Monotonous period function for equivariant differential equations with homogeneous nonlinearities

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V Symposium on Planar Vector Fields
January 12th 2026

This work was financially supported by the Ministry of Science and Innovation-State Research Agency of the Spanish Government through grants PID2022-136613NB-I00 and PID2023-146424NB-I00. It is also supported by the grant 2021-SGR-00113 from AGAUR of Generalitat de Catalunya.

Introduction

$\mathbb{Z}_k$-equivariant differential equations are equations $\dot{z} = F(z, \overline{z})$, $z \in \mathbb{C}$, that are invariant under a rotation through the angle $2\pi/k$ about the origin.

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We study the simplest family of polynomial $\mathbb{Z}_k$-equivariant differential equations with a non-degenerated center at the origin and homogeneous nonlinearities

$$\dot{z} = iz + (z\overline{z})^n z^{k+1}, \quad n, k \text{ positive integers}$$

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Gasull's 16th problem

Is the period function of the center at the origin of the differential equation $\dot{z} = iz + (z\overline{z})^n z^{k+1}$, with n and k positive integers, monotonous decreasing?

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Answer: YES

Connection with the quadratic reversible family

The period of the center at the origin of $\dot{z} = iz + (z\overline{z})^n z^{k+1}$ is strongly related with the period of the center at the origin of the reversible quadratic family

$$\begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + Dx^2 + Fy^2. \end{cases}$$

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Proposition

The behavior and the number of critical periods of the center at the origin of $\dot{z} = iz + (z\overline{z})^n z^{k+1}$ coincide with the one of the period function of the origin of the quadratic reversible centers with $F = 1 + D$ and $D = -k/(2(k+n)) \in (-1/2, 0)$.

Connection with the quadratic reversible family

Proof. Take polar coordinates $z = re^{i\theta}$ so $\dot{z} = iz + (z\overline{z})^n z^{k+1}$ writes

$$\frac{dr}{dt} = r^{2n+k+1}\cos(k\theta), \quad \frac{d\theta}{dt} = 1 + r^{2n+k}\sin(k\theta).$$

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The change $R = r^{2n+k}$ and $\Theta = k\theta$ and the constant rescaling of time $\tau = kt$, $k > 0$ brings the previous to

$$\frac{dR}{d\tau} = bR^2\cos(\Theta), \quad \frac{d\Theta}{d\tau} = 1 + R\sin(\Theta), \quad b := 1 + 2n/k.$$

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To Cartesian coordinates again, $X + iY = Re^{i\Theta}$,

$$\frac{dX}{d\tau} = -Y + bX^2 - Y^2, \quad \frac{dY}{d\tau} = X + (1+b)XY.$$

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Finally the change $x = -(1+b)Y, y = -(1+b)X$ and $s = -\tau$ brings to

$$\frac{dx}{ds} = -y + xy, \quad \frac{dy}{ds} = x - \frac{1}{1+b}x^2 + \frac{b}{1+b}y^2.$$

The monotonicity criterion

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is a center.Let the period function be parametrized by the energy level $h \mapsto \gamma(h) \subset \{(x,y) \in \mathbb{R}^2 : H(x,y) = h\}$. Thus the period function can be expressed an an Abelian integral,

$$T(h) := \int_{\gamma(h)} \frac{\ell(x)}{y}\,\mathrm{d}x.$$

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Denote by $\mathcal{I} = (x_\ell, x_r)$ the projection on the x-axis of the period annulus, $x_\ell < 0 < x_r$. The potential V defines an analytic involution σ on $\mathcal{I}$ by $V(\sigma(x)) = V(x)$ for all $x \in \mathcal{I} \setminus \{0\}$ and $\sigma(0) = 0$.

The monotonicity criterion

Criterion

Let us consider an Abelian integral of the form
$I(h) = \int_{\gamma(h)} g(x)/y \, \mathrm{d}x$, where $g(x)$ is an analytic function on $\mathcal{I}$.
Define

$$\Pi_\sigma(g)(x) := \frac{f(x) - f(\sigma(x))\sigma'(x)}{2}, \quad f(x) := -\frac{g(x)}{2} + \left(\frac{g(x)V(x)}{V'(x)}\right)'.$$

If $\Pi_\sigma(g)(x)$ has no zeros in $(0, x_r)$ then $I'(h)$ has no zeros.

