Growth estimate for the number of crossing limit cycles in planar piecewise polynomial vector fields

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Teoria Qualitativa e Aplicações

Hilbert's 16th Problem

Given a polynomial differential system

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

$\pi(P,Q)$ denote the number of limit cycles of this system. The **Hilbert number** is defined as $\mathcal{H}(n) := \sup\{\pi(P,Q) : \mathrm{degree}(P), \mathrm{degree}(Q) \leq n\}$.
We know that...

- $\mathcal{H}(1) = 0$
- Some lower bounds for $\mathcal{H}(n)$ for small values of n: $\mathcal{H}(2) \geq 4$, $\mathcal{H}(3) \geq 13$, $\mathcal{H}(4) \geq 28, \mathcal{H}(5) \geq 37, \mathcal{H}(6) \geq 53, \mathcal{H}(7) \geq 74, \mathcal{H}(8) \geq 96, \mathcal{H}(9) \geq 120\ldots$
  Prohens–Torregrosa, Nonlinearity, 2019.
- $\mathcal{H}(n)$ grows as fast as $K\,n^2 \log n$ for some $K > 0$ or, more specifically,

  $$\liminf_{n\to\infty} \frac{\mathcal{H}(n)}{(n+2)^2 \log(n+2)} \geq \frac{1}{2\log 2}.$$

  Christopher–Lloyd, Proc. Roy. Soc. London Ser. A, 1995. Before that, the growth estimate was quadratic.
- $\mathcal{H}(n)$ is **strictly** increasing. More specifically, $\mathcal{H}(n+1) \geq 1 + \mathcal{H}(n)$, if $\mathcal{H}(n) < \infty$.
  Gasull–Santana, Proc. Am. Math. Soc., 2025.

Hilbert's 16th Problem in Piecewise Polynomial Setting

Consider the piecewise polynomial differential system

$$Z : \begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

where

$$P(x,y) = \begin{cases} P^+(x,y), & x > 0, \\ P^-(x,y), & x < 0, \end{cases} \qquad Q(x,y) = \begin{cases} Q^+(x,y), & x > 0, \\ Q^-(x,y), & x < 0, \end{cases}$$

Define $\deg P := \max\{\deg P^+, \deg P^-\}$ (analogously for $\deg Q$) and denote by $\pi_c(P,Q)$ the number of crossing limit cycles of Z. The analogue of the Hilbert number in this discontinuous setting is defined as

$$\mathcal{H}_c(n) := \sup\{\, \pi_c(P,Q) : \deg P, \deg Q \leq n \,\}.$$

We know that...

- Some lower bounds for $\mathcal{H}(n)$ for small values of n:
  - $\mathcal{H}_c(1) \geq 3$ Huan–Yang, Discrete Contin. Dyn. Syst., 2012. Llibre–Ponce. Dyn. Contin. Discrete Impuls. Syst. Ser. B, 2012.
  - $\mathcal{H}_c(2) \geq 12$ Braun–Cruz–Torregrosa, Nonlinear Anal. Real World Appl., 2024.
  - $\mathcal{H}_c(3) \geq 24$ Gouveia–Torregrosa, Appl. Math. Lett., 2020.
- $\mathcal{H}_c(1) < \infty$. Carmona–F.Sánchez–Novaes, Appl. Math. Lett., 2023.
- $\mathcal{H}_c(n) \geq 2n + 1$. Buzzi–Lima–Torregrosa, Physica D, 2018.

We want to show that...

Theorem A

$\mathcal{H}_c(n)$ grows at least as fast as $n^2/4$. Furthermore, if $\mathcal{H}_c(n) < \infty$ for some $n \in \mathbb{N}$, then $\mathcal{H}_c(n+1) \geq 1 + \mathcal{H}_c(n)$. In other words, $\mathcal{H}_c(n)$ is strict increasing if finite.

Hilbert's 16th Problem in Piecewise Polynomial Hamiltonian Setting

There are many works dealing with the piecewise polynomial Hamiltonian case, that is, assuming that (P^+, Q^+) and (P^-, Q^-) are Hamiltonian vector fields. In this setting, obtaining upper bounds is a feasible problem, since it reduces to an algebraic problem involving the Hamiltonian functions. The difficult issue is the optimality of these upper bounds, which is related to obtaining sharp lower bounds.

$$\widehat{\mathcal{H}}_c(n) := \sup\{\pi_c(P, Q) : (P^{\pm}, Q^{\pm}) \text{ are Hamiltonian v.f. and } \deg P, \deg Q \leq n\}.$$

We know that...

