} 0 & \frac{4\pi}{n^2} \\ -\pi & 0 \end{pmatrix}.$$

The first derivative w.r.t. parameter $\varepsilon_1 = \delta - \delta_n$ at $\varepsilon = (0,0)$, becomes

$$\boxed{M_\delta(\delta_n, 0) = M_1(0)} = U_1(2\pi) = \begin{pmatrix} \cos\frac{nt}{2} & \frac{2}{n}\sin\frac{nt}{2} \\ -\frac{n}{2}\sin\frac{nt}{2} & \cos\frac{nt}{2} \end{pmatrix}\Bigg|_{t=2\pi} \cdot V_1(2\pi) = \boxed{(-1)^n \begin{pmatrix} 0 & \frac{4\pi}{n^2} \\ -\pi & 0 \end{pmatrix}}.$$

Practical bifurcation analysis: Hill's equation

Computations for V_2 are similar, but we have now

$$V_2(2\pi) = \int_0^{2\pi} \begin{pmatrix} \frac{1}{n}\sin nt & \frac{4}{n^2}\sin^2\frac{nt}{2} \\ -\cos^2\frac{nt}{2} & -\frac{1}{n}\sin nt \end{pmatrix} p(t)\,dt.$$

Assuming for the even function $p(t)$ the Fourier series $p(t) = \sum_{n=1}^{\infty} \alpha_n \cos nt$, we have that the diagonal entries of $V_2(2\pi)$ vanish, while for the anti-diagonal entries we obtain

$$\int_0^{2\pi} \frac{4}{n^2}\frac{1-\cos nt}{2} p(t)\,dt = -\frac{2\pi\alpha_n}{n^2}, \qquad -\int_0^{2\pi} \frac{1+\cos nt}{2} p(t)\,dt = -\frac{\pi\alpha_n}{2}.$$

Thus, we see that $M_\epsilon(\delta_n, 0) = M_2(0) = (-1)^n \begin{pmatrix} 0 & -\frac{2\pi\alpha_n}{n^2} \\ -\frac{\pi\alpha_n}{2} & 0 \end{pmatrix}.$

Practical bifurcation analysis: Hill's equation

We conclude that, at the points $(\delta, \epsilon) = (\delta_n, 0)$ of the parameter plane, we have for the gradients of anti-diagonal entries

$$\nabla b_M(0) = \left(\frac{4\pi}{n^2}, -\frac{2\pi\alpha_n}{n^2}\right)^\top, \quad \nabla c_M(0) = \left(-\pi, -\frac{\pi\alpha_n}{2}\right)^\top,$$

becoming linearly independent whenever $\alpha_n \neq 0$.

For instance, in the case of the Mathieu's equation, we have $p(t) = \cos t$, and then these gradients are independent only for $n = 1$, while for $n > 1$ we must expect double cusps points.

Practical bifurcation analysis: Hill's equation

The point $(\delta, \epsilon) = (0, 0)$ of the parameter plane needs a special analysis. We now have

$$A(t;0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \frac{\partial A}{\partial \varepsilon_1}(t;0) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \frac{\partial A}{\partial \varepsilon_2}(t;0) = \begin{pmatrix} 0 & 0 \\ -p(t) & 0 \end{pmatrix},$$

so that

$$\exp\left(-\int_0^t A(s;0)\,ds\right) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \quad X(t;0) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Practical bifurcation analysis: Hill's equation

Thus, for the point $(\delta, \epsilon) = (0,0)$ of the parameter plane, it turns out that

$$M_1(0,0) = \frac{\partial M}{\partial \varepsilon_1}(0,0) = \begin{pmatrix} -2\pi^2 & -4\pi^3/3 \\ -2\pi & -2\pi^2 \end{pmatrix}, M_2(0,0) = \frac{\partial M}{\partial \varepsilon_2}(0,0) = \begin{pmatrix} 0 & 4\pi\sigma \\ 0 & 0 \end{pmatrix},$$

where $\sigma = \sum_{n=1}^{\infty} \frac{\alpha_n}{n^2}$. Therefore, regarding $c_M(\varepsilon)$, we see that $c_1 = -2\pi$, and $c_2 = 0$. We look for second order terms.

Practical bifurcation analysis: Hill's equation

Lemma (only c_{22} matters) Assuming the expansion

$$c_M(\delta, \epsilon) = c_1\delta + c_2\epsilon + c_{11}\delta^2 + c_{12}\delta\epsilon + c_{22}\epsilon^2 + \cdots$$

where $c_1 \neq 0$ and $c_2 = 0$, then the curve $c_M(\delta, \epsilon) = 0$ is locally described by the graph of the function $\delta = -\frac{c_{22}}{c_1}\epsilon^2 + O(\epsilon^3)$.

