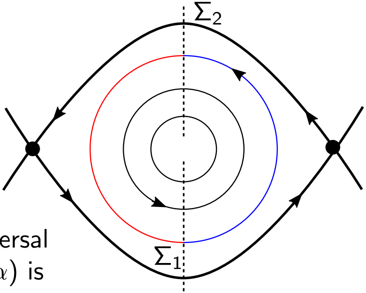
On the difference map around heteroclinic saddle connections in piecewise linear systems

R. Prohens, A.E. Teruel, J. Torregrosa

Universitat de les Illes Balears
IAC3 Institut d'Aplicacions Computacionals de Codi Comunitari
MINISTERIO DE CIENCIA E INNOVACIÓN
AGENCIA ESTATAL DE INVESTIGACIÓN
PID2023- 151974NB-I00
EUROPEAN UNION
EUROPEAN REGIONAL DEVELOPMENT FUND
"A way to make Europe"

Introduction



- Consider a one-parameter family

$$\dot{\mathbf{x}} = X(\mathbf{x};\alpha)$$

of planar centers enclosed by heteroclinic connections of hyperbolic saddles.

- By defining and parameterizing two transversal sections Σ_1, Σ_2, the difference map $\Delta(h;\alpha)$ is

$$\Delta(h;\alpha) = \Pi^+(h) - \Pi^-(h) = 0, \quad h \in (0, h_0).$$

the difference by following the flow in forward and backward time.

- After a generic ε-perturbation of the center

$$\dot{\mathbf{x}} = X(\mathbf{x};\alpha) + \varepsilon X_1(\mathbf{x};\alpha,\lambda,\varepsilon), \quad \lambda \in \mathbb{R}^p$$

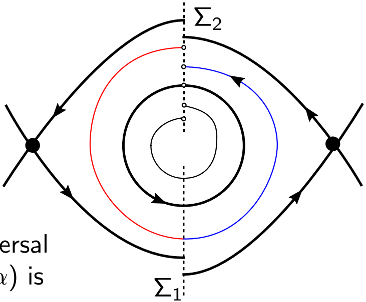
the difference map writes as a power series in ε as

$$\Delta(h;\alpha,\lambda,\varepsilon) = \Pi^+_\varepsilon - \Pi^-_\varepsilon = \Delta_1(h;\alpha,\lambda)\varepsilon + \Delta_2(h;\alpha,\lambda)\varepsilon^2 + \dots$$

with $\Delta_1(h;\alpha,\lambda)$ and $\Delta_2(h;\alpha,\lambda)$ the 1st. and 2nd. Melnikov funct.

- Zeros of $\Delta(h;\alpha,\lambda,\varepsilon)$ corresponds with periodic orbits.

Introduction



- Consider a one-parameter family

$$\dot{\mathbf{x}} = X(\mathbf{x}; \alpha)$$

of planar centers enclosed by heteroclinic connections of hyperbolic saddles.

- By defining and parameterizing two transversal sections Σ_1, Σ_2, the difference map $\Delta(h;\alpha)$ is

$$\Delta(h;\alpha) = \Pi^{+}(h) - \Pi^{-}(h) = 0, \quad h \in (0, h_0).$$

the difference by following the flow in forward and backward time.

- After a generic ε-perturbation of the center

$$\dot{\mathbf{x}} = X(\mathbf{x};\alpha) + \varepsilon X_1(\mathbf{x};\alpha,\lambda,\varepsilon), \quad \lambda \in \mathbb{R}^p \tag{1}$$

the difference map writes as a power series in ε as

$$\Delta(h;\alpha,\lambda,\varepsilon) = \Pi^{+}_{\varepsilon} - \Pi^{-}_{\varepsilon} = \Delta_1(h;\alpha,\lambda)\varepsilon + \Delta_2(h;\alpha,\lambda)\varepsilon^2 + \dots$$

with $\Delta_1(h;\alpha,\lambda)$ and $\Delta_2(h;\alpha,\lambda)$ the 1st. and 2nd. Melnikov funct.

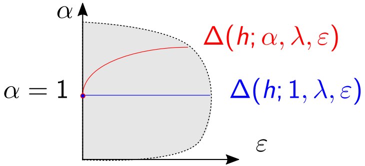
- Zeros of $\Delta(h;\alpha,\lambda,\varepsilon)$ corresponds with periodic orbits of (1).

Introduction

- By the **(IFT)**, simple zeros of the first Melnikov function $\Delta_1(h;\alpha,\lambda)$ extend to zeros of the difference map for sufficiently small ε.
- Let $\bar{\Delta}_1(h;\alpha,\lambda)$ denote an asymptotic expansion of $\Delta_1(h;\alpha,\lambda)$ in terms of h,h^α and $\log(h)$:

$$\Delta_1(h;\alpha,\lambda) = \bar{\Delta}_1(h;\alpha,\lambda) + O(h^{n+\alpha m}, h^{p+\alpha q}\log^r(h)).$$

  For any small simple zero of $\bar{\Delta}_1(h;\alpha,\lambda)$, there exists a nearby simple zero of the difference map $\Delta(h;\alpha,\lambda,\varepsilon)$.
- The structure of $\bar{\Delta}_1(h;\alpha,\lambda)$ provides a lower bound of the number of limit cycles of (1) near the heteroclinic connection for ε small.



