Discrete Malkin functions

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Periodic discrete systems

Let $f_k : \mathbb{R}^d \to \mathbb{R}^d$. We consider the d-dimensional difference equation

$$x_{k+1} = f_k(x_k), \quad k \in \mathbb{N}.$$

The solution with $x_0 = z$ is

$$\psi_0 = z,\ \psi_1 = f_0(z),\ \psi_2 = f_1(f_0(z)),\ \dots .$$

Let $\omega \in \mathbb{N}^*$. We assume ω-periodicity, i.e.

$$f_{k+\omega} = f_k \quad \text{for all} \quad k \in \mathbb{N}.$$

Given a solution $\psi : \mathbb{N} \to \mathbb{R}^d$, we say that ψ is ω-periodic when

$$\psi_{k+\omega} = \psi_k \quad \text{for all} \quad k \in \mathbb{N}.$$

Lemma

A solution ψ is ω-periodic if and only if $\psi_\omega = \psi_0$.

Persistent periodic solutions

We consider the perturbed ω-periodic equation

$$x_{k+1} = f_k(x_k) + \varepsilon g_k(x_k, \varepsilon), \quad k \in \mathbb{N}. \tag{1}$$

We say that an ω-periodic solution

$$\varphi\colon \mathbb{N} \to \mathbb{R}^d \quad \text{of} \quad x_{k+1} = f_k(x_k)$$

persists in (1) if there exists $\rho > 0$ such that for any $\varepsilon \in (-\rho, \rho) \setminus \{0\}$ there exists an ω-periodic solution

$$\psi(\varepsilon)\colon \mathbb{N} \to \mathbb{R}^d \quad \text{of} \quad (1)$$

such that

$$\lim_{\varepsilon \to 0} \psi(\varepsilon)_0 = \varphi_0.$$

Discrete Malkin functions

Let $1 \leq \delta \leq d$, $V \subset \mathbb{R}^\delta$ be open and $\xi \colon V \to \mathbb{R}^d$ be smooth.
A discrete Malkin function is

$$\mathcal{M} \colon V \subset \mathbb{R}^\delta \to \mathbb{R}^\delta$$

such that

for each simple zero $\alpha^* \in V$ of $\mathcal{M}$

we have that

$\xi(\alpha^*)$ is the initial value of an ω-periodic solution of $x_{k+1} = f_k(x_k)$

that persists in $x_{k+1} = f_k(x_k) + \varepsilon g_k(x_k, \varepsilon)$.

A. Gasull and C. Valls, *Discrete Melnikov functions*, Journal of Difference Equations and Applications, 2022. (the case $\delta = d$)

Existence of periodic solutions

A. Halanay, *Solutions périodique et presque-périodique des systèmes d'équations aux difféerences finies*, Arch. Rational Mech. Anal., 1963.

A. Halanay and V. Răsvan, *Periodic and almost periodic solutions for a class of systems described by coupled delay differential and difference equations*, Nonlinear Analysis TMA, 1977.

J. M. Cushing, Periodically forced nonlinear systems of difference equations, Journal of Difference Equations and Applications, 1998.

F. Dannan, S. Elaydi and P. Liu, *Periodic solutions of difference equations*, Journal of Difference Equations and Applications, 2000.

C. Pötzsche, *Bifurcations in a periodic discrete-time environment*, Nonlinear Analysis: Real World Applications, 2013.

Continuous Malkin functions

Let $1 \le \delta \le d$, $V \subset \mathbb{R}^\delta$ be open and $\xi\colon V \to \mathbb{R}^d$ be smooth.
A continuous Malkin function is

$$\mathcal{M}\colon V \subset \mathbb{R}^\delta \to \mathbb{R}^\delta$$

such that

for each simple zero $\alpha^* \in V$ of $\mathcal{M}$

we have that

$\xi(\alpha^*)$ is the initial value of an ω-periodic solution of $x' = f_t(x)$

that persists in $x' = f_t(x) + \varepsilon g_t(x, \varepsilon)$.

I.G. Malkin, *On Poincaré's theory of periodic solutions*, Akad. Nauk SSSR. Prikl. Mat. Meh. **13** (1949), 633–646 (in Russian).