Practical bifurcation analysis: Hill's equation

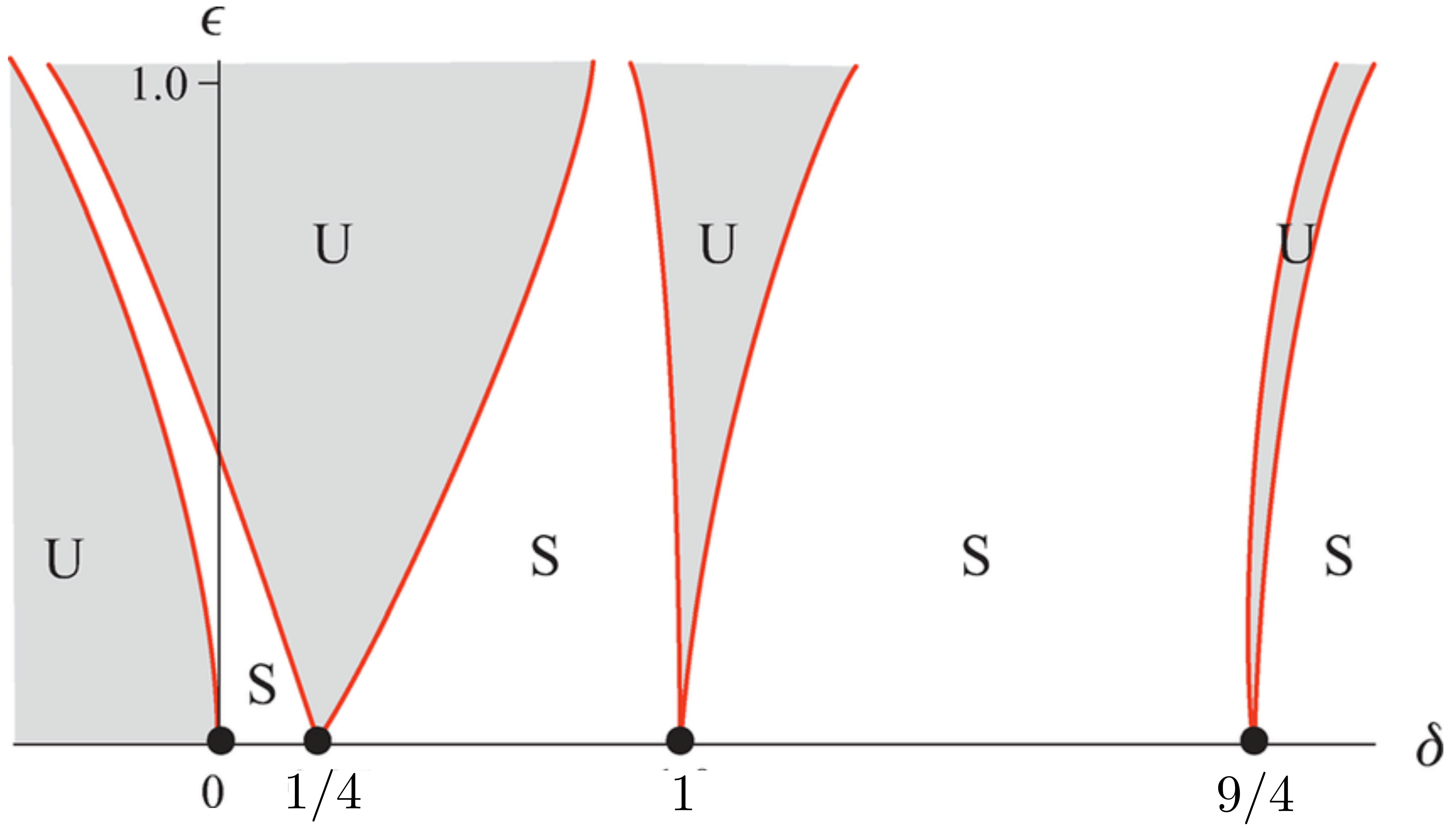
The initial value problem for U_{22} produces

$$c_{22} = -\int_0^{2\pi} P(t)^2\,dt = -\sum_{n=1}^{\infty} \frac{\pi\alpha_n^2}{n^2}, \text{ where } P(t) = \int_0^t p(s)\,ds.$$

Therefore, the curve $c_M(\delta, \epsilon) = 0$ is locally described by the graph of the function

$$\delta = -\frac{c_{22}}{c_1}\epsilon^2 + O(\epsilon^3) = -\frac{\hat{\sigma}}{2}\epsilon^2 + O(\epsilon^3), \text{ with } \widehat{\sigma} = \sum_{n=1}^{\infty} \frac{\alpha_n^2}{n^2}.$$

Practical bifurcation analysis: Hill's equation



For instance, in the case of the Mathieu's equation, we have $p(t) = \cos t$, and then these gradients are independent only for $n = 1$, while for $n > 1$ we must expect double cusps points.

Reversible Hill's equation: main result

Theorem. Consider the Hill's equation $\ddot{x} + (\delta + \epsilon\, p(t))\, x = 0$, where $p(t)$ is an **even** 2π-periodic function with **zero mean value**, being δ and ϵ constant parameters. The following statements hold.

(a) As in any R-reversible system, the diagonal entries of the monodromy matrix $M(\delta, \epsilon)$ are equal. Moreover, for the values $(\delta, \epsilon) = (\delta_n, 0)$ with $\delta_n = n^2/4$, $n \in \mathbb{N}$, such monodromy matrix is equal to $(-1)^n I$.

Reversible Hill's equation: main result

Theorem. Consider the Hill's equation $\ddot{x} + (\delta + \epsilon\, p(t))\, x = 0$, where $p(t)$ is an **even** 2π-periodic function with **zero mean value**, being δ and ϵ constant parameters. The following statements hold.