- Coll, B., Dumortier, F., Prohens, R. *Alien limit cycles in Liénard equations*. JDE, 254(3), 2013

PWL framework

- In this work we use the PWL setting to describe the structure of $\bar{\Delta}_1(h;\alpha,\lambda)$ and consequently of $\Delta(h;\alpha,\lambda,\varepsilon)$.
- Consider $\mathcal{V}_L=\{x<-1\}$, $\mathcal{V}^-=\{x=-1\}$, $\mathcal{V}_C=\{-1<x<1\}$, $\mathcal{V}^+=\{x=1\}$, and $\mathcal{V}_R=\{1<x\}$.
- Consider the following ε-perturbation of the α-family of PWL systems

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=A_j\begin{pmatrix}x\\ y\end{pmatrix}+\mathbf{b}_j+\sum_{i=1}^{n}\varepsilon^i\left(A_{ij}\begin{pmatrix}x\\ y\end{pmatrix}+\mathbf{b}_{ij}\right), \qquad (2)$$

when $(x,y)\in\mathcal{V}_j$, being $j\in\{L,C,R\}$,

$$A_L=A_C=\begin{pmatrix}\frac{1-\alpha}{2} & 1+\alpha\\ \frac{1+\alpha}{4} & \frac{1-\alpha}{2}\end{pmatrix}, \qquad \mathbf{b}_L=\mathbf{b}_C=3\begin{pmatrix}\frac{1-\alpha}{2}\\ \frac{1+\alpha}{4}\end{pmatrix},$$

$$A_R=\begin{pmatrix}2(1-\alpha) & 1+\alpha\\ 4(1+\alpha) & -2(1-\alpha)\end{pmatrix}, \qquad \mathbf{b}_R=4\begin{pmatrix}1-\alpha\\ -2(1+\alpha)\end{pmatrix},$$

- and A_{ij}, $\mathbf{b}_{ij}$ arbitrary matrices and vectors.

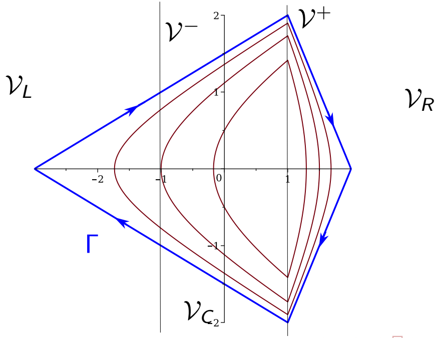
PWL framework

Theorem

Non-perturbed PWL system, i.e. system (2) with $\varepsilon = 0$ and $\alpha \in \mathbb{R}^+$.

a) Exibits a two hyperbolic saddle connection Γ with saddles at $(-3, 0)$, $(2, 0)$ and hyperbolicity ratios α and $1/\alpha$.

b) Γ bounds a period annulus with 2 and 3 zones periodic orbits. The boundary periodic orbit is tangent at $(-1, -\frac{1-\alpha}{1+\alpha})$.



PWL framework

Proof:

- For $\varepsilon = 0$ and $\alpha \in \mathbb{R}^+$, the **continuous** function

$$H = \begin{cases} H_L = H_C = (\frac{1}{2}x - y + \frac{3}{2})(\frac{1}{2}x + y + \frac{3}{2})^\alpha & x \leq 1, \\ H_R = (-2x - y + 4)(-2x + y + 4)^\alpha & x \geq 1, \end{cases}$$

is constant along orbits:

$$\frac{\partial H_j}{\partial x}\dot{x} + \frac{\partial H_j}{\partial y}\dot{y} = 0 \text{ in } \mathcal{V}_j, \text{ with } j \in \{L, C, R\}$$

$$H_L(1, y) = H_C(1, y) = H_R(1, y) \text{ for } y \in \mathbb{R}$$

H is a **first integral** of the non-perturbed system:

a) Straight lines intersect transversaly at $(-3, 0)$, $(2, 0)$.
Straight lines intersect $\mathcal{V}^+$ at $(1, -2)$, $(1, 2)$.

b) Consider u, v such that $-2 \leq u < v \leq 2$. The period annulous follows by checking that

$$H_L(1, u) - H_L(1, v) = H_R(1, u) - H_R(1, v).$$

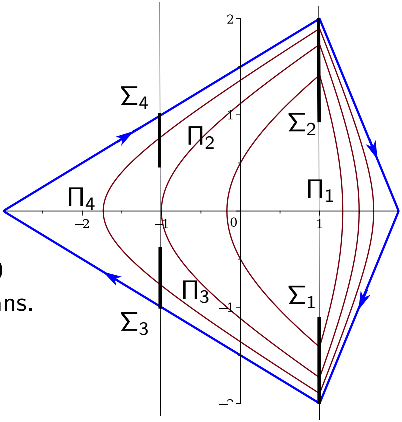
Difference map

- Consider the parametrized transversal sections:
  $\Sigma_1 = \{(1, -2+h) : h \in (0, h_0)\}, \quad \Sigma_2 = \{(1, 2-h) : h \in (0, h_0)\},$
  $\Sigma_3 = \{(-1, -1+h) : h \in (0, h_0)\}, \Sigma_4 = \{(-1, 1-h) : h \in (0, h_0)\},$
- and the transition maps:

  $\Pi_1, \Pi_2, \Pi_3, \Pi_4.$
- The difference map $\Delta : \Sigma_1 \to \Sigma_4$

  $\Delta(h; \alpha) = \Pi_2^{-1} \circ \Pi_1^{-1} - \Pi_4 \circ \Pi_3$

  is identically zero.