- $\widehat{\mathcal{H}}_c(n) \geq n - 1$. Yang–Han–Huang, J. Differ. Equ., 2011. Li–Llibre, J. Dyn. Differ. Equ., 2023.

We want to show that...

Theorem B

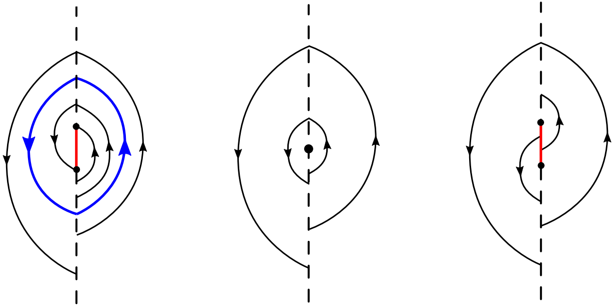
$\widehat{\mathcal{H}}_c(n)$ *grows at least as fast as* $(n \log n)/(2 \log 2)$.

Pseudo-Hopf Bifurcation

Consider the one-parameter family of $\mathcal{C}^1$ piecewise vector fields

$$Z_b(x,y) = \begin{cases} Z^+(x, y-b), & x > 0, \\ Z^-(x,y), & x < 0, \end{cases}$$

Assume that, for $b = 0$, the origin is a monodromic equilibrium of Z_0. We set $\ell = -1$ if the origin is attracting, $\ell = 1$ if it is repelling, and $\ell = 0$ otherwise. Also, $a = 1$ if the trajectories near the origin rotate counterclockwise, and $a = -1$ otherwise. Then, if $a\,b\,\ell < 0$ and b is small, Z_b admits a limit cycle surrounding a sliding segment:



$b < 0$ $\quad$ $b = 0$ $\quad$ $b > 0$

Outline of the Proof of Theorem A

Theorem A

$\mathcal{H}_c(n)$ grows at least as fast as $n^2/4$, that is,

$$\liminf_{n\to\infty} \frac{\mathcal{H}_c(n)}{n^2} \geq \frac{1}{4}. \quad (1)$$

Moreover, if $\mathcal{H}_c(n) < \infty$ for some $n \in \mathbb{N}$, then $\mathcal{H}_c(n+1) \geq 1 + \mathcal{H}_c(n)$.

Outline of the Proof of Theorem A

Proposition 1

There exists a sequence of piecewise polynomial vector fields $(Z_k)_{k\in\mathbb{N}}$ with degree $n_k = 2^k - 1$ and having $(n_k^2 - 1)/4$ crossing limit cycles.

- Let $(Z_k)_{k\in\mathbb{N}}$ be the sequence of polynomial vector fields provided by Christopher and Lloyd. For each k, Z_k has degree $n_k = 2^k - 1$ and possesses at least $(2^{k-1} - 1)^2$ hyperbolic limit cycles crossing the y-axis.
- More precisely, Z_k has $2^{k-1} - 1$ foci distributed along the y-axis, each surrounded by $2^{k-1} - 1$ small-amplitude hyperbolic limit cycles.
- Moreover, Z_k has the following form

$$Z_k(x,y) = \Big(X_k(y) + \varepsilon\big(P_{k-1}(x^2 - a_k, y^2 - b_k) + P_k(x)\big), Y_k(x)\Big),$$

where $(X_k(y), Y_k(x)) = (-\partial_y H_k(x,y), \partial_x H_k(x,y))$, H_k, P_{k-1}, and P_k are polynomials, a_k and b_k are suitably chosen parameters, and $\varepsilon \neq 0$ is a small parameter.

Outline of the Proof of Theorem A

- Computing the divergence of Z_k along the y-axis yields $\mathrm{div} Z_k(0, y) = \varepsilon P_k'(0)$. Hence, each of the $2^{k-1} - 1$ foci is hyperbolic, and all share the same stability.
- The independence of the second coordinate of $Z_k(x, y)$ from the variable y ensures that the orbits near all foci on the y-axis rotate in the same direction.
- Now, for each k, consider the one-parameter family of piecewise polynomial vector fields

$$Z_k^b(x,y) = \begin{cases} Z_k(x, y-b), & x > 0, \\ Z_k(x,y), & x \leq 0. \end{cases}$$

  Since, for $b = 0$, all foci of $Z_k^0 = Z_k$ on the y-axis have the **same stability** and the **nearby orbits rotate in the same direction**, they undergo a simultaneous **pseudo-Hopf bifurcation** as the parameter b crosses zero. This bifurcation creates **additional** $2^{k-1} - 1$ crossing limit cycles, each surrounding a sliding segment arising from the foci on the y-axis, while the original $(2^{k-1} - 1)^2$ hyperbolic ones **persist** for b sufficiently small as crossing limit cycles.
- Consequently, for each k, there exists a sufficiently small $b_k \neq 0$ such that $Z_k^{b_k}(x, y)$ exhibits $2^{k-1}(2^{k-1} - 1) = (n_k^2 - 1)/4$ crossing limit cycles.