Jinming Li, Chengzhi Li, Changjian Liu, Dechen Wang.
The period function of reversible Lotka-Volterra quadratic centers, J. Differential Equations 307, 556-–579 (2022)

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Francesc Mañosas and Jordi Villadelprat. *Criteria to bound the number of critical periods*, J. Differential Equations 246, 2415–2433 (2009)

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Antonio Garijo and Jordi Villadelprat. *Algebraic and analytical tools for the study of the period function*, J. Differential Equations 257, 2464–2484 (2014)

Application of the criterion

The change of variables $u = x, v = y(1-x)^{-(D+1)}$ transforms the reversible system with $F = D+1$ to

$$\dot{u} = -v(1-u)^{D+2}, \quad \dot{v} = u(1+Du)(1-u)^{-D+1},$$

with first integral $H(u,v) = \frac{1}{2}v^2 + V(u)$, where

$$V(u) := \frac{1}{2}u^2(1-u)^{-2(D+1)}, \quad \ell(u) := (1-u)^{-(D+2)}.$$

When $D \in (-1/2, 0)$ the projection of the period annulus is $(u_\ell, u_r) = (-\infty, 1)$.

Problem: The period function is an Abelian integral of the type requested by the Criterion, BUT the direct application is not possible, $\Pi_\sigma(\ell)(u)$ changes sign in $(0, u_r)$.

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The exterior derivative of the 1-form $\alpha := \ell(u) v \, \mathrm{d}u$ is $\mathrm{d}\alpha = -\ell(u) \, \mathrm{d}u \wedge \mathrm{d}v$ and it admits a Gelfand-Leray form with respect to H given by

$$\frac{\mathrm{d}\alpha}{\mathrm{d}H} = -\frac{\ell(u)}{V'(u)} \, \mathrm{d}v.$$

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$$\frac{\mathrm{d}\alpha}{\mathrm{d}H} = -\frac{\ell(u)}{V'(u)} \, \mathrm{d}v.$$

Thus, the derivative of the Abelian integral $A(h)$ can be writen as

$$A'(h) = \int_{\gamma(h)} \frac{\mathrm{d}\alpha}{\mathrm{d}H} = -\int_{\gamma(h)} \frac{\ell(u)}{V'(u)} \, \mathrm{d}v = \int_{\gamma(h)} \frac{\ell(u)}{v} \, \mathrm{d}u = T(h)$$

on account that $\mathrm{d}H = 0$ along $\gamma(h)$.

Application of the criterion

From $H(u,v) = \frac{1}{2}v^2 + V(u) = h$ we have $v = \frac{2}{v}(h - V(u))$ and

$$A(h) = \int_{\gamma(h)} \ell(u) v \, \mathrm{d}u = 2\int_{\gamma(h)} \frac{\ell(u)(h - V(u))}{v} \, \mathrm{d}u$$

$$= 2hT(h) - 2\int_{\gamma(h)} \frac{\ell(u)V(u)}{v} \, \mathrm{d}u.$$

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Also, integrating by parts,

$$A(h) = \int_{\gamma(h)} \ell(u) v \,\mathrm{d}u = \int_{\gamma(h)} v\,\mathrm{d}(L(u)) = -\int_{\gamma(h)} L(u)\,\mathrm{d}v$$
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$$=\int_{\gamma(h)}\frac{L(u)V'(u)}{v}\,\mathrm{d}u,\quad \text{with } L'(u)=\ell(u).$$

Adding both together and defining

$$I(h):=\int_{\gamma(h)}(L(u)V'(u)-2\ell(u)V(u))\frac{\mathrm{d}u}{v},$$

we have $2A(h)=2hT(h)+I(h)$ and deriving
$2hT'(h)+I'(h)=0.$

Application of the criterion

In our case:

$$I(h) = \int_{\gamma(h)} \frac{u(1-u)^{-3(D+1)}}{v}\,\mathrm{d}u,$$

so we will apply the Criterion with $g(u) := u(1-u)^{-3(D+1)}$.

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$$\sigma'(u) = \frac{V'(u)}{V'(\sigma(u))}.$$

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Therefore,

$$\Pi_\sigma(g)(u) = \frac{f(u) - f(\sigma(u))\sigma'(u)}{2} = \frac{V'(u)}{2}(\phi(u) - \phi(\sigma(u)))$$

with $\phi(u) = f(u)/V'(u)$.

