On the integrability of the projective equations

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V Symposium on Planar Vector Fields

Main class of dynamical systems

We consider the following class of equations

$$y_{xx} + a_3(x,y)y_x^3 + a_2(x,y)y_x^2 + a_1(x,y)y_x + a_0(x,y) = 0. \qquad (*)$$

We develop methods for establishing integrability of equations for this class via

- constructing equivalence classes for integrable members of (*);
- classifying transcendental invariant sets and first integrals for (*);

The motivation is diverse:

- Unparametrized geodesics and integrability of (pseudo)Riemannian metrics.
- Integrability of nonlinear oscillators and dynamical systems that are particular cases of (*): generalized Lotka-Voltera system, Rayleigh-like oscillators...
- Integrability of symmetry reductions of partial differential equations.

The talk is based on the works:

- J. Giné, D. Sinelshchikov CNSNS 131 (2024) 107875;
- J. Giné, D. Sinelshchikov, QTDS 24 (2025) 26.
- J. Giné, D. Sinelshchikov, submitted to JGP (2025).

Geodesics flow of projective connections

Let M be a smooth manifold and Γ^i_{jk} is a torsion-free projective connection, i.e.

$$\Gamma^i_{jk} = \Gamma^i_{kj}.$$

This connection defines the covariant derivative of one vector field along another

$$\nabla_{\partial_j}\partial_k = \Gamma^i_{jk}\partial_i.$$

Projective connection is defined up an arbitrary one form

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{kj} + \delta^i_j\omega_k + \delta^i_k\omega_j, \quad \omega = \omega_l dx^l.$$

The curve $\gamma = (x^1(t), \ldots, x^n(t))$ on M is called geodesic if

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

In coordinates

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0.$$

and excluding the parameter in the two-dimensional case

$$y_{xx} - \Gamma^1_{22}y_x^3 + (\Gamma^2_{22} - 2\Gamma^1_{12})y_x^2 + (2\Gamma^2_{12} - \Gamma^1_{11})y_x + \Gamma^2_{11} = 0.$$

Geodesics flow on Riemannian manifolds

Let $\mathfrak{g} = (g_{ij})$ be a (pseudo)Riemannian metric on a manifold M. Then

$$\Gamma^i_{jk} = \frac{g^{il}}{2}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right), \quad g^{ik} g_{kj} = \delta^i_j$$

The curve $\gamma = (x^1(t), \ldots, x^n(t))$ minimizes local distances on M

$$ds^2 = g_{ij} dx^i dx^j.$$

The geodesics satisfy the equation on TM

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0,$$

and on T^*M

$$\dot{x}^i = H_{p_i}, \quad \dot{p}^i = -H_{x_i}, \quad H = \frac{1}{2} g^{ij} p_i p_j,$$

which are connected via the Legendre transformation

$$p_k = g_{ki} \dot{x}^i, \quad \dot{x}^j = g^{jk} p_k.$$

Unparametrized geodesics

Suppose that $n = 2$ and $(x^1, x^2) = (x, y)$. Then, we can eliminate t via

$$y_x = \frac{y_t}{x_t}, \quad y_{xx} = \frac{x_t y_{tt} - y_t x_{tt}}{x_t^3},$$

which yields

$$y_{xx} - \Gamma^1_{22} y_x^3 + (\Gamma^2_{22} - 2\Gamma^1_{12}) y_x^2 + (2\Gamma^2_{12} - \Gamma^1_{11}) y_x + \Gamma^2_{11} = 0.$$

In terms of metric's component we have

$$\begin{aligned}
a_0 &= \frac{1}{2} \frac{2g_{11}g_{12,x} - g_{11}g_{11,y} - g_{12}g_{11,x}}{g_{11}g_{22} - g_{12}^2},\\
a_1 &= \frac{1}{2} \frac{2g_{11}g_{22,x} - 3g_{12}g_{11,y} - g_{22}}{g_{11}g_{22} - g_{12}^2},\\
a_2 &= \frac{1}{2} \frac{g_{11}g_{22,y} + 3g_{12}g_{22,x} - 2g_{12}g_{12,y} - 2g_{22}g_{11,y}}{g_{11}g_{22} - g_{12}^2},\\
a_3 &= \frac{1}{2} \frac{g_{12}g_{22,y} + g_{22}g_{22,x} - 2g_{22}g_{12,y}}{g_{11}g_{22} - g_{12}^2}.
\end{aligned}$$

Not every projective equation is connected to a metric: **4** coefficients can be presented in terms of **3** metric components.