(b) In the parameter plane (δ, ϵ), from each point $(\delta, \epsilon) = (\delta_n, 0)$ there emerge two curves where the value $\mathrm{trace}(M) = (-1)^n\, 2$ remains constant, acting as stability boundaries for the system. These curves form a saddle point whenever the coefficient α_n in the Fourier series $p(t) = \sum_{n=1}^{\infty} \alpha_n \cos nt$, does not vanish.

Reversible Hill's equation: main result

Theorem. Consider the Hill's equation $\ddot{x} + (\delta + \epsilon\, p(t))\, x = 0$, where $p(t)$ is an **even** 2π-periodic function with **zero mean value**, being δ and ϵ constant parameters. The following statements hold.

(c) When $\alpha_n = 0$ in the above Fourier series, we must generically expect a double cusp point, but other more degenerate configurations are possible.

Reversible Hill's equation: main result

Theorem. Consider the Hill's equation $\ddot{x} + (\delta + \epsilon\, p(t))\, x = 0$, where $p(t)$ is an **even** 2π-periodic function with **zero mean value**, being δ and ϵ constant parameters. The following statements hold.

(d) From the origin $(\delta, \epsilon) = (0, 0)$, where $a_M = 1$, $b_M = 2\pi$, and $c_M = 0$, only one stability curve emerges, which is locally described by the graph of the function

$$\delta = -\frac{\hat{\sigma}}{2}\epsilon^2 + O(\epsilon^3), \text{ with } \widehat{\sigma} = \sum_{n=1}^{\infty} \frac{\alpha_n^2}{n^2}.$$

Application to Meissner's equation

In a previous work, see [FOP], we studied the case of Meissner equations with two steps in each period, obtaining the geometry of its parameter stability regions. In particular, in the elliptic-elliptic case for the quoted two step Meissner equation, we considered the T-periodic equation

$$\ddot{x} + q(t)x = 0, \quad q(t) = \begin{cases} \omega_a^2 > 0, & t \in [0, \tau), \\ \omega_b^2 > 0, & t \in [\tau, T). \end{cases}$$

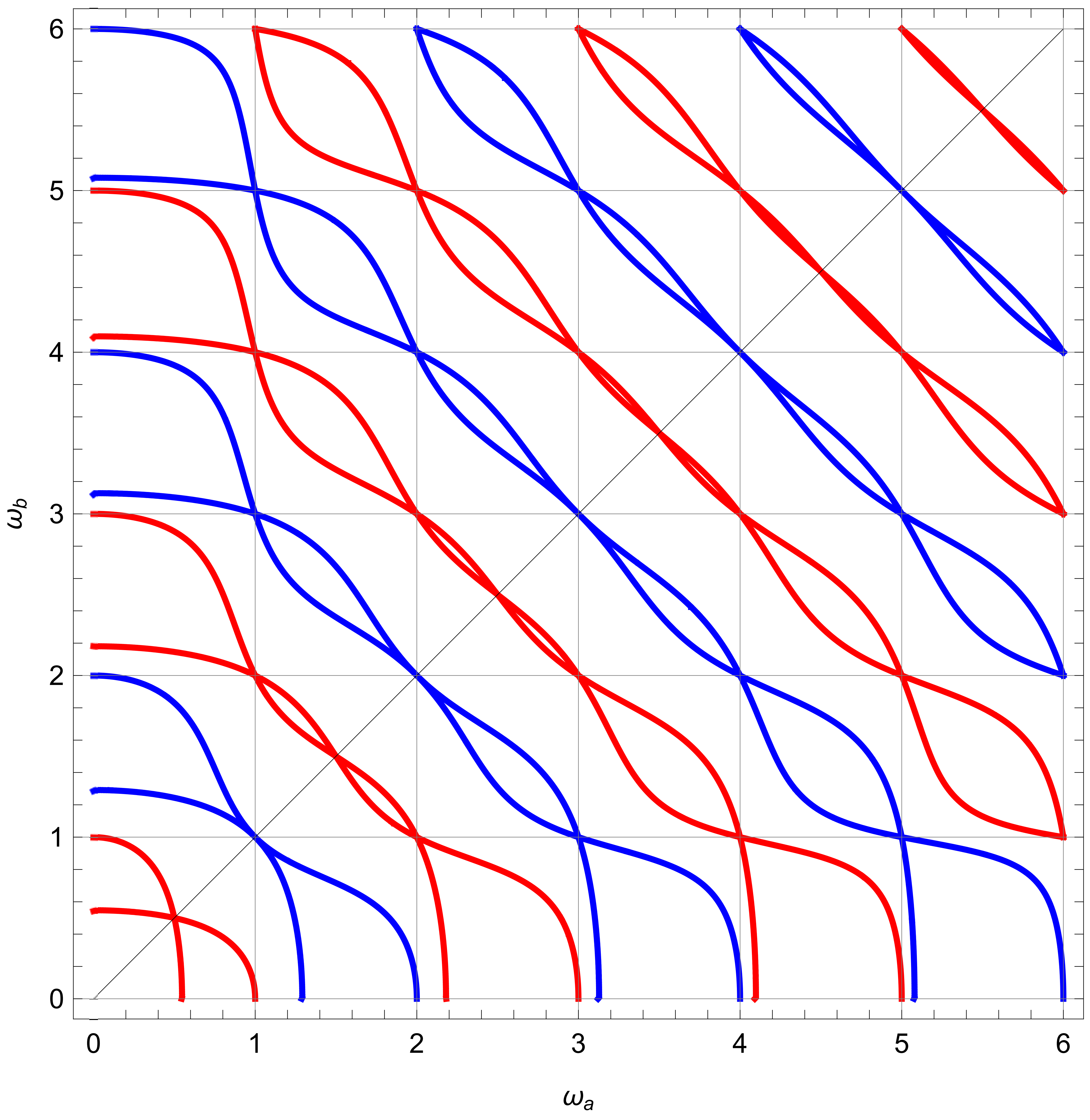
Here, we will restrict our attention to the case $T = 2\pi$, and $\tau = \pi$.