- After ε-perturbation:
  for any $h \in (0, h_0)$, there exists $\varepsilon > 0$ small enough such that perturbed trans. maps $\Pi_{1\varepsilon}, \Pi_{2\varepsilon}, \Pi_{3\varepsilon}, \Pi_{4\varepsilon}$ are defined

  and also the difference map

$$\Delta(h; \alpha, \lambda, \varepsilon) = \Pi_{2\varepsilon}^{-1} \circ \Pi_{1\varepsilon}^{-1}(h) - \Pi_{4\varepsilon} \circ \Pi_{3\varepsilon}(h).$$

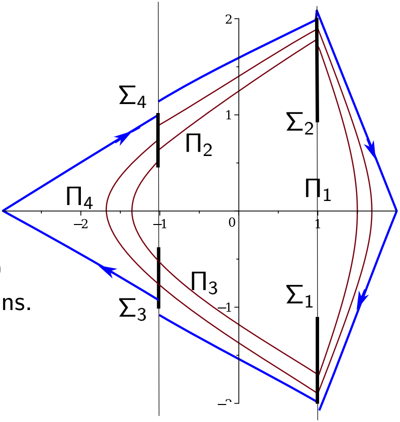
Difference map

- Consider the parametrized transversal sections:
  $\Sigma_1 = \{(1, -2+h) : h \in (0, h_0)\}$, $\Sigma_2 = \{(1, 2-h) : h \in (0, h_0)\}$,
  $\Sigma_3 = \{(-1, -1+h) : h \in (0, h_0)\}$, $\Sigma_4 = \{(-1, 1-h) : h \in (0, h_0)\}$,
- and the transition maps:

  $\Pi_1, \Pi_2, \Pi_3, \Pi_4$.
- The difference map $\Delta : \Sigma_1 \to \Sigma_4$

  $\Delta(h;\alpha) = \Pi_2^{-1} \circ \Pi_1^{-1} - \Pi_4 \circ \Pi_3$

  is identically zero.



- After ε-perturbation:
  for any $h \in (0, h_0)$, there exists $\varepsilon > 0$ small enough such that perturbed trans. maps $\Pi_{1\varepsilon}, \Pi_{2\varepsilon}, \Pi_{3\varepsilon}, \Pi_{4\varepsilon}$ are defined

  and also the difference map

$$\Delta(h;\alpha,\lambda,\varepsilon) = \Pi_{2\varepsilon}^{-1} \circ \Pi_{1\varepsilon}^{-1}(h) - \Pi_{4\varepsilon} \circ \Pi_{3\varepsilon}(h).$$

Difference map in the PWL framework

- Consider $H(x,y,\varepsilon) = \begin{cases} H_L(x,y,\varepsilon) & \text{if } x \in \mathcal{V}_L \cup \mathcal{V}^-, \\ H_C(x,y,\varepsilon) & \text{if } x \in \mathcal{V}_C, \\ H_R(x,y,\varepsilon) & \text{if } x \in \mathcal{V}^+ \cup \mathcal{V}_R, \end{cases}$

$$H_L(x,y,\varepsilon) = \left(\mathbf{r}_{0L}^{[1]} + \sum_{i=1}^{n} \varepsilon^i \mathbf{r}_{iL}^{[1]}\right) \left(\mathbf{r}_{0L}^{[2]} + \sum_{i=1}^{n} \varepsilon^i \mathbf{r}_{iL}^{[2]}\right)^{\alpha + \sum_{i=1}^{n} \varepsilon^i \alpha_{iL}},$$

$$H_C(x,y,\varepsilon) = \left(\mathbf{r}_{0C}^{[1]} + \sum_{i=1}^{n} \varepsilon^i \mathbf{r}_{iC}^{[1]}\right) \left(\mathbf{r}_{0C}^{[2]} + \sum_{i=1}^{n} \varepsilon^i \mathbf{r}_{iC}^{[2]}\right)^{\alpha + \sum_{i=1}^{n} \varepsilon^i \alpha_{iC}},$$

$$H_R(x,y,\varepsilon) = \left(\mathbf{r}_{0R}^{[1]} + \sum_{i=1}^{n} \varepsilon^i \mathbf{r}_{iR}^{[1]}\right) \left(\mathbf{r}_{0R}^{[2]} + \sum_{i=1}^{n} \varepsilon^i \mathbf{r}_{iR}^{[2]}\right)^{\alpha + \sum_{i=1}^{n} \varepsilon^i \alpha_{iR}},$$

- being $\mathbf{r}_{ij}^{[k]} = c_{ij}^{[k]} x + d_{ij}^{[k]} y + e_{ij}^{[k]}$ and $H(x,y,0) = H(x,y)$.

