Derivatives of the Separation Function of Generalized Saddle Connections

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joint work with
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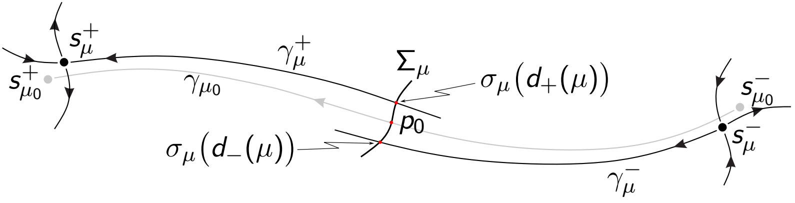
Classical setting

Separation function $d(\mu) = d_+(\mu) - d_-(\mu)$ of a saddle connection $\gamma_{\mu_0}(t)$ between $s^{\pm}_{\mu_0} = \lim_{t\to\pm\infty} \gamma_{\mu_0}(t)$ measured on the parametrized transverse sections $\sigma_\mu : (-\varepsilon, \varepsilon) \to \Sigma_\mu$ with $\sigma_{\mu_0}(0) = \gamma_{\mu_0}(0) = p_0$.



A classical formula asserts that the partial derivative $\partial_{\mu_j} d(\mu_0)$ can be computed by means of the (convergent) improper integral

$$\frac{1}{\partial_r \sigma_{\mu_0}(0) \wedge X_{\mu_0}(p_0)} \int_{-\infty}^{+\infty} e^{-\int_0^t \mathrm{div}(X_{\mu_0})(\gamma_{\mu_0}(s))ds} \left(X_{\mu_0} \wedge \partial_{\mu_j} X_{\mu_0}\right)(\gamma_{\mu_0}(t))dt,$$

where $u \wedge v = u_1 v_2 - u_2 v_1$.

Derivatives of separation function of a saddle connection

$$\partial_{\mu_j} d(\mu_0) = \frac{1}{\partial_r \sigma_{\mu_0}(0) \wedge X_{\mu_0}(p_0)} \left(\lim_{\tau \to +\infty} M_j(\tau) - \lim_{\tau \to -\infty} M_j(\tau) \right),$$

where

$$M_j(\tau) := \int_0^\tau e^{-\int_0^t \operatorname{div}(X_{\mu_0})(\gamma_{\mu_0}(s))ds} \left(X_{\mu_0} \wedge \partial_{\mu_j} X_{\mu_0} \right)(\gamma_{\mu_0}(t))dt.$$

Remark: If $s_\mu^\pm = \lim_{t \to \pm\infty} \gamma_\mu(t)$ are not hyperbolic saddles then $\lim_{\tau \to \pm\infty} M_j(\tau)$ may be infinite.

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Under some hypothesis on $s_\mu^\pm$ there exist a function $R_j^\pm(\tau)$ such that $L_\pm = \lim_{\tau \to \pm\infty} (M_j(\tau) - R_j^\pm(\tau))$ are finite and

$$\partial_{\mu_j} d(\mu_0) = \frac{L_+ - L_-}{\partial_r \sigma_{\mu_0}(0) \wedge X_{\mu_0}(p_0)}.$$

We can apply this formula for connections between some semi-hyperbolic saddles and many degenerate singularities $s_\mu^\pm$.

Generalized saddle separatrices

Let $\mathfrak{X} = \{X_\mu\}_{\mu\approx\mu_0}$ be a family of vector fields on $\Omega \subset \mathbb{R}^2$. A family $\Gamma = \{\Gamma_\mu\}_{\mu\approx\mu_0}$ of curves in Ω are *generalized stable/unstable saddle separatrices* for $\mathfrak{X}$ if Γ_μ is an orbit of X_μ and there exist

- a diffeomorphism $\Phi : U \subset \Omega \times \mathbb{R}^N \to \Phi(U) \subset \mathbb{R}^2 \times \mathbb{R}^N$ of the form $\Phi(x, y; \mu) = (\phi_\mu(x, y); \mu)$,
- a smooth positive function $g : \Phi(U) \to \mathbb{R}$, and
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such that, setting $U_\mu = \{p \in \mathbb{R}^2 : (p, \mu) \in U\}$, they satisfy