[FOP] Freire E., Ordóñez M., Ponce E. (2022). Hidden regularity of stability boundaries in two-step Hill's equations, *W. Lacarbonara et al. (eds.), Advances in Nonlinear Dynamics, NODYCON Conference Proceedings Series*, : 723–733.

Application to Meissner's equation

$$\ddot{x}+q(t)x=0, \quad q(t)=\begin{cases} \omega_a^2>0, & t\in[0,\,\pi), \\ \omega_b^2>0, & t\in[\pi,\,2\pi). \end{cases}$$

The stability boundaries appear building instability pockets, thanks to the intersections between the curves where $\mathrm{trace}(M)=2$ (in blue), or between the curves with $\mathrm{trace}(M)=-2$ (in red).



Application to Meissner's equation

$$\ddot{x}+q(t)x=0, \quad q(t)=\begin{cases} \omega_a^2>0, & t\in[0,\,\pi), \\ \omega_b^2>0, & t\in[\pi,\,2\pi). \end{cases}$$

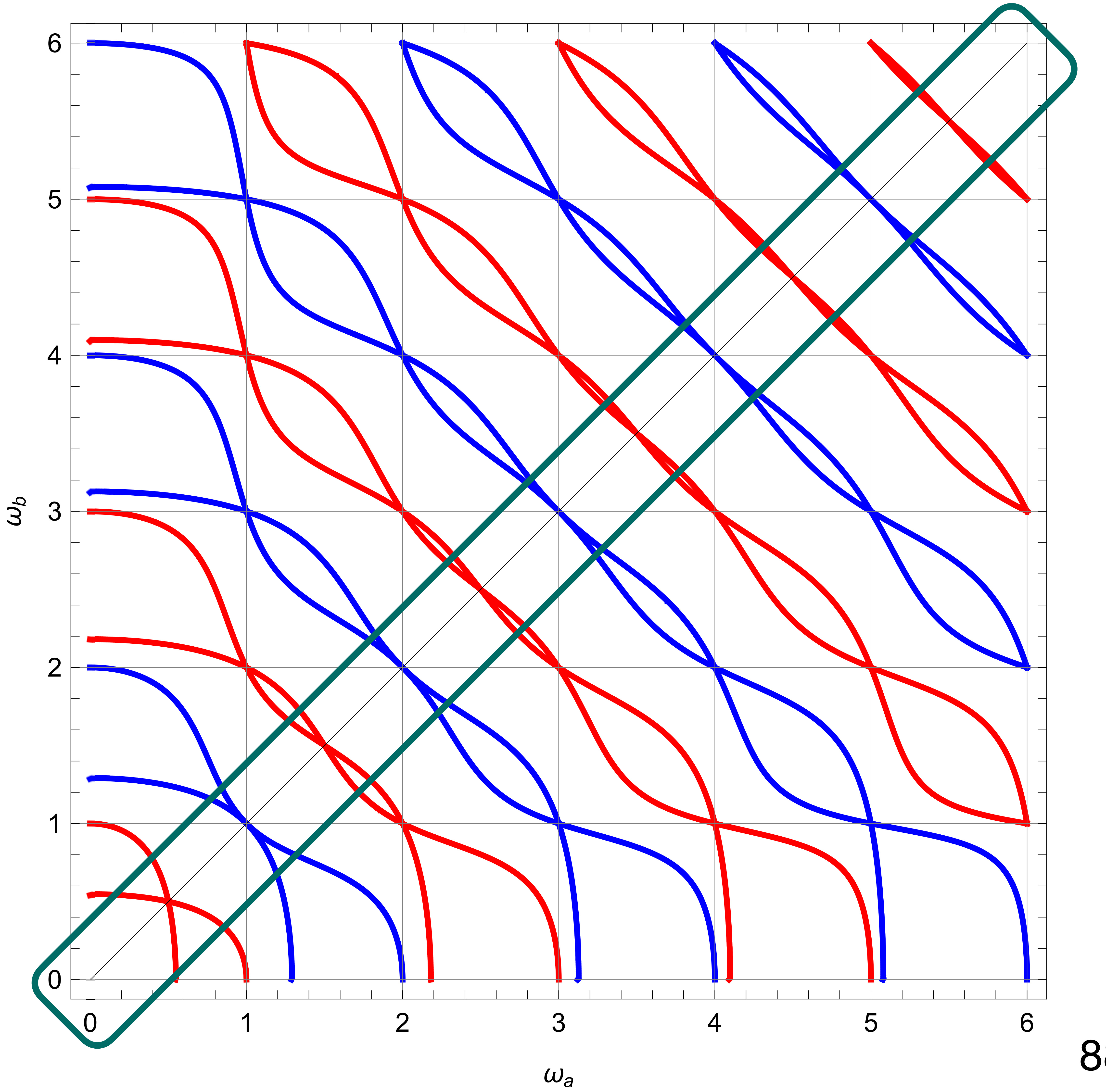
After the translation in time $\tau = t-\pi/2$ we achieve a R-reversible equation with

$$\tilde{q}(\tau)=\underbrace{\frac{\omega_a^2+\omega_b^2}{2}}_{\delta}+\underbrace{\frac{\omega_a^2-\omega_b^2}{2}}_{\epsilon}\sum_{1}^{\infty}\underbrace{\frac{4}{n\pi}\sin\frac{n\pi}{2}}_{\alpha_n}\cos n\tau.$$