Metrisability problem

Theorem

Projective equation is metrsable iff [a]

$$\psi_{1,x} = 2a_0\psi_2 - \frac{2}{3}a_1\psi_1, \quad \psi_{3,y} = \frac{2}{3}a_2\psi_3 - 2a_3\psi_2,$$
$$\psi_{1,y} + 2\psi_{2,x} = \frac{2}{3}a_1\psi_2 + 2a_0\psi_3 - \frac{4}{3}a_2\psi_1,$$
$$\psi_{3,x} + 2\psi_{2,y} = \frac{4}{3}a_1\psi_3 - 2a_3\psi_1 - \frac{2}{3}a_2\psi_2.$$

Here $g_{11} = \psi_1/\Delta^2$, $g_{12} = \psi_2/\Delta^2$, $g_{22} = \psi_3/\Delta^2$ *and* $\Delta = \psi_1\psi_3 - \psi_2^2$.
Transformations to the Liouville system are

$$\psi_1 = \Delta^2 g_{11}, \quad \psi_2 = \Delta^2 g_{12}, \quad \psi_3 = \Delta^2 g_{22}, \quad \Delta = \psi_1\psi_3 - \psi_2^2 \neq 0,$$

[a] R. Liouville, 1889

Compatibility was demonstrated in *R. Bryant et al J. Diff. Geom (2009)*.

First integrals and invariants

Defenition

A function which is constant on the trajectories of the considered dynamical system is called a first integral.

$$XI = 0, \quad X = \partial_x + z\partial_y + z_x\partial_z, \quad z = y_x, \quad z_x = -a_3z^3 - a_2z^2 - a_1z - a_0,$$
$$XI = 0, \quad X = v^i\partial_{x^i} - \Gamma^i_{jk}v^jv^k\partial_{v^i}, \quad v^i = \dot{x}^i,$$
$$X_HI = 0, \quad \{\cdot, H\} = \partial_{x^i}\cdot\partial_{p_i}H - \partial_{p_i}\cdot\partial_{x^i}H.$$
$$\{I, H\}|_{p_k=g_{ki}v^i} = v^i\partial I_{x^i} - \Gamma^i_{jk}v^jv^kI_{v^i}$$

Defenition

A function R is called an invariant of a vector field if $XR|_{R=0} = 0$.

$$XR = LR,$$
$$-X_HK = \{H, K\} = LK.$$

Integrability of metrics

Defenition

We say that a metric is integrable if its equations for geodesics are integrable.

Defenition

We say that a dynamical system is integrable if it has 'sufficient' amount of functionally independent first integrals.

Maupertuis' principle:

$$H = \frac{1}{2} g^{ij} p_i p_j + U(x), \quad H(p, x) = h_0 \to \tilde{g}_{ij} = (h_0 - U) g_{ij}$$

Open problems:[*]

- Existence and classification of metrics with polynomial/rational first integrals.
- Existence of explicit analytical representations of geodesics.
- Superintegrable metrics with polynomial/rational first integrals.
- Extension of existing integrability theories.