- $g(p; \mu)\big((\phi_\mu)_* X_\mu\big)(p) = \hat{X}_\mu(p)$ for all $p \in \phi_\mu(U_\mu) \cap \hat{\Omega}$,
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- $\hat{X}_{\mu_0}$ has a hyperbolic saddle $\hat{s}_{\mu_0}$ unfolding in a family of hyperbolic saddles $\{s_\mu\}_{\mu\approx\mu_0}$ of $\hat{\mathfrak{X}}$,
- $\exists q_\mu \in \Gamma_\mu \cap \phi_\mu^{-1}(\hat{\Omega})$ whose positive/negative semiorbit by X_μ is contained in U_μ and $\phi_\mu(q_\mu)$ belongs to a stable/unstable separatrix of $\hat{X}_\mu$ at $\hat{s}_\mu$ (all of them on the same side of the corresponding invariant manifold).

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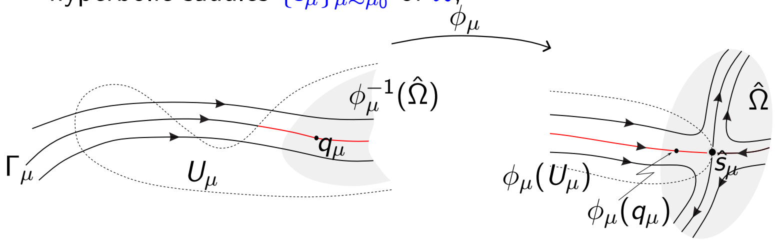

Examples of generalized saddle separatrices

- Classical setting: saddle separatrices, $\Phi = \mathrm{id}$, $g \equiv 1$, $\hat{\mathfrak{X}} = \mathfrak{X}$.
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Lemma: Assume that $\{s_\mu\}_{\mu\approx\mu_0}$ is a smooth family of singular points of $\mathfrak{X} = \{X_\mu\}_{\mu\approx\mu_0}$ such that the eigenvalues λ_μ, λ'_μ of $DX_\mu(s_\mu)$ satisfy $\lambda_{\mu_0}(\lambda_{\mu_0} - \lambda'_{\mu_0}) > 0$. Then there are exactly two trajectories of X_μ arriving to s_μ tangent to the eigenspace of λ_μ and, as μ varies, each one forms a family of generalized saddle separatrices for $\mathfrak{X}$ taking $g \equiv 1$ and $\phi_\mu(p) = \varphi(p - s_\mu)$, where φ is a directional blow-up of the origin.

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Remark:

- If $\lambda_{\mu_0} > \lambda'_{\mu_0} > 0$ then s_{μ_0} is a *node* of X_{μ_0}.
- If $\lambda_{\mu_0} > \lambda'_{\mu_0} = 0$ then s_{μ_0} is a *semi-hyperbolic singularity*.

Generalized saddle connection and its separation function

$\Gamma^{\pm} = \{\Gamma^{\pm}_{\mu}\}_{\mu\approx\mu_0}$ family of generalized $\pm$ saddle separatrices for $\mathfrak{X}$.
$\sigma = \{\sigma_{\mu}\}_{\mu\approx\mu_0}$ family of parametrized transverse sections to $\mathfrak{X}$ with $\sigma_{\mu_0}(0) \in \Gamma^{\pm}_{\mu_0}$ and $\sigma_{\mu} : (-\varepsilon, \varepsilon) \to \Sigma_{\mu} \subset \Omega$. Then there exist smooth functions $d_{\pm}(\mu)$ such that, for $\mu \approx \mu_0$,

$$\Sigma_{\mu} \cap \Gamma^{\pm}_{\mu} = \{\sigma_{\mu}(d_{\pm}(\mu))\}.$$

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If $\Gamma^{+}_{\mu_0} = \Gamma^{-}_{\mu_0} =: \Gamma_{\mu_0}$ we say that the families Γ^{+} and Γ^{-} have a *generalized saddle connection* at Γ_{μ_0} and its *separation function* measured on the family σ is

$$d(\mu) = d_{+}(\mu) - d_{-}(\mu).$$

Notice that $d_{\pm}(\mu_0) = 0$ and $d(\mu_0) = 0$.