Application of the criterion

To prove that $\Pi_\sigma(g)(u)$ has no zeros on $(0, u_r) = (0, 1)$ it is enough to show that the two curves defined by

$$V(u) - V(w) = 0 \quad \text{and} \quad \phi(u) - \phi(w) = 0$$

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The first can be written as

$$V(u) - V(w) = u^2(1-u)^{-2(D+1)} - w^2(1-w)^{-2(D+1)}$$

which vanishes if and only if

$$F_1(u, w; D) := u(1-u)^{-(D+1)} + w(1-w)^{-(D+1)} = 0$$

Application of the criterion

The second writes, after some manipulation using $F_1(u, w; D) = 0$, as

$$F_2(u, w; D) := Q(u) - Q(w)$$

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$$Q(u) = \frac{(1-u)(1 + 2Du + D(1+2D)u^2)}{u(1+Du)^3}.$$

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Technicality: Let $\Gamma_i(D) := \{(u, w) \in (0, 1) \times (-\infty, 0) : F_i(u, w; D) = 0\}$. Each $\Gamma_i(D)$ is the graphic of a function $w = \psi_i(u; D)$.

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Strategy of the proof: treat it as a bifurcation problem. Let $\mathcal{Z}(D)$ be the number of intersections of the two curves. We will see that $\mathcal{Z}(-1/3) = 0$ and that $\mathcal{Z}(D)$ is maintained for $D \in (-1/2, 0)$.

A toy model

$$P_{a,b}(x) = x^2 + ax + b, \quad a, b \in \mathbb{R}$$

Goal: Determine $\mathcal{Z}(a, b)$ the number of zeros of $P_{a,b}$ on $(0, 1)$.

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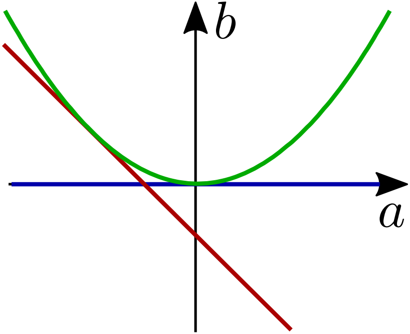
- $P_{a,b}(0) = 0$. Zeros emerging/disappearing from $x = 0$.
- $P_{a,b}(1) = 0$. Zeros emerging/disappearing from $x = 1$.
- $\Delta(P_{a,b}) = 0$. Collapsing of two zeros.

A toy model



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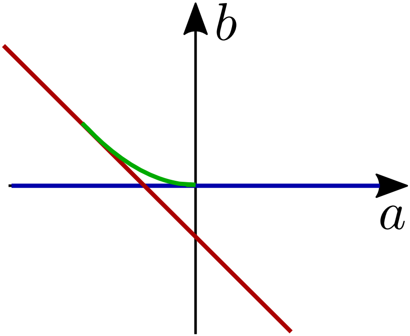

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From the intersection of curves POV: we need to control that no intersection bifurcates near $(u, w) = (0, 0)$, no intersection bifurcates near $(u, w) = (1, -\infty)$ and no intersection bifurcates near any $(u, w) \in (0, 1) \times (-\infty, 0)$.

Bifurcations from the inner boundary

Those are controlled by the local expansion of the curves ψ_1 and ψ_2 near $u = 0$.

Easily can be verified by

$$\psi_2(u; D) - \psi_1(u; D) \approx \frac{16}{3} D(D+1)(2D+1)u^4 + O(u^5)$$

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$$\psi_2(u; D) - \psi_1(u; D) \approx \frac{16}{3} D(D+1)(2D+1)u^4 + O(u^5)$$

Thus no intersection of the curves arbitrary close to $(u, w) = (0, 0)$.

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On account of the expression of $F_2(u,w;D) = Q(u) - Q(w)$ with

$$Q(u) = \frac{(1-u)(1+2Du+D(1+2D)u^2)}{u(1+Du)^3},$$

the sign of the polynomial $1+2Dw+D(1+2D)w^2$ implies that $F_2(u,w;D)=0$ cannot hold for $w < w^*(D) < 0$ with

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$$w^*(D) := \frac{-D + \sqrt{-D(1+D)}}{D(1+2D)}.$$

Therefore, for any fixed $D^* \in (-1/2, 0)$, no intersection near $(u, w) = (1, -\infty)$ can occur.

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$$F_1(u, w; -1/3) = \frac{u}{(1-u)^{2/3}} + \frac{w}{(1-w)^{2/3}},$$

and

$$F_2(u, w; -1/3) = \frac{3Q_2(u, w)}{uw(u-3)^3(w-3)^3},$$

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Thus we want to show that Q_1 and Q_2 have no common roots.