[*] *A. Bolsinov et al. Philos. Trans. R. Soc. A (2018), K. Burns, V.S. Matveev Ergod. Theory Dyn. Syst. (2019)*

First integrals and invariants

Proposition

If a projective equation has a first integral or invariant then the corresponding geodesic flow also has a first integral and an invariant.
The lift from a projective equation is as follows [a]

$$I(x,y,z) \to I\left(x,y,\frac{\dot{y}}{\dot{x}}\right), \quad I(x,y,z) \to I\left(x,y,\frac{H_{p_2}}{H_{p_1}}\right),$$

$$R(x,y,z) \to R\left(x,y,\frac{\dot{y}}{\dot{x}}\right), \quad L(x,y,z) \to \dot{x}L\left(x,y,\frac{\dot{y}}{\dot{x}}\right),$$

$$R(x,y,z) \to R\left(x,y,\frac{H_{p_2}}{H_{p_1}}\right), \quad L(x,y,z) \to (g^{11}p_1 + g^{12}p_2)L\left(x,y,\frac{H_{p_2}}{H_{p_1}}\right).$$

[a] *J. Giné, D. Sinelshchikov submitted to JGP (2025)*

First integrals and invariants

The relative Killing tensors* and quasi-polynomial invariants † are

$$K = A^{i_1 \dots i_d}(x, y) p_{i_1} \dots p_{i_d}, \quad R = B_k(x, y) y_x^k.$$

Proposition

Suppose that a vector field has k invariants R_k with the cofactors L_k. Then the function $I = R_1^{q_1} \dots R_k^{q_k}$ is a first integral iff $\sum_{j=1}^{k} q_j L_j = 0$, $q_k \in \mathbb{C}$.
Suppose that the Hamiltonian H has k relative Killing tensors K_k with the cofactors L_k. Then the function $I = K_1^{q_1} \dots K_k^{q_k}$ is a first integral iff $\sum_{j=1}^{k} q_j L_j = 0$.

Defenition

We will call a Hamiltonian system D integrable if is has a first integral of the form $I = K_1^{q_1} \dots K_k^{q_k}$ where K_j are relative Killing tensors.

*B. Kruglikov Russ. J. Math. Phys 32(2025)

† *J. Giné, D. Sinelshchikov QTDS (2025), J. Giné, D. Sinelshchikov submitted to JGP (2025)*

Projective Lie algebra

Defenition

A projective Lie algebra is the Lie algebra of point symmetries $X = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ *of the projective equation:*

$$\boldsymbol{pr}X^{(2)}E = 0 \quad when \quad E = 0.$$

We use the notation $\mathfrak{p}(c)$ and $\mathfrak{p}(\mathfrak{g})$ for this Lie algebra.

Theorem

We have that[a]

$$\dim \mathfrak{p} \in \{0,1,2,3,8\}.$$

Iff $\dim \mathfrak{p} = 8$ *then there exist local coordinates* $w = F(x,y)$, $\xi = G(x,y)$ *where the projective equation is* $w_{\xi\xi} = 0$.
All these equations are trivially merisable to a metric with a constant curvature.

[a] *Tressé, Lie, Cartan*

Differential invariants of point symmetries

Defenition

We say that a function $F(x, y, y_x, y_{xx}, \ldots, y_{n,x})$ is an absolute or relative invariant of the equivalence group of the point transformations, if

$$F(\boldsymbol{pr}X^{(n)}(x, y, y_x, y_{xx}, \ldots, y_{n,x})) = F(x, y, y_x, y_{xx}, \ldots, y_{n,x}).$$

Theorem

We have that $\dim \mathfrak{p}(c) = \{8\}$*, iff the Liouville invariants* $\mathcal{L}_1 = \mathcal{L}_2 = 0$*.*

$$\mathcal{L}_1 = \frac{1}{3}a_{2,xx} - \frac{2}{3}a_{1,xy} + a_{0,yy} + a_0 a_{2,y} + a_2 a_{0,y} - a_3 a_{0,x} - 2a_0 a_{3,x} - \frac{2}{3}a_1 a_{1,y} + \frac{1}{3}a_1 a_{2,x},$$
$$\mathcal{L}_2 = \frac{1}{3}a_{1,yy} - \frac{2}{3}a_{2,xy} + a_{3,xx} - a_3 a_{1,x} - a_1 a_{3,x} + a_0 a_{3,y} + 2a_3 a_{0,y} + \frac{2}{3}a_2 a_{2,x} - \frac{1}{3}a_2 a_{1,y}.$$

In order to find admitted projective Lie algebra we compute basis differential invariants *.