Main result

Let $\{\Gamma_\mu^\pm\}_{\mu\approx\mu_0}$ be generalized $\pm$ separatrices for $\{X_\mu\}_{\mu\approx\mu_0}$ having a generalized saddle connection at $\Gamma_{\mu_0} = \gamma_{\mu_0}\big((T_-, T_+)\big)$. Let $d(\mu)$ be its separation function measured on the parametrized transverse sections $\{\sigma_\mu : (-\varepsilon, \varepsilon) \to \Sigma_\mu\}_{\mu\approx\mu_0}$ with $p_0 := \sigma_{\mu_0}(0) = \gamma_{\mu_0}(0)$.

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If $g_\mu^\pm\cdot\big((\phi_\mu^\pm)_*X_\mu\big)=\hat{X}_\mu^\pm$ define

$$R_j^\pm(\tau):=\Big(X_{\mu_0}\wedge(D\phi_{\mu_0}^\pm)^{-1}(\partial_{\mu_j}\phi_{\mu_0}^\pm)\Big)(\gamma_{\mu_0}(\tau))e^{-\int_0^\tau \operatorname{div}(X_{\mu_0})(\gamma_{\mu_0}(t))dt}.$$

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Then

$$\partial_{\mu_j} d(\mu_0) = \frac{1}{\sigma'_{\mu_0}(0)\wedge X_{\mu_0}(p_0)}\Big(\lim_{\tau\to T_+}(M_j(\tau)+R_j^+(\tau)) - \lim_{\tau\to T_-}(M_j(\tau)+R_j^-(\tau))\Big),$$

where recall that

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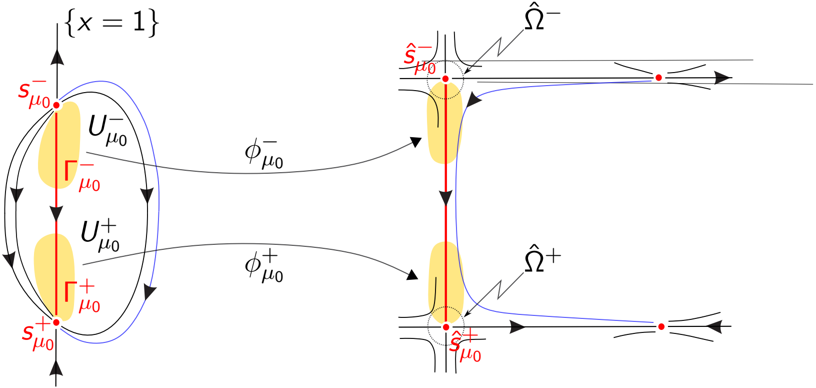
Remark: If $\partial_{\mu_j}\phi^{\pm}_{\mu_0} \equiv 0$ then $R^{\pm}_{j} \equiv 0$.

Example 1: heteroclinic connection between two nodes

$$X_\mu = (\varepsilon_0 - y + xy)\partial_x + (x + Dx^2 + Fy^2 + \varepsilon_1 y + \varepsilon_2 xy)\partial_y,$$

$\mu = (D, F, \varepsilon_0, \varepsilon_1, \varepsilon_2) \approx \mu_0 = (D_0, F_0, 0, 0, 0)$ with $D_0 < -1$ and $F_0 > \frac{1}{2}$.



The eigenvalues $\lambda_1^\pm$ and $\lambda_2^\pm$ of $A_\pm = DX_{\mu_0}(s^\pm_{\mu_0})$ satisfy

$$\mp\lambda_1^\pm > \mp\lambda_2^\pm > 0 \text{ and } \ker(A_\pm - \lambda_1^\pm \mathrm{Id}) = \langle(0, 1)\rangle$$

and we can apply the previous Lemma.