Proposition 2

Assume that $\mathcal{H}_c(n) < \infty$ for some $n \in \mathbb{N}$. Then, there exists a piecewise polynomial vector field of degree n possessing exactly $\mathcal{H}_c(n)$ hyperbolic crossing limit cycles.

- Let $Z(x,y) = (P(x,y), Q(x,y))$ be a piecewise polynomial vector field, where

$$P(x,y) = \begin{cases} P^+(x,y), & x > 0, \\ P^-(x,y), & x \le 0, \end{cases} \qquad Q(x,y) = \begin{cases} Q^+(x,y), & x > 0, \\ Q^-(x,y), & x \le 0, \end{cases}$$

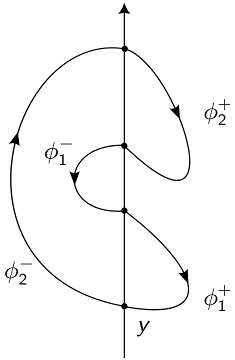
$P^\pm, Q^\pm$ are polynomials of degree n, and Z has exactly $\mathcal{H}_c(n) < \infty$ crossing limit cycles. Denote $Z^+ = Z|_{x>0}$ and $Z^- = Z|_{x \le 0}$, and consider the perturbation of Z, for b close to 0,

$$Z_b(x,y) = \begin{cases} Z^+(x, y-b), & x > 0, \\ Z^-(x,y), & x \le 0. \end{cases}$$

Outline of the Proof of Theorem A

- Let C be a crossing limit cycle of $Z_0 = Z$. At each point where C intersects the discontinuity set, a first return map is locally defined. Let $\pi_C : S_C \to S_C$, $S_C \subset \Sigma$, denote one of such maps and y_C its fixed point. Accordingly, $\{(0, y_C)\} = C \cap S_C$.
- π_C can be written as the composition of finitely many half-return maps determined by Z^+ and Z^-, namely,

$$\pi_C(y) = \phi_1^+ \circ \phi_1^- \circ \cdots \circ \phi_k^+ \circ \phi_k^-(y).$$



- Each of these maps is analytic and also strictly decreasing, thus $(\phi_i^{\pm})'(y) \leq 0$.

Outline of the Proof of Theorem A

- From transversality, the perturbed first return map remains well defined for $b \neq 0$ sufficiently small:

$$\pi_C(y;b) = \phi^+_{1,b} \circ \phi^-_1 \circ \cdots \circ \phi^+_{k,b} \circ \phi^-_k(y),$$

  where $\phi^+_{i,b}(y) = \phi^+_i(y-b)+b$, for $i = 1, \ldots, k$.
- In addition,

$$\begin{aligned}\frac{\partial}{\partial b}\pi_C(y;b) =& 1 - (\phi^+_1)' + (\phi^+_1)'(\phi^-_1)' - (\phi^+_1)'(\phi^-_1)'(\phi^+_2)' + \cdots \\ & - [(\phi^+_1)'(\phi^-_1)' \cdots (\phi^+_{k-1})'(\phi^-_{k-1})'(\phi^+_k)'] + [(\phi^+_1)'(\phi^-_1)' \cdots (\phi^+_k)'(\phi^-_k)'] \geq 1\end{aligned}$$

- Consider the displacement function associated with the crossing limit cycle C,

$$\delta_C(y,b) := \pi_C(y;b) - y.$$

  Since $\delta_C(\cdot,0)$ is analytic, expanding it at $y = y_C$ we get

$$\delta_C(y,0) = a_C(y-y_C)^{\ell_C} + (y-y_C)^{\ell_C+1}R_C(y),$$

  where $a_C \neq 0$, ℓ_C is a positive integer, and the function R_C is analytic.