*Y.Y. Bagderina, J. Phys. A. 49 (2016) 155202

Approach

The family of projective equations

$$y_{xx} + a_3(x,y)y_x^3 + a_2(x,y)y_x^2 + a_1(x,y)y_x + a_0(x,y) = 0$$

is closed with respect to point

$$w = F(x,y), \quad \xi = G(x,y),$$

and nonlocal transformations

$$w = F(x,y), \quad d\xi = [G_1(x,y)y_x + G_2(x,y)]dx, \quad F_y G_2 G_1 \neq 0.$$

Thus we can construct factor spaces for integrable projective equations modulo point and nonlocal transformations.
Nonlocal transformations preserve first integrals and invariants

$$I(w, w_\xi) \to I\left(F, \frac{F_x + F_y y_x}{G_2 + G_1 y_x}\right), \quad R(w, w_\xi) \to R\left(F, \frac{F_x + F_y y_x}{G_2 + G_1 y_x}\right).$$

Equivalence problems

- Linearization via point transformations, classification of Lie algebras admitted by the considered family of equations (*S. Lie 1891, M.A. Tressé 1894, E. Cartan ,1984*):

  $\Longrightarrow$ leads only to the metrics with constant curvature: everything is known about (super)integrability.

- Equivalence to Painlevé equations (*N. Karpman, 1985, W. Sarlet et al. 1987, V.S. Dryma 2002–2019, Yu.Yu. Bagderina 2007–2019*)

  $\Longrightarrow$ Painlevé equations generically are not metrizable

- Linearization under nonlocal transformations (*L.G.S. Duarte et al. 1994, S.V. Meleshko et al. 2010, 2011, 2013, D.S. 2020, 2021, 2023*)

  $\Longrightarrow$ leads to families of (super)integrable metrics.

- Equivalence to Painlevé-type equations under nonlocal transformations (*D.S. 2021, A.I., D.S. 2023*)

  $\Longrightarrow$ leads to families of (super)integrable metrics.

Linearizability via nonlocal transformations

We consider projective equations

$$y_{xx} + a_3(x, y)y_x^3 + a_2(x, y)y_x^2 + a_1(x, y)y_x + a_0(x, y) = 0,$$

nonlocal transformations

$$w = F(x, y), \quad d\xi = [G_1(x, y)y_x + G_2(x, y)]dx, \quad F_y G_2 G_1 \neq 0.$$

and linear equation

$$w_{\xi\xi} + \beta w_\xi + \alpha w = 0.$$

where α, β are arbitrary parameters.
In the autonomous case we assume that

$$F_x = G_{1,x} = G_{2,x} = a_{i,x} = 0, \quad i = 0, 1, 2, 3.$$

and use the notation

$$y_{xx} + k(y)y_x^3 + h(y)y_x^2 + f(y)y_x + g(y) = 0.$$

Nonlocally linearizable metrics

Theorem

Suppose that the projective equation

$$y_{xx} + a_3(x, y)y_x^3 + a_2(x, y)y_x^2 + a_1(x, y)y_x + a_0(x, y) = 0$$

is metrisable. Then the following statements are equivalent: 1) projective vector field can be linearized via

$$w = F(x, y), \quad d\xi = [G_1(x, y)y_x + G_2(x, y)]dx, \quad F_y G_2 G_1 \neq 0.$$

2) the correspondong geodesic flow is integrable

$$I = K_1^{\beta+\rho} K_2^{\beta-\rho} K_3^{-2\rho},$$

where

$$K_{1,2} = \left[(2F_y + (\beta \pm \rho)FG_1)g^{2i} + (2F_x + (\beta \pm \rho)FG_2)g^{1i}\right] p_i,$$
$$K_3 = (G_1 g^{2i} + G_2 g^{1i})p_i, \quad \rho = \sqrt{\beta^2 - 4\alpha}, \quad \alpha\beta \neq 0.$$