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The separation function $d(\mu)$ of the generalized saddle connection $\Gamma_{\mu_0} \subset \{x = 1\}$ measured on the transverse sections $\sigma_\mu(r) = (1 + r, 0)$ satisfies $\boxed{d(\mu) = \partial_{\varepsilon_0} d(\mu_0)\varepsilon_0 + \mathrm{o}(\varepsilon_0)}$ ($x = 1$ is invariant by $X_\mu|_{\varepsilon_0 = 0}$).

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$\gamma_{\mu_0}(t) = (1, y(t))$ with $\frac{dy}{dt} = D_0 + 1 + F_0 y(t)^2,\ T_\pm = \pm\infty$.

$$M_3(\tau) = -\int_0^{y(\tau)} \left(1 + \frac{F_0}{D_0 + 1} y^2\right)^{-1-\frac{1}{2F_0}} dy,$$

$$R_3^\pm(\tau) = \mp\sqrt{-F_0(D_0 + 1)} \left(1 + \frac{F_0}{D_0 + 1} y(\tau)^2\right)^{-\frac{1}{2F_0}}$$

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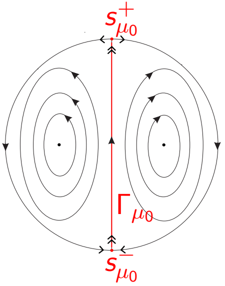
diverge as $\tau \to \pm\infty$ because $y(\tau)^2 \to -\frac{D_0+1}{F_0}$ and $F_0 > 0$, but

$$\partial_{\varepsilon_0} d(\mu_0) = \sqrt{\frac{\pi(D_0 + 1)}{-F_0}} \frac{\Gamma(-\frac{1}{2F_0})}{\Gamma(-\frac{1}{2F_0} + \frac{1}{2})} \neq 0 \quad \text{for } F_0 \neq 1.$$

Example 2: heteroclinic connection between saddle-nodes

$X_\mu = (\varepsilon - y + xy)\partial_x + (x + Dx^2 + y^2)\partial_y$ with $\mu = (D, \varepsilon) \approx \mu_0 = (D_0, 0)$ and $D_0 > -1$.



$(u, v) = \psi_\pm(x, y) = (\pm\frac{1-x}{y}, \pm\frac{1}{y})$ satisfies $(\psi_\pm)_* X_\mu = \pm\frac{1}{v}\bar{X}_\mu$,

$\bar{X}_\mu = (-\varepsilon v^2 - Du^3 + (2D+1)u^2 v - (D+1)uv^2)\partial_u + (-1 - Du^2 + (2D+1)uv - (D+1)v^2)v\partial_v.$

The eigenvalues of $D\bar{X}_\mu(0,0)$ are $\lambda_\mu = -1$ and $\lambda'_\mu = 0$ with eigenspaces $u = 0$ and $v = 0$. In this case the diffeomorphisms $\phi_\pm = \varphi \circ \psi_\pm$ do not depend on μ.

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$\gamma_{\mu_0}(t) = (1, y(t))$ with $y'(t) = D_0 + 1 + y(t)^2$, $t \in (T_-, T_+)$,
$\mathrm{div}(X_\mu) = 3y$, $\exp\left(-\int_0^t \mathrm{div}(X_{\mu_0})(\gamma_{\mu_0}(s))ds\right) = \left(1 + \frac{y(t)^2}{D_0+1}\right)^{-\frac{3}{2}}$,
$(X_{\mu_0} \wedge \partial_\varepsilon X_{\mu_0})(\gamma_{\mu_0}(t)) = -(1 + D_0 + y(t)^2)$.