Proposition

If $\mathbf{r}_{0j}^{[1]}$ and $\mathbf{r}_{0j}^{[2]}$ are not parallel, there exists a choice of the coefficients of $\mathbf{r}_{ij}^{[k]}$ and α_{ij} such that H_j is a first integral of system (2), restricted to $\mathcal{V}_j$.

Difference map in the PWL framework

- Transition maps $\Pi_{1\varepsilon}^{-1}(h), \Pi_{2\varepsilon}^{-1}(h), \Pi_{3\varepsilon}(h)$ and $\Pi_{4\varepsilon}(h)$ satisfy

$$
\begin{aligned}
H_R(1,-2+h,\varepsilon) - H_R(1,2-\Pi_{1\varepsilon}^{-1}(h),\varepsilon) &= 0\\
H_C(1,2-h,\varepsilon) - H_C(-1,1-\Pi_{2\varepsilon}^{-1}(h),\varepsilon) &= 0\\
H_C(1,-2+h,\varepsilon) - H_C(-1,-1+\Pi_{3\varepsilon}(h),\varepsilon) &= 0\\
H_L(-1,-1+h,\varepsilon) - H_L(-1,1-\Pi_{4\varepsilon}(h),\varepsilon) &= 0
\end{aligned}
$$

- Maps $\Pi_{2\varepsilon}^{-1}(h)$ and $\Pi_{3\varepsilon}(h)$ are analytic in h, and the coefficients can be obtained from previous equations.

- Maps $\Pi_{1\varepsilon}^{-1}(h)$ and $\Pi_{4\varepsilon}(h)$ are Dulac maps which satisfy

Proposition

The perturbed Dulac function associated to the flow of a linearizable saddle of hyperbolicity ratio α writes as follows:

$$D(h,\varepsilon) = h^\alpha f_{00}(h,h^\alpha) + \sum_{i=1}^{\infty} \varepsilon^i \sum_{j=0}^{i} \frac{\log^j h}{h^{i-j}} f_{ij}(h,h^\alpha),$$

where f_{ij} are analytic functions and where f_{00} does not vanish at $h=0$.

Dulac maps and regular maps

$$\Pi_{1\varepsilon}^{-1}(h) = h^{\alpha} f_{00}^{R}(h, h^{\alpha}) + \varepsilon \left[\frac{1}{h} f_{10}^{R}(h, h^{\alpha}) + \log(h) f_{11}^{R}(h, h^{\alpha}) \right] + \mathcal{O}(\varepsilon^2),$$

$$f_{00}^{R}(h, h^{\alpha}) = \frac{4}{\kappa^2} - \frac{1}{\kappa^2} h + \frac{4\alpha}{\kappa^4} h^{\alpha} - \frac{2\alpha}{\kappa^4} h^{1+\alpha} + \frac{2\alpha\,(3\alpha+1)}{\kappa^6} h^{2\alpha} + \mathcal{O}(h^3)$$

$$\begin{aligned} f_{10}^{R}(h, h^{\alpha}) = & - r_{1R}^{[1]} h + \frac{4\alpha r_{1R}^{[2]}}{\kappa^2} h^{\alpha} - \frac{8\ln(2)\,\alpha_{1R} + (\alpha-1) r_{1R}^{[1]} + 2\alpha r_{1R}^{[2]}}{\kappa^2} h^{1+\alpha} \\ & + \frac{8\alpha^2 r_{1R}^{[2]}}{\kappa^4} h^{2\alpha} + \frac{8\ln(2)\,\alpha_{1R} + \alpha r_{1R}^{[1]} + \alpha r_{1R}^{[2]}}{4\kappa^2} h^{2+\alpha} \\ & - \frac{4\,(4\ln(2)\,\alpha - 1)\,\alpha_{1R} + \alpha\,(2\alpha-1)\, r_{1R}^{[1]} + \alpha\,(6\alpha+1)\, r_{1R}^{[2]}}{\kappa^4} h^{1+2\alpha} \\ & + \frac{6\alpha^2(3\alpha+1) r_{1R}^{[2]}}{\kappa^6} h^{3\alpha} + \mathcal{O}(h^4), \end{aligned}$$

$$f_{11}^{R}(h, h^{\alpha}) = \frac{4\alpha_{1R}}{\kappa^2} h^{\alpha} - \frac{\alpha_{1R}}{\kappa^2} h^{1+\alpha} + \frac{8\alpha\alpha_{1R}}{\kappa^4} h^{2\alpha} + \mathcal{O}(h^3),$$

$$\kappa = 2^{\alpha}, \quad r_{1R}^{[1]} = c_{1R}^{[1]} + 2d_{1R}^{[1]} + e_{1R}^{[1]}, \quad r_{1R}^{[2]} = c_{1R}^{[2]} - 2d_{1R}^{[2]} + e_{1R}^{[2]}.$$