Outline of the Proof of Theorem A

- Since,

$$\frac{\partial \delta_C}{\partial b}(y,b) = \frac{\partial \pi_b}{\partial b}(y) \geq 1.$$

  the isolated zero y_C unfolds differently in three distinct scenarios, namely:

  (O) ℓ_C is odd: one can find $\beta_C > 0$ for which the function $\delta_C(\cdot, b)$, for $b \in (-\beta_C, \beta_C)$, admits a curve of zeros $y_0 : (-\beta_C, \beta_C) \to \mathbb{R}$ satisfying $y_0(0) = y_C$. Moreover, for each $b \neq 0$, this zero is simple.

  (E^+) ℓ_C is even and $a_C > 0$: one can find $\beta_C > 0$ for which $\delta_C(\cdot, b)$, for $b \in (-\beta_C, \beta_C)$, admits two curves of zeros $y_1, y_2 : (-\beta_C, 0] \to \mathbb{R}$ satisfying $y_1(0) = y_2(0) = y_C$. Moreover, for each $b \neq 0$, these zeros are simple.

  (E^-) ℓ_C is even and $a_C < 0$: one can find $\beta_C > 0$ for which $\delta_C(\cdot, b)$ admits two curves of zeros $y_1, y_2 : (0, \beta_C] \to \mathbb{R}$ satisfying $y_1(0) = y_2(0) = y_C$. Moreover, for $b \neq 0$, these zeros are simple.

- We say that C is an O, E^+, or E^- crossing limit cycle of Z according to whether y_C is a zero of $\delta_C(\cdot, 0)$ of type O, E^+, or E^-, respectively.

Outline of the Proof of Theorem A

- Finally, denote by o, e^+, and e^- the numbers of O, E^+, and E^- crossing limit cycles of $Z_0 = Z$, respectively. Notice that $o + e^+ + e^- = \mathcal{H}_c(n)$. We may assume that $e^+ \geq e^-$. Taking the unfolding scenarios into account, we select the smallest value β^* among the values β_C corresponding to the O and E^+ crossing limit cycles C of Z_0. Hence, for each $b \in (-\beta^*, 0)$, the function $\delta_C(\cdot, b)$ associated with each O crossing limit cycle C possesses a simple zero, while for each E^+ crossing limit cycle C, the function $\delta_C(\cdot, b)$ has two simple zeros. Consequently, the piecewise vector field Z_b admits at least $o + 2e^+ \geq \mathcal{H}_c(n)$ hyperbolic crossing limit cycles.
- Since, by hypothesis, Z_b cannot have more than $\mathcal{H}_c(n) < \infty$ crossing limit cycles, it follows that Z_b has exactly $\mathcal{H}_c(n)$ crossing limit cycles, all of them hyperbolic.

Outline of the Proof of Theorem A

Proposition 3

Assume that $\mathcal{H}_c(n) < \infty$ for some $n \in \mathbb{N}$. Then, $\mathcal{H}_c(n+1) \geq \mathcal{H}_c(n) + 1$.

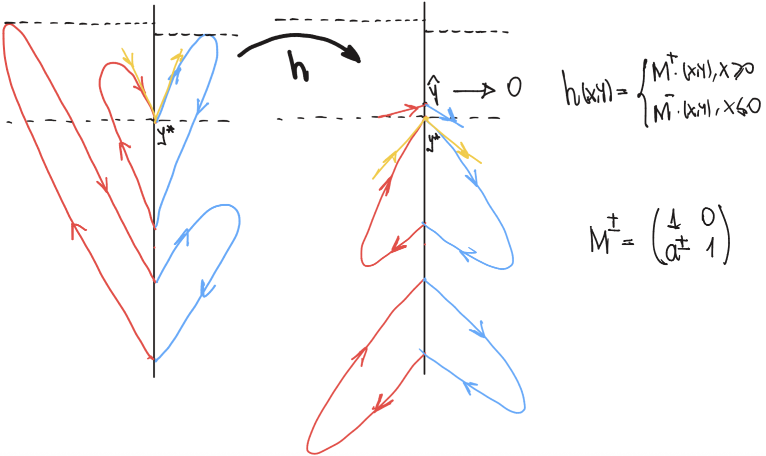
- Assume that $\mathcal{H}_c(n) < \infty$. Let $Z(x,y) = (P(x,y), Q(x,y))$ be the piecewise polynomial vector field

$$P(x,y) = \begin{cases} P^+(x,y), & x \geq 0, \\ P^-(x,y), & x \leq 0, \end{cases} \qquad Q(x,y) = \begin{cases} Q^+(x,y), & x \geq 0, \\ Q^-(x,y), & x \leq 0, \end{cases}$$

where $P^\pm, Q^\pm$ are polynomials of degree n and Z has exactly $\mathcal{H}_c(n)$ crossing limit cycles, al of them hyperbolic.