Linearizability criteria

Theorem

The following statements are equivalent[a]*:*

1) *an autonomous projective equation is linearizable via GNT;*
2) *one of the following correlations holds:*

$$
\begin{aligned}
&(I) \quad \beta^2(2\beta^2-9\alpha)^2(kA_y-AB)^3-A^5(\beta^2-3\alpha)^3=0,\\
&(II) \quad \beta^2-3\alpha=0, \quad kA_y-AB=0,\\
&(III) \quad 2\beta^2-9\alpha=0, \quad A=0,\\
&A=27kg^2-9hfg+2f^3+9gf_y-9fg_y, \quad B=3hg-f^2+3g_y;
\end{aligned}
$$

3) *an autonomous projective equation possesses a first integral*

$$
I=\left(\frac{2F_y y_x}{G_1 y_x+G_2}+(\beta+\sqrt{\rho})F\right)^{\sqrt{\rho}+\beta}\left(\frac{2F_y y_x}{G_1 y_x+G_2}+(\beta-\sqrt{\rho})F\right)^{\sqrt{\rho}-\beta}.
$$

[a] *D. Sinelshchikov PLA (2020)*

Classification of projective Lie algebras

Theorem

Suppose that $A = 0$*. Then* $\dim \mathfrak{p}(c) \in \{1, 2, 3, 8\}$*. In the generic case* $\dim \mathfrak{p}(c) = 1$*.*
We have that $\dim \mathfrak{p}(c) = 2$ *iff*

$$gBB_{yy} + g_y BB_y - 2gB_y^2 = 0, \quad gB_y(9gB_y - 4B^2) \neq 0.$$

We have that $\dim \mathfrak{p}(c) = 3$ *and* $\mathfrak{p}(c) \cong \mathfrak{sl}(2, \mathbb{R})$ *iff*

$$9g^2 B_{yy}^2 + 9g\left(B + 2g_y\right) B_y B_{yy} - 25gB_y^3 - \left(4B + 3g_y\right)\left(B - 3g_y\right) B_y^2 = 0, \;\; gB_y \neq 0,$$
$$2gB_{yy} + BB_y + 2g_y B_y \neq 0.$$

Finally, $\dim \mathfrak{p}(c) = 8$ *and* $\mathfrak{p}(c) \cong \mathfrak{sl}(3, \mathbb{R})$ *iff*

$$B_y = 0, \quad g \neq 0.$$

Integrable metrics: linearizable equations

Theorem

Suppose that $A = 0$. Then autonomous projective equation is metrisable iff

- I $g_{ij,x} = 0$;
- II *Darboux superintegrable metrics, i.e.* $\dim \mathfrak{p}(\mathfrak{g}) = 3$ *and* $\mathfrak{p}(\mathfrak{g}) \cong \mathfrak{sl}(2, \mathbb{R})$
- III *flat metrics, i.e.* $\dim \mathfrak{p}(\mathfrak{g}) = 8$ *and* $\mathfrak{p}(\mathfrak{g}) \cong \mathfrak{sl}(3, \mathbb{R})$

Corollary

All these metrics are superintegrable. Their geodesics are formed by the curves

$$(x, x(y)) = \left(x, c_2 - \int_0^y \left(\frac{f(\xi)}{3g(\xi)} - \sqrt{\frac{C'(\xi)}{g(\xi)[c_1 - 2C(\xi)]}} \right) d\xi \right),$$

where c_1, c_2 are arbitrary constants. Here $C = \int g \exp\left\{ 2 \int \left(h - \frac{f^2}{3g} \right) dy \right\} dy$.