Dulac maps and regular maps

$$\Pi_{2\varepsilon}^{-1}(h) = f_0^u(h) + \varepsilon f_1^u(h) + \mathcal{O}(\varepsilon^2)$$

$$f_0^u(h) = \kappa h + \frac{\kappa(2\kappa-1)\alpha}{4}h^2 + \frac{\kappa(2\kappa-1)\alpha}{32}((6\kappa-1)\alpha + 2\kappa + 1)h^3 + \mathcal{O}(h^4),$$

$$\begin{aligned} f_1^u(h) &= \kappa r_{1C_R}^{[1]} - r_{1C_L}^{[1]} \\ &+ \frac{\kappa}{4}\left(4\ln(2)\alpha_{1C} + \alpha\left(4\kappa r_{1C_R}^{[1]} - 2r_{1C_L}^{[2]} - 2r_{1C_L}^{[1]} - r_{1C_R}^{[1]} + r_{1C_R}^{[2]}\right)\right)h \\ &+ \frac{\kappa}{32}\left(12\kappa^2\alpha(3\alpha+1)r_{1C_R}^{[1]} - 8\left(\left(r_{1C_R}^{[2]} - 2r_{1C_R}^{[1]} - 2r_{1C_L}^{[2]} - 2r_{1C_L}^{[1]}\right)\alpha^2\right.\right. \\ &+ \left.\left(r_{1C_L}^{[1]} + r_{1C_L}^{[2]}\right)\alpha - 2\alpha_{1C}\left(2\alpha\ln 2 + 1\right)\right)\kappa + \left(r_{1C_R}^{[1]} + 4r_{1C_L}^{[1]} - 2r_{1C_R}^{[2]} + 4r_{1C_L}^{[2]}\right)\alpha^2 \\ &+ \left.\left(2r_{1C_R}^{[2]} - r_{1C_R}^{[1]}\right)\alpha - 8\alpha_{1C}\left(\alpha\ln 2 + 1\right)\right)h^2 + \mathcal{O}(h^3), \end{aligned}$$

$$\kappa = 2^\alpha, \quad r_{1C_R}^{[1]} = c_{1C}^{[1]} + 2d_{1C}^{[1]} + e_{1C}^{[1]}, \quad r_{1C_R}^{[2]} = c_{1C}^{[2]} - 2d_{1C}^{[2]} + e_{1C}^{[2]}.$$

$$r_{1C_L}^{[1]} = -c_{1C}^{[1]} + d_{1C}^{[1]} + e_{1C}^{[1]}, \quad r_{1C_L}^{[2]} = -c_{1C}^{[2]} - d_{1C}^{[2]} + e_{1C}^{[2]}.$$

Difference map

Theorem

For α near resonance ($|\alpha - 1| \ll 1$) and $0 < \varepsilon \ll 1$, the difference map

$$\Delta(h; \alpha, \lambda, \varepsilon) = \varepsilon \Delta_1(h; \alpha, \lambda) + \varepsilon^2 \Delta_2(h; \alpha, \lambda) + \mathcal{O}(\varepsilon^2),$$

is as follows.

a) If $\alpha > 1$, then

$$\begin{aligned}
\bar{\Delta}_1 = & \gamma_0(\lambda) 1 \Big(1 + f_0(h, h^\alpha)\Big) + \gamma_1(\lambda) h^{\alpha-1} \Big(1 + f_1(h, h^\alpha)\Big) \\
& + \gamma_2(\lambda) \Big(h^\alpha \log(h) \Big(1 + f_2(h, h^\alpha)\Big) + \mu_0 h^{2\alpha} \Big(1 + f_3(h, h^\alpha)\Big)\Big) \\
& + \gamma_3(\lambda) h^\alpha \Big(1 + f_4(h, h^\alpha)\Big) + \gamma_4(\lambda) h^{\alpha+1} \Big(1 + f_5(h, h^\alpha)\Big) \\
& + \gamma_5(\lambda)(\alpha - 1) h^{2\alpha} P(h, h^\alpha) \Big(1 + f_6(h, h^\alpha)\Big) + \mathcal{O}(h^7),
\end{aligned}$$

with $P(h, h^\alpha) = \mu_1(\alpha - 1) + \mu_2 h^{2-\alpha} + \mu_3(\alpha - 1) h + \mu_4 h^\alpha$, $\mu_i = \mu_i(\alpha)$ being analytic functions, such that $\mu_i(1) \neq 0$, f_i analytic functions, $f_i(0, 0) = 0$, and $\gamma_i(\lambda)$ arbitrary real numbers.

Difference map

Theorem

For α near resonance ($|\alpha - 1| \ll 1$) and $0 < \varepsilon \ll 1$, the difference map

$$\Delta(h; \alpha, \lambda, \varepsilon) = \varepsilon \Delta_1(h; \alpha, \lambda) + \varepsilon^2 \Delta_2(h; \alpha, \lambda) + \mathcal{O}(\varepsilon^2),$$

is as follows.