Outline of the Proof of Theorem A

- The following conditions (A) and (B) can be assumed without loss of generality:
  (A) All crossing limit cycles lie entirely within the open lower half-plane $\{(x, y) : \ y < 0\}$, and the origin is a crossing point, that is, $P^+(0,0)P^-(0,0) > 0$.



  (B) The following nondegeneracy conditions hold,

$$Q^+(0,0)Q^-(0,0) \neq 0, \quad \text{and} \quad P^+(0,0)Q^+(0,0) + P^-(0,0)Q^-(0,0) \neq 0,$$

Outline of the Proof of Theorem A

- Now, consider the following 1-parameter family of piecewise polynomial vector fields of degree $n+1$, $Z_\varepsilon(x,y) = (yP(x,y), Q_\varepsilon(x,y))$ where

$$Q_\varepsilon(x,y) = \begin{cases} Q_\varepsilon^+(x,y) = \left(y - \varepsilon P^+(0,0)Q^+(0,0)\right) Q^+(x,y), & x>0, \\ Q_\varepsilon^-(x,y) = \left(y + \varepsilon P^-(0,0)Q^-(0,0)\right) Q^-(x,y), & x<0. \end{cases}$$

- Notice that, for $\varepsilon = 0$, $Z_0(x,y) = yZ(x,y)$ and, then, $Z_0|_{y<0}$ corresponds to a time reparametrization of $Z|_{y<0}$. Taking condition (A) into account, this implies that $Z_0|_{y<0}$ has exactly $\mathcal{H}_c(n)$ hyperbolic crossing limit cycles and, therefore, for $\varepsilon > 0$ sufficiently small Z_ε has at least $\mathcal{H}_c(n)$ hyperbolic crossing limit cycles contained the open lower half-plane $\{(x,y) : \; y<0\}$.

Outline of the Proof of Theorem A

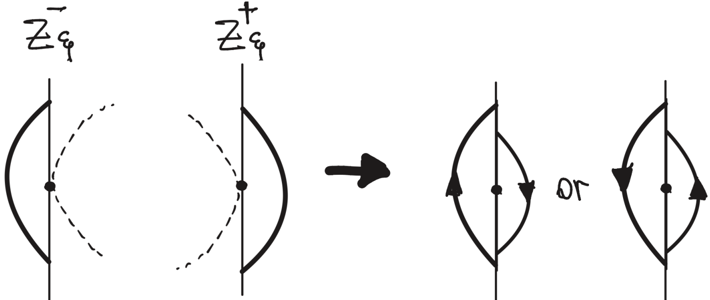
- For $\varepsilon > 0$, the origin is a **monodromic two-fold singularity**. Indeed, denoting $Z_\varepsilon^+ = Z_\varepsilon|_{x\geq 0}$, $Z_\varepsilon^- = Z_\varepsilon|_{x\leq 0}$, and $h(x,y) = x$, from conditions (A) and (B) we get

$$Z_\varepsilon^+ h(0,0) = Z_\varepsilon^- h(0,0) = 0,$$
$$(Z_\varepsilon^+)^2 h(0,0) = -\varepsilon (P^+(0,0))^2 (Q^+(0,0))^2 < 0,$$
$$(Z_\varepsilon^-)^2 h(0,0) = \varepsilon (P^-(0,0))^2 (Q^-(0,0))^2 > 0.$$

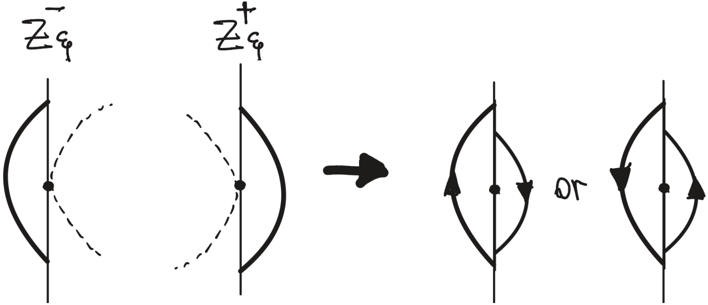
Thus, the origin is an invisible two-fold. In addition, from (A),

$$Q_\varepsilon^+(0,0) Q_\varepsilon^-(0,0) = -\varepsilon^2 P^+(0,0) P^-(0,0) (Q^+(0,0))^2 (Q^-(0,0))^2 < 0,$$

which guarantees the monodronicity around the origin.