Examples of metrisable oscillators

Corollary

If $A = 0$ there is the following solution of the Liouville system

$$\psi_1 = C\left(\frac{g}{C_y}\right)^{2/3}, \quad \psi_2 = \frac{fC}{3(gC_y^2)^{1/3}}, \quad \psi_3 = \frac{9gC_y + f^2C}{18(g^2C_y)^{2/3}}, \quad gC_y \neq 0.$$

Equation	Results
$y_{xx} + \alpha y_x + \omega_0 y + \delta y^n = 0$ $y_{xx} + (y^m + \mu)y_x + \alpha y^{2m+1} + \delta y^{m+1} + \gamma y = 0$	superintegrable metric with a linear and a (transcendental) polynomial first integral of an arbitrary degree;
$y_x = y(\alpha_0 - \alpha_1 y - \alpha_2 z - \alpha_3 y^2)$ $z_x = z(\beta_0 - \beta_1 z - \beta_2 y - \beta_3 y^2 - \beta_4 z^2)$	superintegrable metrics with a linear and a transcendental (rational) first integrals;
$y_{xx} + d_3 y_x^3 + (c_1 y + c_0) y_x^2 + (b_1 y + b_0) y_x + a_1 y + a_0 = 0$	integrable metric with transcendental first integral

The Duffing oscillator

Consider the equation*

$$y_{xx} + \alpha y_x + \omega_0 y + \delta y^n = 0, \quad \omega_0 = \frac{2(n+1)}{(n+3)^2}, \quad n \neq -1, -3.$$

The normal form modulo point transformations

$$y_{xx} + \delta(n+1) y^n = 0.$$

This equation possess two explicit first integrals: autonomous one

$$I_1 = y_x^2 + 2\delta y^{n+1},$$

and non-autonomous one

$$I_2 = x - \frac{y y_x}{y_x^2 + 2\delta y^{n+1}} {}_2F_1\left(\frac{n+3}{2n+2}, 1; \frac{n+2}{n+1}; \frac{2\delta y^{n+1}}{y_x^2 + 2\delta y^{n+1}}\right).$$

*Integrability established in T. Stachowiak J Differential Equations 2019;266:5895–911.

The Duffing oscillator

The corresponding Hamiltonian for the geodesics flow

$$H = y^{n+1}\left(p_1^2 + 2\delta y^{n+1} p_2^2\right).$$

This Hamiltonian is superintegrable with a linear first integral $L = p_1$ and

$$T = x - \frac{y p_1 p_2}{K}\,{}_2F_1\left(\frac{n+3}{2n+2}, 1; \frac{n+2}{n+1}, \frac{p_1^2}{K}\right), \quad K = p_1^2 + 2y^{n+1}\delta p_2^2.$$

If $n = -(2k+3)/(2k+1)$, $k \in \mathbb{N}^0$ then

$$P_k = x y^{-\frac{2(k+1)}{2k+1}} K^{k+1} - p_2 p_1^{2k+1} y^{-\frac{1}{2k+1}} \sum_{s=0}^{k} \frac{(-1)^s s! \binom{k}{s}}{\left(\frac{1}{2}-k\right)_s} p_1^{2(s-k)} K^{k-s}.$$

This provides an example of superintegrable metric with linear and polynominal of degree $2k+2$ first integral.

Conformal metrics

One can choose isothermal coordinates such that $g_{11} = g_{22}$ and $g_{12} = 0$:

$$ds^2 = \Lambda(x, y)(dx^2 + dy^2) = \exp\{2\lambda(x, y)\}(dx^2 + dy^2).$$

Theorem

Autonomous projective equation is metrisable for the conformal metrics iff

$$k = f, \quad h = g, \quad f_y = 0.$$

The solution of the Liouville system is

$$\psi_1 = \psi_3 = \exp\left\{\frac{2}{3}\left(\int g dy - \epsilon x\right)\right\}, \quad f = k = \epsilon \neq 0.$$

We intersect these metrisability conditions with linearizability ones to obtain new families of integrable conformal metrics.