c) If $\alpha < 1$, then

$$\begin{aligned} \bar{\Delta}_1 = & \gamma_0(\lambda) h^{\alpha-1} \Big(1 + f_0(h, h^\alpha)\Big) + \gamma_1(\lambda) 1 \Big(1 + f_1(h, h^\alpha)\Big) \\ & + \gamma_2(\lambda) \Big(h^\alpha \log(h) \Big(1 + f_2(h, h^\alpha)\Big) + \mu_0 h^{2\alpha}(1 + f_3(h, h^\alpha))\Big) \\ & + \gamma_3(\lambda) h^\alpha \Big(1 + f_4(h, h^\alpha)\Big) + \gamma_4(\lambda) h^{2\alpha} \Big(1 + f_5(h, h^\alpha)\Big) \\ & + \gamma_5(\lambda)(\alpha - 1) h^{\alpha+1} Q(h, h^\alpha) \Big(1 + f_6(h, h^\alpha)\Big) + \mathcal{O}(h^7), \end{aligned}$$

with $Q(h, h^\alpha) = \mu_1(\alpha - 1) + \mu_2 h^{2\alpha-1} + \mu_3(\alpha - 1) h^\alpha + \mu_4 h$, $\mu_i = \mu_i(\alpha)$ being analytic functions, such that $\mu_i(1) \neq 0$, f_i analytic functions, $f_i(0, 0) = 0$, and $\gamma_i(\lambda)$ arbitrary real numbers.

Difference map

Theorem

For α near resonance ($|\alpha - 1| \ll 1$) and $0 < \varepsilon \ll 1$, the difference map

$$\Delta(h; \alpha, \lambda, \varepsilon) = \varepsilon \Delta_1(h; \alpha, \lambda) + \varepsilon^2 \Delta_2(h; \alpha, \lambda) + \mathcal{O}(\varepsilon^2),$$

is as follows.

b) If $\alpha = 1$, then

$$\begin{aligned}\Delta_1 =& \gamma_0(\lambda) 1 \Big(1 + f_0(h)\Big) + \gamma_2(\lambda) h \log(h) \Big(1 + f_1(h)\Big) + \gamma_3(\lambda) h \Big(1 + f_2(h)\Big) \\ &+ \gamma_4(\lambda) h^2 \Big(1 + f_3(h)\Big),\end{aligned}$$

with f_i analytic functions, $f_i(0) = 0$, and $\gamma_i(\lambda)$ arbitrary real numbers. Moreover, when $\Delta_1 \equiv 0$, then

$$\begin{aligned}\bar{\Delta}_2 =& \gamma_0(\lambda) \log h \Big(1 + g_0(h)\Big) + \gamma_1(\lambda) 1 \Big(1 + g_2(h)\Big) + \gamma_2(\lambda) h \log(h) \Big(1 + g_3(h)\Big) \\ &+ \gamma_3(\lambda) h \Big(1 + g_4(h)\Big) + \gamma_4(\lambda) h^2 \Big(1 + g_5(h)\Big) + \gamma_5(\lambda) h^3 \Big(1 + g_6(h)\Big) + \mathcal{O}(h^7),\end{aligned}$$

with g_i analytic functions, $g_i(0) = 0$, and $\gamma_i(\lambda)$ arbitrary real numbers.

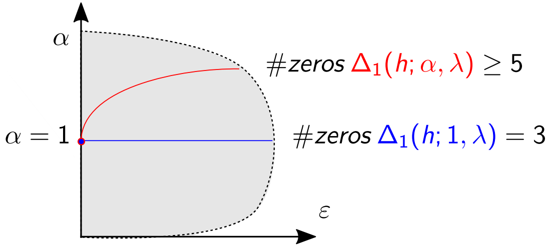
Difference map

Corollary

Consider PWL system (2) with $\varepsilon > 0$ small enough.

a) If $\alpha \neq 1$, the system has at least 5 limit cycles near the heteroclinic connection, all of them are obtained as zeros of the first Melnikov function.

b) If $\alpha = 1$, the system has at least 5 limit cycles near the heteroclinic connection, but it has only 3 as zeros of the first Melnikov function.



Roussarie-Ecalle compensator

- Let us analyze the transition of $\bar{\Delta}_1(h;\alpha,\lambda)$ through $\alpha = 1$.
- Let us consider the **Ecalle-Roussarie compensator**

$$\omega(h,\alpha) = \begin{cases} \frac{h^{\alpha-1}-1}{\alpha-1} & \alpha \neq 1, \\ \log h & \alpha = 1. \end{cases}$$

- For $\alpha \neq 1$, since $h^{\alpha-1} = 1 + (\alpha-1)\omega$, let us use a (h,ω) expansion of $\bar{\Delta}_1(h;\alpha,\lambda)$

$$\begin{aligned}\bar{\Delta}_1 =& \gamma_0(\lambda)h^{\alpha-1} + \gamma_1(\lambda)1 + \gamma_2(\lambda)\left(h^{\alpha}\log(h) + \mu_0 h^{2\alpha}\right) \\ &+ \gamma_3(\lambda)h^{\alpha} + \gamma_4(\lambda)h^{2\alpha} + \gamma_5(\lambda)(\alpha-1)h^{\alpha+1}Q(h,h^{\alpha}) + \mathcal{O}(h^7)\end{aligned}$$

$$\begin{aligned} =& \Big(\gamma_0(\lambda) + \gamma_1(\lambda)\Big)1 + \gamma_2(\lambda)h\log(h) + \gamma_3(\lambda)h + (\gamma_2(\lambda)\mu_0 + \gamma_4(\lambda))h^2 \\ &+(\alpha-1)\Big(\gamma_0(\lambda)\omega + \gamma_2(\lambda)(\omega h\log(h) + 2\mu_0\omega h^2) + \gamma_3(\lambda)\omega h \\ &\qquad + 2\gamma_4(\lambda)\omega h^2 + \gamma_5(\lambda)h^2 Q\Big) \\ &+(\alpha-1)^2\Big(\gamma_2(\lambda)\mu_0\omega^2 h^2 + \gamma_4(\lambda)\omega^2 h^2 + \gamma_5(\lambda)\omega h^2 Q\Big) + \mathcal{O}(h^7)\end{aligned}$$