Locally around the straightline $\omega_a=\omega_b$, we can think of a Hill's equation, where

$$\delta(\omega_a,\omega_b)=\frac{\omega_a^2+\omega_b^2}{2}, \quad \epsilon(\omega_a,\omega_b)=\frac{\omega_a^2-\omega_b^2}{2},$$

taking advantage of our previous computations.



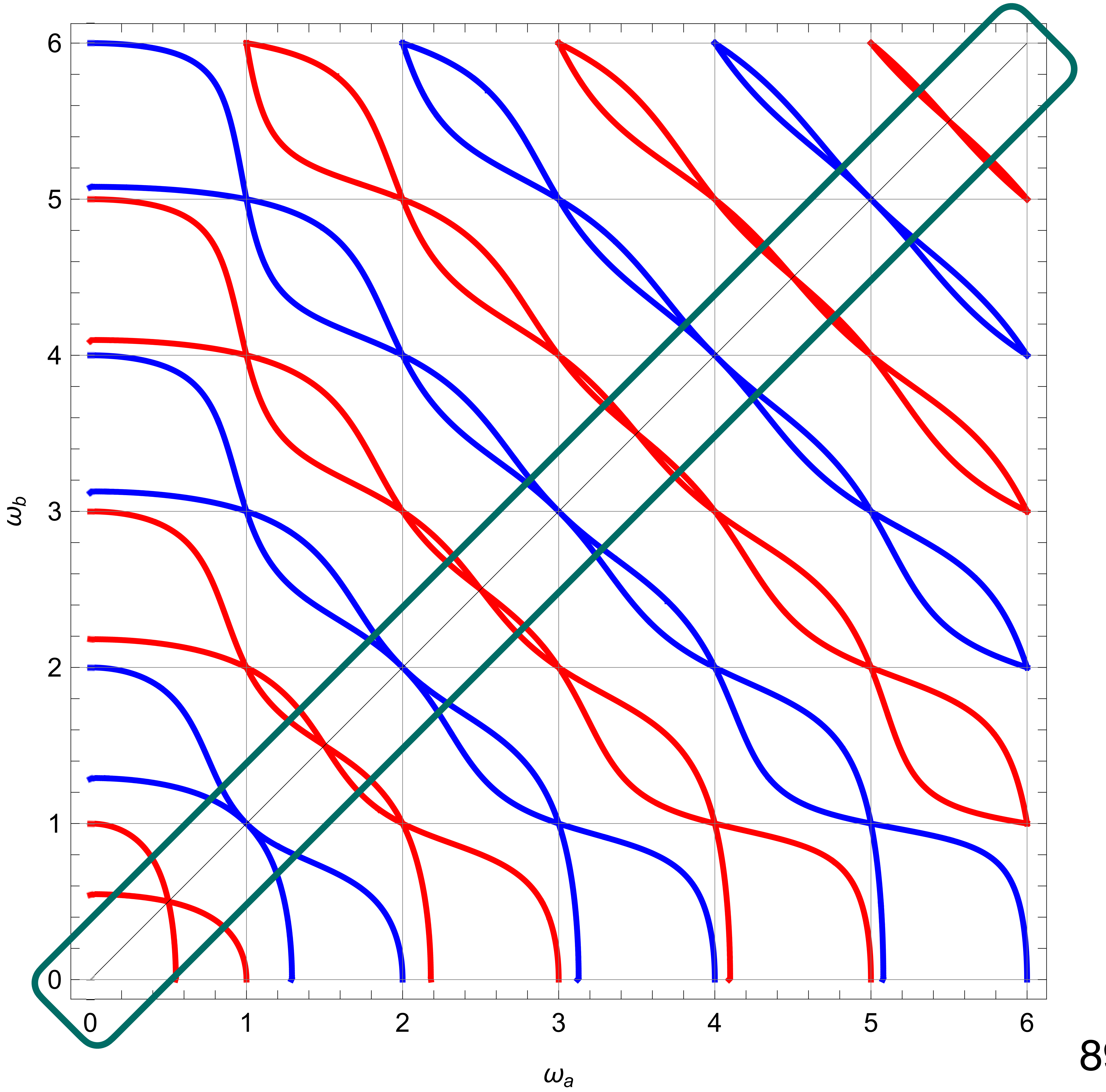
Application to Meissner's equation

$$\ddot{x}+q(t)x=0, \quad q(t)=\begin{cases} \omega_a^2>0, & t\in[0,\,\pi), \\ \omega_b^2>0, & t\in[\pi,\,2\pi). \end{cases}$$