Integrable conformal metrics I

Theorem

The geodesic flow with the Hamiltonian

$$H = \frac{1}{2}\exp\left\{2\left(\int g dy - x\right)\right\}\left(p_1^2 + p_2^2\right),$$

is integrable with the transcendental first integral

$$T = K_1^{1-i\sqrt{3}} K_2^{1+i\sqrt{3}} K_3^{-2} K_4,$$

iff

$$9gI_y + I^4 - 27Ig^2 + I = 0, \quad I^3 = A = 18g^2 - 9g_y + 2,$$

and K_j are relative Killing tensors

$$K_{1,2} = 12gp_1 + (1 \mp \sqrt{3}i)(1 - 2I \pm \sqrt{3}i)p_2,$$

$$K_3 = 3gp_1 + (I+1)p_2, \quad K_4 = \exp\left\{\frac{2}{3}\int \frac{I^2}{g} dy\right\}.$$

Integrable conformal metrics II

Theorem

The geodesic flow for the conformal metric is integrable with the first integral

$$T = K_1^{\rho+\beta} K_2^{\rho-\beta} K_3^{-2\rho} K_4,$$

iff the function $g \neq 0$ *is a solution of the equation*

$$\beta^2(2\beta^2 - 9\alpha)^2(gA_y - BA)^3 - A^5(\beta^2 - 3\alpha)^3 = 0,$$

and K_j *are relative Killing tensors*

$$K_{1,2} = (3\rho \mp \beta)\, lp_2 \mp 2(\delta\beta)^{\frac{1}{3}}\,(3gp_1 + \epsilon p_2)\,, \quad \delta = 2\beta^2 - 9\alpha,$$

$$K_3 = \beta^{\frac{2}{3}}\, lp_2 - \delta^{\frac{1}{3}}(3gp_1 + \epsilon p_2), \quad K_4 = \exp\left\{2\alpha\rho(\delta\beta)^{-\frac{2}{3}} \int l^2 g^{-1} dy\right\}.$$

If $4\alpha = \beta^2(1 - r^2)$, $r = p/q \in \mathbb{Q}$, $r \neq \pm 1$, $r \neq \pm(1/3)$ *then*

$$T = K_1^{p+q} K_2^{p-q} K_3^{-2p} K_4^{\frac{q}{\beta}}, \quad p \in \mathbb{Z}, \quad q \in \mathbb{N}.$$

Properties of integrability conditions

Proposition

Conditions for the existence of a first integral are integrable as differential equations for the function g.

For example, for the first case of integrable conformal metrics we have that there is an explicit integrating factor

$$M = \frac{g}{\left(9g_y^2\epsilon^2 - 6\epsilon^2\left(\epsilon^2 - 3g^2\right)g_y + \left(\epsilon^2 + 9g^2\right)\left(\epsilon^2 - 3g^2\right)^2\right)\left(9g_y - 2\epsilon^2 - 18g^2\right)^{\frac{1}{3}}}.$$

Proposition

In the generic cases $\dim \mathfrak{p}(\mathfrak{g}) = 1$ *for the metrics above.*

Rational integrals I

Theorem

The geodesic flow for the conformal metric is integrable with the first integral

$$I = F^2 K_1^{-2} K_2,$$

where $K_{1,2}$ are relative Killing tensors

$$K_1 = 6\lambda_y^2 p_1 - m p_2, \quad K_2 = 36\lambda_y^4 p_1^2 - 12\lambda_y^2 m p_1 p_2 + (36 G_2^2 \lambda_y^2 + m^2) p_2^2$$

iff the function λ is a solution of the equation

$$11m^3 - 18\lambda_y\lambda_x m^2 + 6\lambda_y(12\lambda_y\lambda_{yy} - 6\lambda_y^3 - m_x)m + 36\lambda_y^3(6\lambda_x\lambda_y^2 - m_y) = 0, \quad m = 2\lambda_{xy} + 2\lambda_x\lambda_y.$$

and the corresponding projective equation is linearizable via

$$w = F(y), \quad d\xi = [G_1(x,y)y_x + G_2(x,y)]dx,$$

to

$$w_{\xi\xi} + w = 0.$$

Example

Consider the following non-autonomous cubic Liénard oscillator

$$y_{xx} + (b_1(x) + b_0(x))y_x + c_3(x)y^3 + c_2(x)y^2 + c_1(x)y + c_0(x) = 0.$$