Outline of the Proof of Theorem A



- Computing the Lyapunov coefficient at the origin, we get

$$V_2(\varepsilon) = \frac{2\delta}{3\varepsilon} \frac{P^+(0,0)Q^+(0,0) + P^-(0,0)Q^-(0,0)}{P^+(0,0)Q^+(0,0)P^-(0,0)Q^-(0,0)} + \mathcal{O}(1), \;\; \delta = \mathrm{sign}(P^+(0,0)).$$

  Under condition (B), $V_2(\varepsilon) \neq 0$ for $\varepsilon > 0$ sufficiently small. The origin is attracting (resp. repelling), provided that $V_2(\varepsilon) < 0$ (resp. $V_2(\varepsilon) > 0$).

- Hence, take $\varepsilon = \overline{\varepsilon} > 0$ small in order that $Z_{\overline{\varepsilon}}$ has $\mathcal{H}_c(n)$ hyperbolic crossing limit cycles and $V_2(\overline{\varepsilon}) \neq 0$. Notice $\ell = \mathrm{sign}(V_2(\overline{\varepsilon}))$ and $a = -\delta$.

Outline of the Proof of Theorem A

- Finally, consider the perturbation

$$\widetilde{Z}_b(x,y) = \begin{cases} Z^{+}_{\overline{\varepsilon}}(x, y-b), & x>0,\\ Z^{-}_{\overline{\varepsilon}}(x,y), & x\le 0.\end{cases}$$

  For $b=0$ we have $\widetilde{Z}_0 = Z_{\overline{\varepsilon}}$, which possesses $\mathcal{H}_c(n)$ hyperbolic crossing limit cycles contained in the open lower half-plane $\{(x,y): \ y<0\}$. By hyperbolicity, these cycles **persist** for all sufficiently small $|b|$. Hence, $\widetilde{Z}_b$ has at least $\mathcal{H}_c(n)$ hyperbolic crossing limit cycles lying in $\{(x,y): \ y<0\}$ whenever $|b|$ is small.
- Moreover, $\widetilde{Z}_b$ undergoes a pseudo-Hopf bifurcation at $b=0$ and an additional hyperbolic crossing limit cycle surrounding S, whenever $\mathrm{sign}(b) = \mathrm{sign}(\delta V_2(\overline{\varepsilon}))$ $(a\, b\, \ell < 0)$. Therefore, there exists $\overline{b}$ arbitrarily close to 0 such that $\widetilde{Z}_{\overline{b}}$ has at least $1+\mathcal{H}_c(n)$ hyperbolic crossing limit cycles.
- Since $\widetilde{Z}_{\overline{b}}$ has degree $n+1$, we conclude that $\mathcal{H}_c(n+1) \ge 1+\mathcal{H}_c(n)$.

Outline of the Proof of Theorem B

Theorem B

$\widehat{\mathcal{H}}_c(n)$ grows at least as fast as $(n \log n)/(2 \log 2)$, that is

$$\liminf_{n \to \infty} \frac{\widehat{\mathcal{H}}_c(n)}{n \log n} \geq \frac{1}{2 \log 2}. \tag{2}$$

Outline of the Proof of Theorem B

Our strategy is to construct a sequence of piecewise polynomial Hamiltonian vector fields $(Z_k)_{k\in\mathbb{N}}$ of degree $n_k = 3\cdot 2^k - 1$, each possessing $c_k = 3\cdot k\cdot 2^{k-1} + 1$ crossing limit cycles. Once such a sequence is obtained, Theorem B follows directly. Indeed, since $k = \log((n_k+1)/3)/\log 2$, we have

$$\widehat{\mathcal{H}}_c(n_k) \geq c_k = 1 + \frac{(n_k+1)\log\left(\frac{n_k+1}{3}\right)}{2\log 2},$$

which yields the asymptotic estimate

$$\liminf_{n\to\infty} \frac{\widehat{\mathcal{H}}_c(n)}{n\,\log\, n} \geq \frac{1}{2\log 2}. \tag{3}$$

Thus, it remains to prove the following result.

Proposition 4

There exists a sequence $(Z_k)_{k\in\mathbb{N}}$ of piecewise polynomial Hamiltonian vector fields with degree $n_k = 3\cdot 2^k - 1$ and having $c_k = 3\cdot k\cdot 2^{k-1} + 1$ crossing limit cycles.

The sequence of vector fields of Proposition 4 will be constructed inductively.