After the translation in time $\tau = t - \pi/2$ we achieve a R-reversible equation with

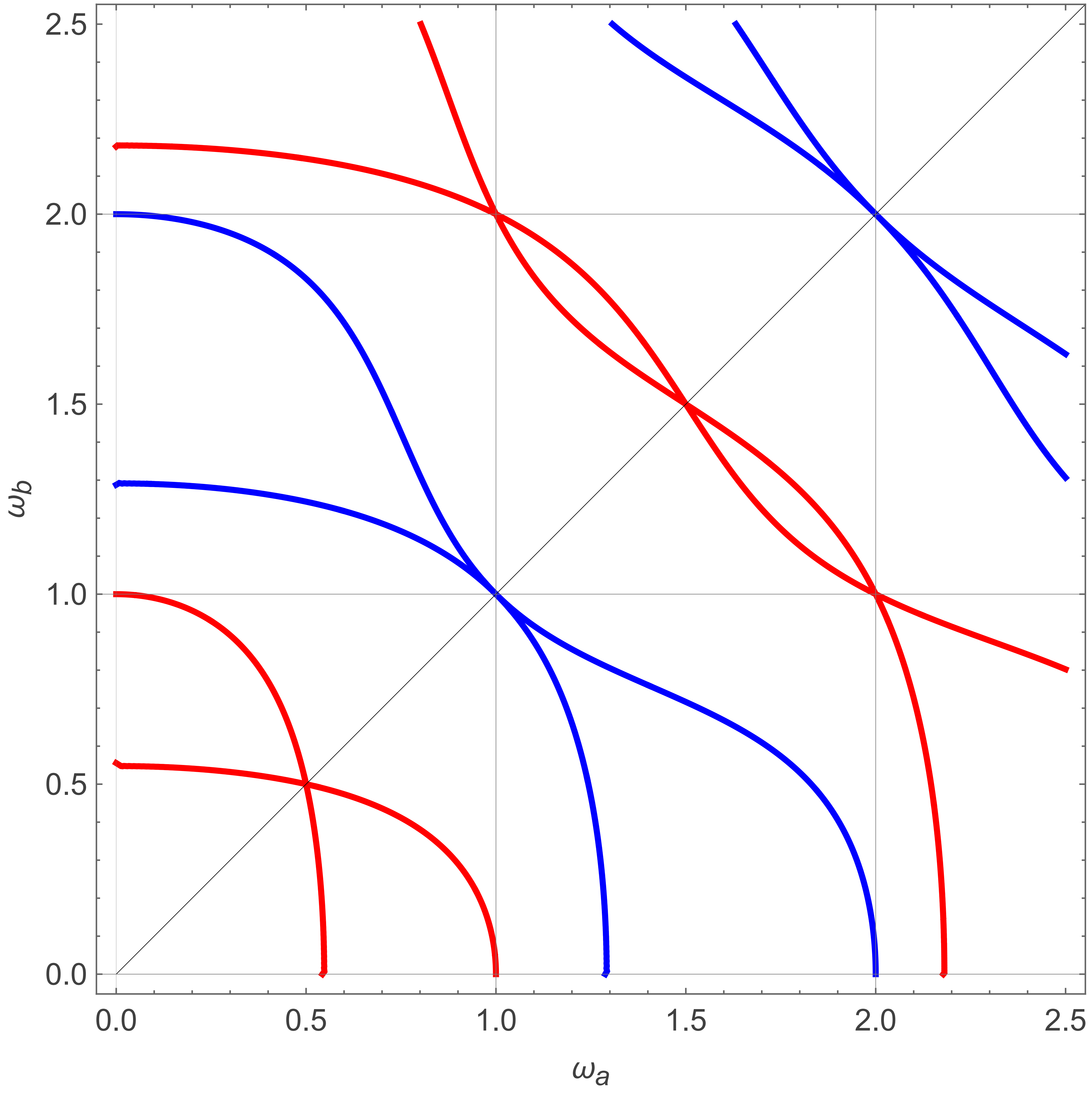
$$\tilde{q}(\tau) = \underbrace{\frac{\omega_a^2+\omega_b^2}{2}}_{\delta} + \underbrace{\frac{\omega_a^2-\omega_b^2}{2}}_{\epsilon} \sum_1^\infty \underbrace{\frac{4}{n\pi}\sin\frac{n\pi}{2}}_{\alpha_n} \cos n\tau.$$

Since α_n vanishes only for n even, at the points $\omega_a = \omega_b = \dfrac{n}{2}$, where $\delta_n = \dfrac{n^2}{4}$ and $\epsilon = 0$, we have saddles for n odd (where $M=-I$) and cusps for n even (where $M=I$).



Application to Meissner's equation

For any point of the form $(\omega_a, \omega_b) = (n_a, n_b) \in \mathbb{N}^2$ with $n_a \neq n_b$, we have $b_M = c_M = 0$, and direct computations show that $b_1 c_2 - b_2 c_1 \neq 0$.
In our original setting, we had a third parameter r, associated to the duration for the first step. In such a three-parameter context, we can identify the above points as the origin in the space $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\omega_a - n_a,\ \omega_b - n_b, r - \pi)$.