One can show that its coefficients satisfy compatibility condition if $b_0 = -p_x/p$, $c_3 = s^2/9$, $c_2 = c_0 = 0$, $c_1 = \mu p^2$ and $b_1 = s$, where $p = p(x) \neq 0$ is an arbitrary function and μ is an arbitrary constant.
As a consequence, the equation

$$y_{xx} + \left(py - \frac{p_x}{p}\right) y_x + \frac{p^2}{9} y(y^2 + 9\mu) = 0,$$

possesses a first integral

$$I = \frac{6y_x + py^2 + 9p\mu}{(3y_x + py^2 + 9p\mu)^2}.$$

This equation is metrisable for an arbitrary function $p(x)$. When $p(x) = \mathrm{e}^{-\nu x}$ the corresponding metric is

$$ds^2 = \mathrm{e}^{-2\nu x} \left(\frac{1}{3}(y^2 + 9\mu)dx^2 + 2dxdy\right).$$

Quasi-polynomial invariants

- We assume that an invariant curve and a cofactor are polynomials in one variable[†]:

$$R = \sum_{l} A_l(x) y^l, \quad \lambda = \sum_{l} B_l(x) y^l$$

- If it is possible, we find an upper bound on the degree of an invariant curve.
- We obtain a system of *linear* ordinary differential and algebraic equations for the coefficients of an invariant curve and its cofactor.

For example, the quadratic system

$$w_\tau = u, \quad u_\tau = -2wu + u + w^2 - \beta^2,$$

has invariant curves

$$H_4 = u + w^2 - \beta^2, \quad \lambda_4 = 1,$$

$$H_5 = (w-\beta) I_{2\beta}\{2\sqrt{H_4}\} - \sqrt{H_4} I_{2\beta+1}\{2\sqrt{H_4}\}, \quad \lambda_5 = -w,$$

$$H_6 = (w-\beta) K_{2\beta}\{2\sqrt{H_4}\} + \sqrt{H_4} K_{2\beta+1}\{2\sqrt{H_4}\}, \quad \lambda_6 = -w.$$

[†] I. García,J. Giné Appl. Math. Lett 16 (2003); J. Giné, M. Grau, J. Llibre Nonlinearity 26 (2013).

Example

Consider the following dynamical system

$$w_t = w(w-1)(w^2+u) + (b-1)w - a,$$
$$u_t = -(2w^3 - 2w^2 - 1)(w^2+u) - 2w(bw - w - a).$$

Here $a, b \notin \mathbb{Z}$ and $a - b \neq -(1+q)$, $q \in \mathbb{N} \cup \{0\}$.
Using variables

$$z = w, \quad y = u + w^2,$$

we obtain

$$y_t = y, \quad z_t = yz^2 + (b-1-y)z - a.$$

Since the maximal degree of an invariant curve with respect to z is 2 we look for

$$R = A(y)z^2 + B(y)z + C(y).$$

Example continued

One can construct quasi-polynomial invariant curves

$$\begin{aligned} R_1 =& (z-1)\mathcal{M}(a,b,y) - z\mathcal{M}(a+1,b,y), \\ R_2 =& y^{1-b}\left\{([y+a-b]z+a)\mathcal{M}(a-b+1,2-b,y) - \right. \\ & \left. (a-1)z\mathcal{M}(a-b,2-b,y)\right\}. \end{aligned}$$

The first integral is

$$I = R_1 R_2^{-1}.$$

The invariants have essential singularity at ∞.

Proposition

This dynamical system is not globally Puiseux integrable up to an arbitrary algebraic change of the variables at the values of the parameters a and b for which the Kummer function does not degenerate.

Summary

- Equivalence criteria for projective equations can be used study integrability of two-dimensional Riemannian metrics.
- There is new class of generalized-Darboux integrable geodesic flows.
- There are families of metrics with explicit geodesics, some of which correspond to applied nonlinear oscillators.
- Using quasi-polynomial invariants one can extend exiting integrability theory.

Thank you for your attention!*

Instituto Biofisika Institutua

ikerbasque Basque Foundation for Science

Funded by the European Union

*Funded by the European Union [H2020-MSCA-COFUND-2020-101034228-WOLFRAM2]. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or European Research Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.