As the first step of the induction, we consider the perturbed polynomial Hamiltonian function of degree $d_0 = 3$,

$$H_0^{\pm}(x, y, \varepsilon) = \frac{(\pm x - 1)^2 - y^2}{2} + \varepsilon P_0^{\pm}(y), \tag{4}$$

where $P_0^{\pm} = a_{0,0}^{\pm} + a_{0,1}^{\pm} y + a_{0,2}^{\pm} y^2 + a_{0,3}^{\pm} y^3$ and $\varepsilon > 0$ is a small parameter. The corresponding piecewise polynomial Hamiltonian vector field of degree $n_0 = 2$ is defined by

$$Z_{0,\varepsilon} = \begin{cases} Z_{0,\varepsilon}^{+} \ , \text{ if } x > 0 \\ Z_{0,\varepsilon}^{-} \ , \text{ if } x < 0, \end{cases} \quad \text{where} \quad Z_{0,\varepsilon}^{\pm} = \left(\frac{\partial}{\partial y} H_0^{\pm}(x, y, \varepsilon), -\frac{\partial}{\partial x} H_0^{\pm}(x, y, \varepsilon) \right). \tag{5}$$

Outline of the Proof of Theorem B

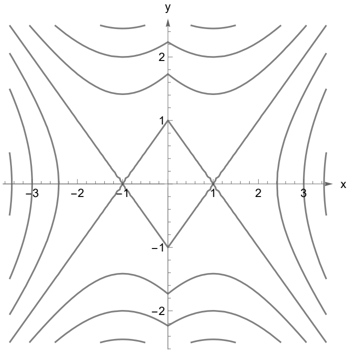


Figure: Level curves of H_0 for $\varepsilon = 0$.

Lemma 1

There exists coefficients of P_0 and $\bar{\varepsilon} > 0$ such that $Z_{0,\varepsilon}$ has a hyperbolic crossing limit cycle contained in $\mathbb{R} \times (-1, 1)$ for every $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$.

Now, starting from the polynomial Hamiltonian function H_0 of degree $d_0 = 3$ given in (4), we recursively define a sequence of Hamiltonian functions $(H_k)_{k\in\mathbb{N}}$. To this end, consider the singular transformation

$$\Phi : (x, y) \mapsto (x, y^2 - 2).$$

For each $k \in \mathbb{N}$, we define a polynomial Hamiltonian function $H_k^{\pm}$ of degree $d_k = 2d_{k-1}$ by

$$H_k^{\pm}(x, y, \varepsilon, E_k) = H_{k-1}^{\pm}(\Phi(x, y), \varepsilon, E_{k-1}) + \varepsilon\, \varepsilon_k\, P_k^{\pm}(y), \tag{6}$$

where $E_k = (\varepsilon_1, \ldots, \varepsilon_k) \in \mathbb{R}^k$ and

$$P_k^{\pm}(y) = \sum_{j=0}^{d_k} a_{k,j} y^j.$$

Finally, for each $k \in \mathbb{N}$, define the piecewise polynomial Hamiltonian vector field of degree $n_k := d_k - 1$ by

$$Z_{k,\varepsilon,E_k} = \begin{cases} Z^{+}_{k,\varepsilon,E_k}, & \text{if } x > 0 \\ Z^{-}_{k,\varepsilon,E_k}, & \text{if } x < 0, \end{cases} \quad \text{where } Z^{\pm}_{k,\varepsilon,E_k} = \left(\frac{\partial H_k^{\pm}}{\partial y}(\cdot, \varepsilon, E_k), -\frac{\partial H_k^{\pm}}{\partial x}(\cdot, \varepsilon, E_k) \right). \tag{7}$$

The transformation $\Phi(x,y) = (x, y^2 - 2)$ is not a diffeomorphism on the whole plane $\mathbb{R}^2$. Nevertheless, the restrictions $\Phi|_{\mathbb{R}\times(0,2)} : \mathbb{R}\times(0,2) \to \mathbb{R}\times(-2,2)$ and $\Phi|_{\mathbb{R}\times(-2,0)} : \mathbb{R}\times(-2,0) \to \mathbb{R}\times(-2,2)$ are diffeomorphisms. Consequently, the vector field $Z_{k+1,\varepsilon,E_{k+1}}|_{\varepsilon_{k+1}=0}$, when restricted to either $\mathbb{R}\times(0,2)$ or $\mathbb{R}\times(-2,0)$, is equivalent to Z_{k,ε,E_k} restricted to $\mathbb{R}\times(-2,2)$, for all $k \in \mathbb{N}$.