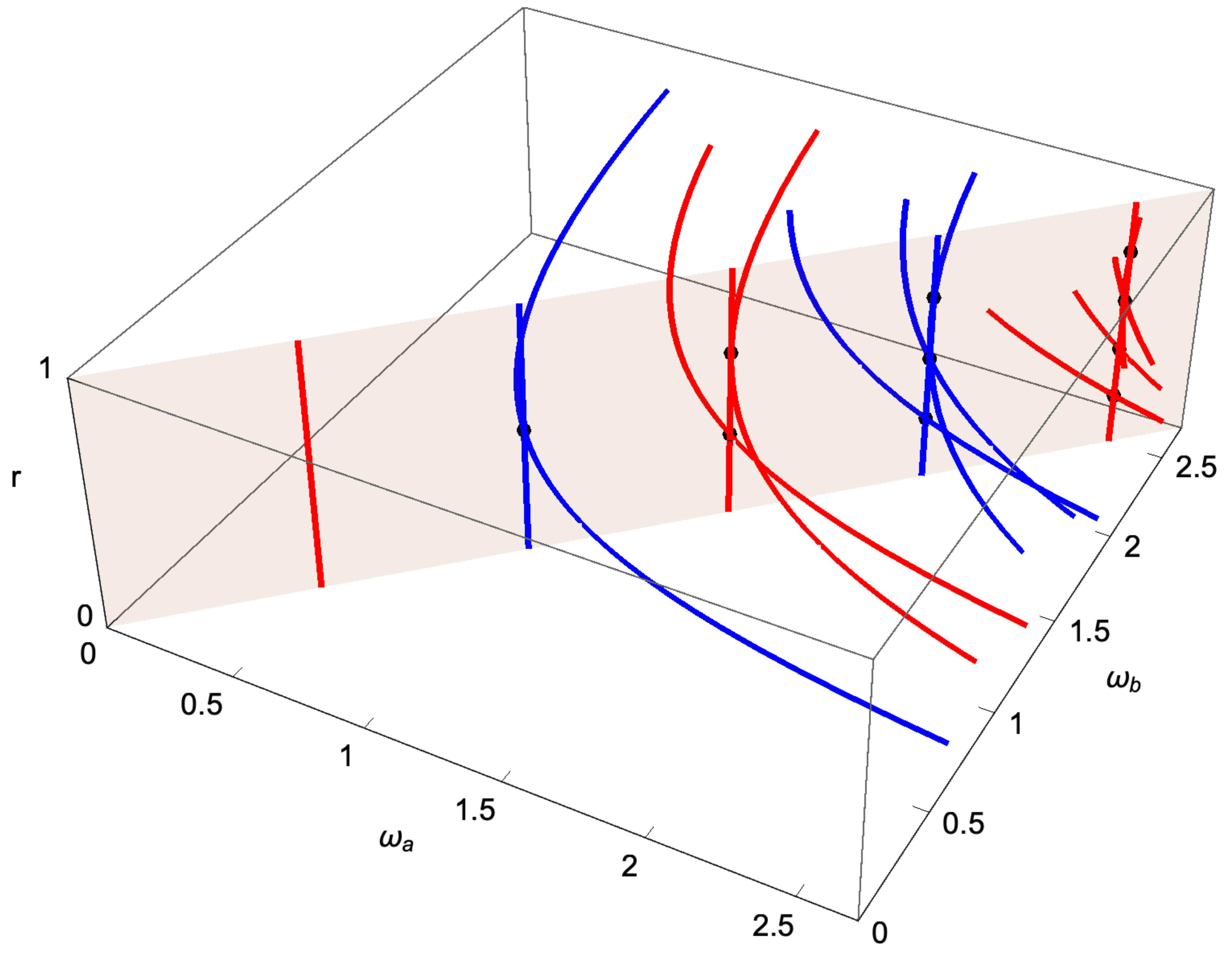


Application to Meissner's equation

Therefore, for the two equations $b_M(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 0$, $c_M(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 0$, we can apply the implicit function theorem at $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0, 0, 0)$ to assure the existence of two smooth functions h_1, h_2, with $h_1(0) = h_2(0) = 0$ that allow us to solve the above equations in terms of ε_3. Thus, along the curve

$$(\varepsilon_1, \varepsilon_2) = (h_1(\varepsilon_3), h_2(\varepsilon_3))$$

the monodromy matrix M is $\pm I$.



Work required on a specific Ince's equation

Richard H. Rand, in his *Lecture Notes on Nonlinear Vibrations*, see p. 60, consider a specific Ince's equation, which can be rewritten as

$$(1+\epsilon\cos t)\,\ddot{x}+\frac{\epsilon}{2}\dot{x}\sin t+\delta\,x=0,$$

or equivalently

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -\dfrac{\delta}{1+\epsilon\cos t} & -\dfrac{\epsilon\sin t}{2(1+\epsilon\cos t)}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