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$X_\mu = (\varepsilon - y + xy)\partial_x + (x + Dx^2 + y^2)\partial_y$ with $\mu = (D, \varepsilon) \approx \mu_0 = (D_0, 0)$ and $D_0 > -1$. The separation function of the generalized saddle connection $\Gamma_{\mu_0} = \{x = 1\}$ measured on the transverse sections $\sigma_\mu(r) = (1 - r, 0)$ satisfies

$$\boxed{d(\mu) = \partial_\varepsilon d(\mu_0)\varepsilon + \mathrm{o}(\varepsilon).}$$

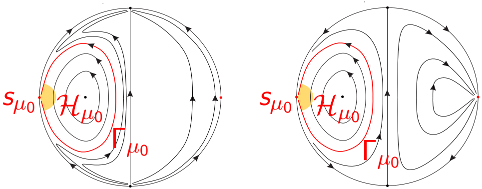
$\gamma_{\mu_0}(t) = (1, y(t))$ with $y'(t) = D_0 + 1 + y(t)^2$, $t \in (T_-, T_+)$, $\mathrm{div}(X_\mu) = 3y$, $\exp\left(-\int_0^t \mathrm{div}(X_{\mu_0})(\gamma_{\mu_0}(s))ds\right) = \left(1 + \frac{y(t)^2}{D_0+1}\right)^{-\frac{3}{2}}$, $(X_{\mu_0} \wedge \partial_\varepsilon X_{\mu_0})(\gamma_{\mu_0}(t)) = -(1 + D_0 + y(t)^2)$. Thus,

$$\partial_\varepsilon d(\mu_0) = \frac{1}{D_0+1}\int_{T_-}^{T_+} \left(1 + \frac{y(t)^2}{D_0+1}\right)^{-\frac{3}{2}} (1 + D_0 + y(t)^2)dt$$

$$= \frac{1}{D_0+1}\int_{-\infty}^{+\infty} \left(1 + \frac{y^2}{D_0+1}\right)^{-\frac{3}{2}} dy = \frac{2}{\sqrt{D_0+1}} \neq 0.$$

Example 3: homoclinic connection of a nilpotent singularity

$X_\mu = -y(1-x)\partial_x + (x + \varepsilon y + Fy^2)\partial_y$ with $\mu = (F, \varepsilon) \approx \mu_0 = (F_0, 0)$ and $F_0 > \frac{1}{2}$.



$F_0 > 1$ $F_0 \in (\frac{1}{2}, 1)$

Homoclinic connection $\Gamma_{\mu_0} = \{x = \frac{1-2F_0}{2} y^2 + \frac{1}{2F_0}\} = \partial \mathcal{H}_{\mu_0}$, $\mathcal{H}_{\mu_0}$ hyperbolic sector of $s_{\mu_0} \in \ell_\infty$ meeting $\{y = 0,\ x < 0\}$.

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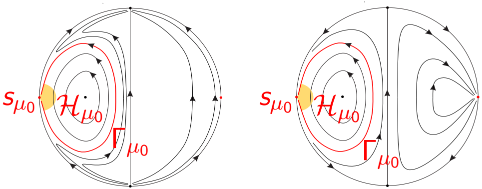

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$$u_1 \cdot (\phi_1)_* X_\mu = -\left(v_1 u_1 \partial_{u_1} + (u_1 - u_1^2 - \varepsilon u_1 v_1 - (F-1)v_1^2)\partial_{v_1}\right)$$

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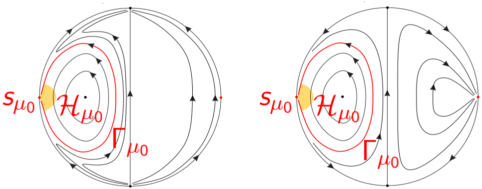

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has a nilpotent singularity at $(u_1, v_1) = (0,0)$, and the blow-ups $(u_2, v_2) = \phi_2(u_1, v_1) = (\frac{u_1}{v_1}, v_1)$ and $(u_3, v_3) = \phi_3(u_2, v_2) = (u_2, \frac{v_2}{u_2})$

$$u_3 v_3 \cdot \phi_* X_\mu = \hat{X}_\mu = (1 - Fv_3 - \varepsilon u_3 v_3 - u_3^2 v_3)u_3\partial_{u_3} + (-2 + (2F-1)v_3 + 2\varepsilon u_3 v_3 + 2u_3^2 v_3)v_3\partial_{v_3},$$

where $\phi = \phi_3 \circ \phi_2 \circ \phi_1$ does not depend on μ.