Roussarie-Ecalle compensator

- Let us analyze the transition of $\bar{\Delta}_1(h;\alpha,\lambda)$ through $\alpha = 1$.
- Let us consider the **Ecalle-Roussarie compensator**

$$\omega(h,\alpha) = \begin{cases} \frac{h^{\alpha-1}-1}{\alpha-1} & \alpha \neq 1, \\ \log h & \alpha = 1. \end{cases}$$

- For $\alpha \neq 1$, since $h^{\alpha-1} = 1 + (\alpha-1)\omega$, let us use a (h,ω) expansion of $\bar{\Delta}_1(h;\alpha,\lambda)$

$$\begin{aligned}\bar{\Delta}_1 =& \gamma_0(\lambda)h^{\alpha-1} + \gamma_1(\lambda)1 + \gamma_2(\lambda)\left(h^{\alpha}\log(h) + \mu_0 h^{2\alpha}\right) \\ &+ \gamma_3(\lambda)h^{\alpha} + \gamma_4(\lambda)h^{2\alpha} + \gamma_5(\lambda)(\alpha-1)h^{\alpha+1}Q(h,h^{\alpha}) + \mathcal{O}(h^7) \\ \\ =& \Big(\gamma_0(\lambda) + \gamma_1(\lambda)\Big)1 + \gamma_2(\lambda)h\log(h) + \gamma_3(\lambda)h + (\gamma_2(\lambda)\mu_0 + \gamma_4(\lambda))h^2 \\ &+ (\alpha-1)\Big(\gamma_0(\lambda)\omega + \gamma_2(\lambda)(\omega h\log(h) + 2\mu_0\omega h^2) + \gamma_3(\lambda)\omega h \\ &\qquad + 2\gamma_4(\lambda)\omega h^2 + \gamma_5(\lambda)h^2 Q\Big) \\ &+ (\alpha-1)^2\Big(\gamma_2(\lambda)\mu_0\omega^2 h^2 + \gamma_4(\lambda)\omega^2 h^2 + \gamma_5(\lambda)\omega h^2 Q\Big) + \mathcal{O}(h^7)\end{aligned}$$

Conclusion

- When $\alpha = 1$, only four independent monomials persist

$$\bar{\Delta}_1 = \Big(\gamma_0(\lambda)+\gamma_1(\lambda)\Big)1+\gamma_2(\lambda)h\log(h)+\gamma_3(\lambda)h+(\gamma_2(\lambda)\mu_0+\gamma_4(\lambda))h^2$$

- Other independent monomials move to higher order terms as $\alpha \to 1$:

$$\omega \to \log(h), \quad h^2 Q \to h^3$$

- This suggests that, starting from $\alpha = 1$, a suitable perturbation allows higher-order terms in ε to recombine and become visible at the first order in ε when $\alpha \neq 1$.
- This phenomenon is not specific to the previous example; other examples have been analyzed and lead to identical conclusions.

PWL framework

- Consider $\mathcal{V}_L = \{x < -1\}$, $\mathcal{V}^- = \{x = -1\}$, $\mathcal{V}_C = \{-1 < x < 1\}$, $\mathcal{V}^+ = \{x = 1\}$, and $\mathcal{V}_R = \{1 < x\}$.
- Consider the following ε-perturbation of the α-family of PWL systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_j \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{b}_j + \sum_{i \geq 1} \varepsilon^i \left(A_{ij} \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{b}_{ij} \right), \qquad (3)$$

when $(x, y) \in \mathcal{V}_j$, being $j \in \{L, C, R\}$,

$$A_L = \begin{pmatrix} \alpha - 2 & -(1+\alpha) \\ -2(1+\alpha) & 2\alpha - 1) \end{pmatrix}, \qquad \mathbf{b}_L = 2 \begin{pmatrix} \alpha - 2 \\ -2(1+\alpha) \end{pmatrix},$$

$$A_C = \begin{pmatrix} 1 & -2 \\ 1/2 & -1 \end{pmatrix}, \qquad \mathbf{b}_C = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$A_R = \begin{pmatrix} 1 - 2\alpha & -(1+\alpha) \\ -2(1+\alpha) & 2 - \alpha \end{pmatrix}, \qquad \mathbf{b}_R = 2 \begin{pmatrix} 2\alpha - 1 \\ 2(1+\alpha) \end{pmatrix},$$

and A_{ij} and $\mathbf{b}_{ij}$ arbitrary matrices and vectors.