Work required on a specific Ince's equation

For $\epsilon = 0$, the system

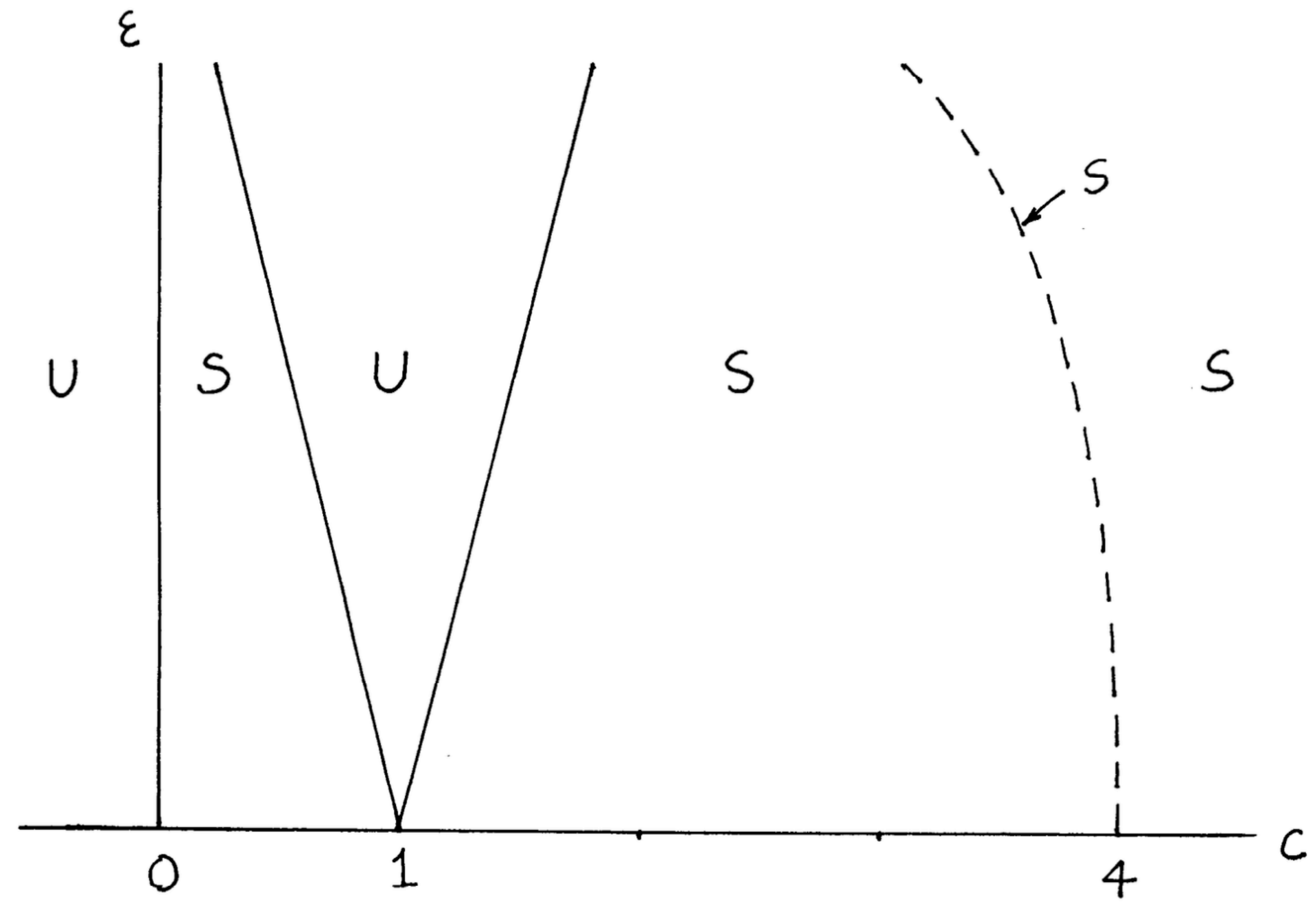
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{\delta}{1+\epsilon\cos t} & -\dfrac{\epsilon\sin t}{2(1+\epsilon\cos t)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

exhibits the points $(\delta, \epsilon) = (\delta_n, 0)$ with $\delta_n = n^2/4$, where $M = \pm I$, as in Hill's equation.

Work required on a specific Ince's equation

It is assured that for $n > 1$ the two curves emerging from the point $(\delta_n, 0)$ coincide, so that, as in any common point for both curves, the condition $M = \pm I$ is maintained.

We have computed up to second order the expansions for b_M and c_M, around the first critical points, which seem proportional, but a different technique is needed to confirm the Rand's statement.



Stability chart for $(1 + \frac{\epsilon}{2}\cos 2t)\,\frac{d^2x}{dt^2} + \frac{\epsilon}{2}\sin 2t\,\frac{dx}{dt} + c\,x = 0$. S=Stable, U=Unstable. Note that there is only a single tongue of instablity. The usual instability regions which emanate from the points $c = n^2$, $n = 2, 3, \cdots$ on the c−axis have zero width (and hence do not exist) due to *coexistence.*

Conclusion: Arnold tongues demystified!

We have reviewed the reversibility implications in periodic linear systems, with emphasis in the planar case.

Its characterization allows to predict the structure of the monodromy matrix, which can be obtained by working only with the semi-period.

The curves delimiting the stability boundaries appear associated to the off-diagonal entries of the monodromy matrix, but also to any entry of the half-period evolution operator.

A general result for reversible Hill's equation has been presented.

There are places I'll remember…

All my life, though some have changed
Some forever, not for better
Some have gone and some remain

All these places had their moments
With lovers and friends, I still can recall
Some are dead, and some are living
In my life, I've loved them all

Reversible periodic linear systems: the planar case

Enrique Ponce (joint work with Emilio Freire and Manolo Ordóñez)

Thanks for your attention!

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