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$$\boxed{d(\mu) = \partial_\varepsilon d(\mu_0)\varepsilon + \mathrm{o}(\varepsilon)}$$

because the parabola $\{x = \frac{1-2F}{2}y^2 + \frac{1}{2F}\}$ is invariant by $X_{(F,0)}$.

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$\gamma_{\mu_0}(t) = \left(\frac{1-2F_0}{2} y(t)^2 + \frac{1}{2F_0}, y(t)\right)$, $t \in (T_-, T_+)$, $y'(t) = \frac{1}{2} y(t)^2 + \frac{1}{2F_0}$,

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$$\exp\left(-\int_0^t \mathrm{div}(X_{\mu_0})(\gamma_{\mu_0}(s))ds\right) = \left(1 + F_0 y(t)^2\right)^{-(1+2F_0)},$$

$$(X_{\mu_0} \wedge \partial_\varepsilon X_{\mu_0})(\gamma_{\mu_0}(t)) = -y(t)^2(1 - x(t)) = (1-2F_0)y(t)^2 y'(t),$$

$$\partial_\varepsilon d(\mu_0) = 2F_0(2F_0-1)\int_{-\infty}^{+\infty}(1+F_0y^2)^{-(1+2F_0)}y^2 dy = \frac{(2F_0-1)\sqrt{\pi}}{\sqrt{F_0}}\frac{\Gamma(2F_0 - \frac{1}{2})}{\Gamma(2F_0+1)} \neq 0.$$

Strategy of the proof of the main result

Direct consequence of the following formula for $\partial_{\mu_j} d_\pm(\mu_0; \mathfrak{X}, \Gamma^\pm, \sigma)$:

$$\partial_{\mu_j} d_\pm(\mu_0) = \frac{1}{\partial_r \sigma_{\mu_0}(0) \wedge X_{\mu_0}(p_0)} \left(\lim_{\tau \to T_\pm} \left(M_j(\tau) + R_j^\pm(\tau) \right) - \partial_{\mu_j} \sigma_{\mu_0}(0) \wedge X_{\mu_0}(p_0) \right),$$

which holds in the classical setting ($T_\pm = \pm\infty$, $R_j^\pm \equiv 0$).

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$$d_\pm(\mu; \mathfrak{X}, \Gamma^\pm, \sigma) = \qquad d_\pm(\mu; g\Phi_*^\pm \mathfrak{X}, \Phi^\pm(\Gamma^\pm), \Phi^\pm \circ \sigma)$$

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$$\begin{aligned} \partial_{\mu_j} d_\pm(\mu; \mathfrak{X}, \Gamma^\pm, \sigma) &= \partial_{\mu_j} d_\pm(\mu; g\Phi_*^\pm \mathfrak{X}, \Phi^\pm(\Gamma^\pm), \Phi^\pm \circ \sigma) \\ &\overset{\text{CS}}{=} \lim_{\tau \to \pm\infty} \Delta_j(\tau; g\Phi_*^\pm \mathfrak{X}, \Phi^\pm \circ \sigma) \\ &\overset{\text{L}}{=} \lim_{\tau \to T_\pm} \left(\Delta_j(\tau; \mathfrak{X}, \sigma) + \frac{R_j^\pm(\tau)}{\partial_r \sigma_{\mu_0}(0) \wedge X_{\mu_0}(p_0)} \right). \end{aligned}$$

Auxiliary lemmas in the proof of the main result

Lemma: $\Delta_j(\tau; g \cdot \mathfrak{X}, \sigma) = \Delta_j(h(\tau); \mathfrak{X}, \sigma)$, where $h'(t) = g(\gamma_{\mu_0}(h(t)))$ and $h(0) = 0$, so that

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$$R_j(\tau) = \Big(X_{\mu_0} \wedge (D\phi_{\mu_0})^{-1}(\partial_{\mu_j}\phi_{\mu_0})\Big)(\gamma_{\mu_0}(\tau))e^{-\int_0^\tau \mathrm{div}(X_{\mu_0})(\gamma_{\mu_0}(s))ds}.$$