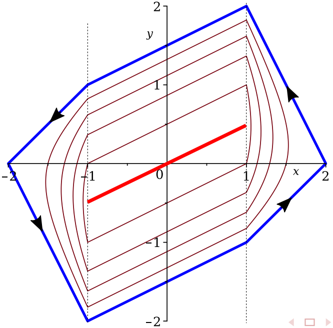
PWK framework

Theorem

The PWL system (3) with $\varepsilon = 0$ and $\alpha \in \mathbb{R}^+$.

a) Exibits a center delimited by a two hyperbolic saddle connection Γ and a segment $\{-x + 2y = 0 : -1 \leq x \leq 1\}$ of equilibria.

b) The hyperbolic saddles are located at $(\pm 2, 0)$ and have hyperbolicity ratio α.



Difference map

Theorem

Difference map $\Delta(h;\alpha,\lambda,\varepsilon) = \Delta_1(h;\alpha,\lambda)\varepsilon + \Delta_2(h;\alpha,\lambda)\varepsilon^2 + \dots$:

- For $\alpha > 1$ there exist independent constants $C_k(\lambda)$, $k = 1,2,3,4$.

$$\bar{\Delta}_1 = C_1(\lambda)1(1 + f_0(h,h^\alpha)) + C_2(\lambda)h^\alpha \log h(1 + f_1(h,h^\alpha))$$
$$+ C_3(\lambda)h^{\alpha-1}(1 + f_2(h.h^\alpha)) + C_4(\lambda)(\alpha-1)h^\alpha(1 + f_3(h,h^\alpha)) + \mathcal{O}(h^7).$$

where f_i analytic functions and $f_i(0,0) = 0$.

- For $\alpha = 1$, there exist independent constants $D_k(\lambda)$, $k = 1,2$

$$\Delta_1(h;1,\lambda) = D_1(\lambda)\sqrt{9-4h} + D_2(\lambda)h\log\left(\frac{3-\sqrt{9-4h}}{3+\sqrt{9-4h}}\right)$$

  - $\Delta_1(h;1,\lambda) = D_1(\lambda)1(1 + g_0(h)) + D_2(\lambda)h\log h(1 + g_1(h))$

    where g_i analytic functions and $g_i(0,0) = 0$.

Assuming $\Delta_1(h;1,\lambda) \equiv 0$, there exist indep. constants $B_k(\lambda)$.

- $\bar{\Delta}_2 = B_1(\lambda)\log(h) + B_2(\lambda)1 + B_3(\lambda)h\log^2 h + B_4(\lambda)h\log h + B_5(\lambda)h$

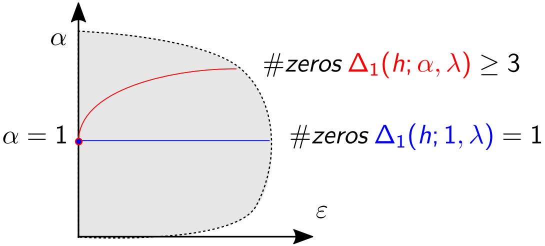
Difference map

Corollary

Consider PWL system (3) with $\varepsilon > 0$ small enough.

a) If $\alpha \neq 1$, the system has at least 3 limit cycles near the heteroclinic connection, all of them are obtained as zeros of the first Melnikov function.

b) If $\alpha = 1$, the system has at least 4 limit cycles near the heteroclinic connection, but it has only one as zero of the first Melnikov function.



Roussarie-Ecalle compensator

- Let us analyze the transition of $\Delta_1(h;\alpha,\lambda)$ through $\alpha = 1$.
- Let us consider the **Ecalle-Roussarie compensator**

$$\omega(h,\alpha) = \begin{cases} \frac{h^{\alpha-1}-1}{\alpha-1} & \alpha \neq 1, \\ \log h & \alpha = 1. \end{cases}$$

- For $\alpha \neq 1$, since $h^{\alpha-1} = 1 + (\alpha-1)\omega$, let us use a (h,ω) expansion of $\bar{\Delta}_1(h;\alpha,\lambda)$

$$\begin{aligned} & C_1 1 + C_2 h^\alpha \log h + C_3 h^{\alpha-1} + C_4(\alpha-1)h^\alpha \\ =& C_1 1 + C_2(h + (\alpha-1)\omega h)\log h + C_3(1 + (\alpha-1)\omega) \\ & + C_4(\alpha-1)(h + (\alpha-1)\omega h) \\ =& (C_1 + C_3)1 + C_2 h \log h + (\alpha-1)(C_3\omega + C_4 h + C_2 h \log h\,\omega) \\ & + (\alpha-1)^2 C_4 \omega h \end{aligned}$$

- Since $\omega \to \log h$ as $\alpha \to 1$
  First order monomials $1, h\log h$
  Second order monomials $\log h, h, h\log^2 h$

On the difference map around heteroclinic saddle connections in piecewise linear systems

R. Prohens, A.E. Teruel, J. Torregrosa

UIB Universitat de les Illes Balears
IAC3 Institut d'Aplicacions Computacionals de Codi Comunitari
MINISTERIO DE CIENCIA E INNOVACIÓN
AGENCIA ESTATAL DE INVESTIGACIÓN
PID2023- 151974NB-I00
EUROPEAN UNION
EUROPEAN REGIONAL DEVELOPMENT FUND
"A way to make Europe"