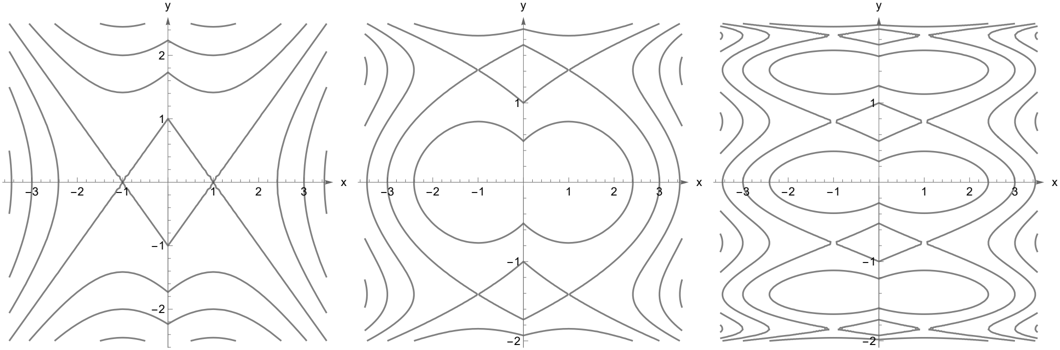


Figure: Level curves of H_0, H_1, and H_2, for $\varepsilon = 0$.

Outline of the Proof of Theorem B

Lemma 2

Assume that there exists $E_k^* \in \mathbb{R}^k$ *and* $\hat{\varepsilon} > 0$ *such that, for every* $\varepsilon \in (0, \hat{\varepsilon}]$, *the piecewise vector field* Z_{k,ε,E_k^*} *has* c_k *hyperbolic crossing limit cycles contained in* $\mathbb{R} \times (-2,\ 2)$ *and satisfying condition* (A_k). *Then, there exist coefficients of* $P_{k+1}^{\pm}$, $\bar{\varepsilon} \in (0, \hat{\varepsilon}]$, *and* $\bar{\varepsilon}_{k+1} > 0$ *such that, for every* $\varepsilon \in (0, \bar{\varepsilon}]$ *and* $\varepsilon_{k+1} \in (0, \bar{\varepsilon}_{k+1})$, *the piecewise vector field* $Z_{k+1,\varepsilon,(E_k^*,\varepsilon_{k+1})}$ *has* $c_{k+1} = 2c_k + d_k - 1$ *hyperbolic crossing limit cycles contained in* $\mathbb{R} \times (-2,\ 2)$ *and satisfying condition* (A_{k+1}).

We say that the c_k hyperbolic crossing limit cycles of Z_{k,ε,E_k^*} satisfy the hypothesis (A_k) provided that

$$\check{y}_i \neq 0 \quad \text{and} \quad \check{y}_i \neq \check{y}_j \text{ for } i \neq j, \qquad (A_k')$$

and

$$\frac{\partial \delta^i}{\partial \varepsilon}(\check{y}_i, 0) = 0, \quad \text{and} \quad \frac{\partial^2 \delta^i}{\partial y \partial \varepsilon}(\check{y}_i, 0) \neq 0. \qquad (A_k'')$$

Here, $(0, y_i(\varepsilon))$ is an intersection points of the i-th limit cycle with the y-axis and $\check{y}_i := y_i(0)$.

By Lemma 1, there exists $\overline{\varepsilon}_0$ such that $Z_{0,\overline{\varepsilon}_0}$ admits $c_0 = 1$ hyperbolic crossing limit cycle. In addition, by Lemma 2, for each k there are a parameter $\overline{\varepsilon}_k$ and a parameter vector E_k^* so that $Z_{k,\overline{\varepsilon}_k,E_k^*}$ admits $c_k = 2c_{k-1} + d_{k-1} - 1$ hyperbolic crossing limit cycles. Thus define the sequence $(Z_k)_{k\in\mathbb{N}}$ of piecewise polynomial Hamiltonian vector fields by

$$Z_0 = Z_{0,\overline{\varepsilon}_0} \quad \text{and} \quad Z_k = Z_{k,\overline{\varepsilon}_k,E_k^*},$$

Solving the recurrence above, we obtain

$$c_k = 3 \cdot k \cdot 2^{k-1} + 1.$$

On the other hand, for each k the vector field Z_k has degree $n_k = 2d_{k-1} - 1$, where $d_k = 2d_{k-1}$ with initial value $d_0 = 3$. Hence,

$$n_k = 3 \cdot 2^k - 1.$$

So we conclude the proof of Proposition 4.

Thank you for your attention!

I also thank

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