Application

Proposition: Family of polynomial vector fields of even degree n:

$$X_\mu = P(x,y)\partial_x + (yq(x,y) + \mu_1 + \mu_2 x + \cdots + \mu_{n+1}x^n)\partial_y,$$

$\mu \in (\mathbb{R}^{n+1}, 0)$, $P(x,0) \neq 0$ and $P_n(1,0)(P_n(1,0) - q_{n-1}(1,0)) < 0$.

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$$\partial_{\mu_j} d(0) = -\int_{-\infty}^{+\infty} \frac{x^{j-1} e^{-\int_0^x \frac{q(u,0)}{P(u,0)} du}}{|P(x,0)|} dx.$$

In particular, $\partial_{\mu_j} d(0) < 0$ if j is odd.

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Corollary: For any $D \in (-1, 0)$ and $F \in (0, 1)$ there exists a sequence $(\varepsilon_{0,n}, \varepsilon_{1,n}, \varepsilon_{2,n}) \in \mathbb{R}^3$ tending to $(0,0,0)$ such that

$$(\varepsilon_{0,n} - y + xy)\partial_x + (x + Dx^2 + Fy^2 + \varepsilon_{1,n} y + \varepsilon_{2,n} xy)\partial_y$$

has two non-nested limit cycles.

Two non-nested limit cycles

Corollary: For any $D \in (-1, 0)$ and $F \in (0, 1)$ there exists a sequence $\mu_n = (\varepsilon_{0,n}, \varepsilon_{1,n}, \varepsilon_{2,n}) \in \mathbb{R}^3$ tending to $(0, 0, 0)$ such that

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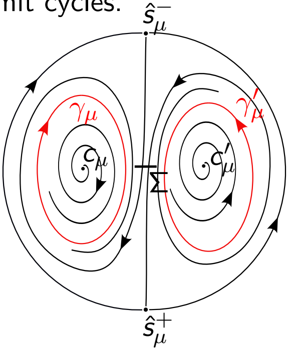


$$d(\mu) = \partial_{\varepsilon_0} d(0)\varepsilon_0 + \cdots, \qquad \partial_{\varepsilon_0} d(0) < 0,$$

$$\mathrm{tr}(DX_\mu)(c_\mu) = (2F+1)\varepsilon_0 + \varepsilon_1 + \cdots,$$

$$\mathrm{tr}(DX_\mu)(c_\mu') = \tfrac{D(2F+1)}{D+1}\varepsilon_0 + \varepsilon_1 - \tfrac{1}{D}\varepsilon_2 + \cdots$$

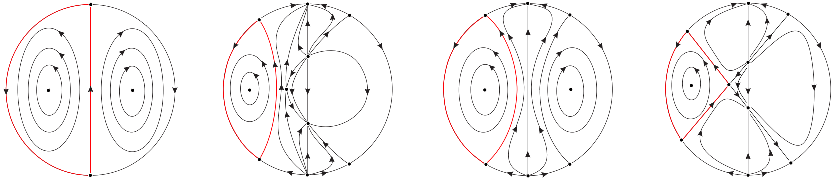
Breaking the outer boundary of Loud centers

Loud family of centers $Y_{D,F} = y(x-1)\partial_x + (x + Dx^2 + Fy^2)\partial_y$ admits the versal quadratic unfolding

$$X_{D,F,\varepsilon_0,\varepsilon_1,\varepsilon_2} = Y_{D,F} + \varepsilon_0\partial_x + (\varepsilon_1 y + \varepsilon_2 xy)\partial_y.$$

Phase portrait in the Poincaré disc of $Y_{D,F}$ for

(1) $D \in (-1,0)$ and $F \in (0,1)$,
(2) $-F < D < -1$,
(3) $D \in (-1,0)$ and $F > 1$,
(4) $-F = D < -1$.



(1) (2) (3) (4)

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Thanks for your attention!