Bifurcations from the interior of the period annulus

The resultant

$$\mathrm{Res}(Q_1(u,w), Q_2(u,w), w) = 32(u-1)^3 u^6 R(u)$$

with $R(u)$ a polynomial of degree 12 with no real roots in $(0,1)$ (by Sturm's theorem).

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Now consider any $D \in (-1/2, 0)$ and assume, with aim of reaching contradiction, that there exists $D^* \in (-1/2, 0)$ and $u^* \in (0,1)$ with $\psi_1(u^*; D^*) = \psi_2(u^*; D^*)$.

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Bifurcations from the interior of the period annulus

Consequently, the result will follow once we show that

$$\psi_1(u;D) - \psi_2(u;D) = 0 \quad \text{and} \quad \psi_1'(u;D) - \psi_2'(u;D) = 0$$

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Equivalently,

$$F_2(u, w; D) = Q(u) - Q(w) = 0$$

and

$$F_3(u, w; D) := \frac{w(w-1)(1+Du)}{u(u-1)(1+Dw)} + \frac{P(u;D)}{P(w;D)} \frac{w^2(1+Dw)^4}{u^2(1+Du)^4} = 0$$

do not intersect on $w^*(D) < w < 0 < u < 1$ for any $D \in (-1/2, 0)$.

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Crucial fact: Both equalities are given in terms of rational functions.

Bifurcations from the interior of the period annulus

We define

$$P_2(u,w;D) := uw(1+Du)^3(1+Dw)^3 F_2(u,w;D), \text{ and}$$

$$P_3(u,w;D) := (u-1)u^2(1+Du)^4(1+Dw)P(w,D)F_3(u,w,D)/w.$$

$P_2(u,w;D)$ is a polynomial of degree 7 and $P_3(u,w;D)$ a polynomial of degree 11.

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$$\mathrm{Res}(P_2(u,w;D), P_3(u,w;D), w) = S(u;D)R(u;D)$$

with $S(u;D)$ not vanishing for $u \in (0,1)$ (easily checked) and $D \in (-1/2, 0)$ and $R(u;D)$ a polynomial of degree 12 in u.

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with $S(u;D)$ not vanishing for $u \in (0,1)$ (easily checked) and $D \in (-1/2, 0)$ and $R(u;D)$ a polynomial of degree 12 in u.

Claim. The polynomial $R(u;D)$ does not have zeros for $(u,D) \in (0,1) \times (-1/2, 0)$.

Bifurcations from the interior of the period annulus

Proof. Again by a bifurcation argument with respect to the parameter D.

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Proof. Again by a bifurcation argument with respect to the parameter D.
$R(0;D) = 54D(D+1)$, $R(1;D) = 4D(1+2D)^4(D+1)^9$ and

$$\Delta_u(D) \propto D^{43}(D+1)^{43}(1+2D)^{32}K_0(D)^3K_1(D)^2K_2(D)^2K_3(D)^2,$$

with $K_0(D) := 22D^2 + 22D + 1$,
$K_1(D) := 128D^4 + 256D^3 + 112D^2 - 16D - 3$,
and K_2 and K_3 polynomials of degree 14 and 8 respectively.

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K_0 and K_1 can be solved explicitly and the only roots in $(-1/2, 0)$ are

$$D_0 := -\frac{1}{2} + \frac{3\sqrt{11}}{22} \quad \text{and} \quad D_1 := -\frac{1}{2} + \frac{\sqrt{5-\sqrt{7}}}{4}.$$

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With Sturm's theorem we can prove that K_2 has a root $D_2 \approx -0.1279963$ and K_3 has no real roots on $(-1/2, 0)$.

Bifurcations from the interior of the period annulus

The three roots D_0, D_1 and D_2 split the interval $(-1/2, 0)$ in four sub-intervals where $R(u; D)$ has the same number of roots on each.

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Final step: dealing with D_i, $i = 0, 1, 2$. Take for instance D_2. To show that $R(u, D_2) < 0$ for all $u \in (0, 1)$ we first locate the root D_2 inside a suitable rational interval

$$D_2 \in [\underline{\delta}, \overline{\delta}] := [-16/125, -267/2086] \approx [-0.128, -0.127996].$$

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Then we construct a polynomial $U(u)$ with rational coefficients satisfying $R(u; D_2) < U(u) < 0$ by upper-bounding each monomial of $R(u; D)$ with $D = \underline{\delta}$ or $D = \overline{\delta}$ depending on the sign of the coefficient and the exponent of D.

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Many thanks for your attention

References:

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