Linear unperturbed system

For each $z \in \mathbb{R}^d$, the solution of

$$x_0 = z, \quad x_{k+1} = A_k x_k \;\text{ is given by}$$

$$\varphi(z)\colon \mathbb{N} \to \mathbb{R}^d, \quad \varphi(z)_k = M_k\, z, \text{ where } M_k = A_{k-1} \cdots A_0,\; k \geq 1,\; M_0 = I_d.$$

For each $z \in \mathbb{R}^d$, the solution of

$$x_0 = z, \quad x' = A_t x \;\text{ is given by}$$

$$\varphi(z)\colon \mathbb{R} \to \mathbb{R}^d, \quad \varphi(z)_t = M_t\, z, \text{ with } \; M_0 = I_d.$$

In both cases M_ω is called *the monodromy matrix.*

Perturbations of a linear system

For each $(z,\varepsilon)\in\mathbb{R}^d\times\mathbb{R}$, the solution of

$$x_0=z,\quad x_{k+1}=A_kx_k+\varepsilon g_k(x_k,\varepsilon)\ \text{ is given by}$$

$$\psi(z,\varepsilon)\colon\mathbb{N}\to\mathbb{R}^d,\quad \psi(z,\varepsilon)_k=\varphi(z)_k+\varepsilon F_k(z,\varepsilon),\quad k\geq 1,$$

where $F_k\colon\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d$ is defined inductively by $F_1(z,\varepsilon)=g_0(z,\varepsilon)$,

$$F_{k+1}(z,\varepsilon)=A_kF_k(z,\varepsilon)+g_k(\psi(z,\varepsilon)_k,\varepsilon),\quad k\geq 1.$$

For each $(z,\varepsilon)\in\mathbb{R}^d\times\mathbb{R}$, the solution of

$$x_0=z,\quad x'=A_tx+\varepsilon g_t(x,\varepsilon)\ \text{ is given by}$$

$$\psi(z,\varepsilon)\colon[0,\omega]\to\mathbb{R}^d,\quad \psi(z,\varepsilon)_t=\varphi(z)_t+\varepsilon F_t(z,\varepsilon),\quad t\in[0,\omega],$$

where $F_0(z,\varepsilon)=0$, and

$$\frac{d}{dt}F_t(z,\varepsilon)=A_tF_t(z,\varepsilon)+g_t(\psi(z,\varepsilon)_t,\varepsilon),\quad t\in[0,\omega].$$

The equation in $\mathbb{R}^d$ in both cases

Find $z = z(\varepsilon) \in \mathbb{R}^d$ solution of the equation

$$\boxed{\psi(z,\varepsilon)_\omega = z}$$

where $\psi(z,\varepsilon)_\omega = \varphi(z)_\omega + \varepsilon F_\omega(z,\varepsilon) = M_\omega\, z + \varepsilon F_\omega(z,\varepsilon)$.
Thus, we have to solve the equation

$$(I_d - M_\omega)z = \varepsilon\, F_\omega(z,\varepsilon)$$

Here

M_ω is the monodromy matrix

which, in the discrete case has the form

$$M_\omega = A_{\omega-1}A_{\omega-2}\cdots A_1 A_0.$$

Since our analysis is at first order in ε, we will need the expression of

$$F_\omega(z,0).$$

$F_\omega(z,0)$

$F_1(z,0) = g_0(z,0)$ if $\omega = 1$, respectively, if $\omega \geq 2$,

$$F_\omega(z,0) = \sum_{i=0}^{\omega-2} A_{\omega-1} \cdots A_{i+1} g_i(\varphi(z)_i, 0) + g_{\omega-1}(\varphi(z)_{\omega-1}, 0)$$

If we assume that the matrices A's are invertible, we can write

$$F_\omega(z,0) = \sum_{i=0}^{\omega-1} M_\omega M_{i+1}^{-1} g_i(\varphi(z)_i, 0).$$

For differential systems we have

$$F_\omega(z,0) = \int_0^\omega M_\omega M_\tau^{-1} g_\tau(\varphi(z)_\tau, 0) d\tau$$

We come back to the equation we have to solve

$$(I_d - M_\omega)z = \varepsilon\, F_\omega(z, \varepsilon)$$

First we analyze the unperturbed part.

Consider the subspaces of $\mathbb{R}^d$,

$$\mathcal{Z} = \ker(I_d - M_\omega), \quad \mathcal{I} = \mathrm{Im}\,(I_d - M_\omega)\,.$$

Then

$$\dim \mathcal{Z} = \dim \mathcal{I}^\perp, \quad \mathcal{I}^\perp = \ker(I_d - M_\omega^T).$$

Periodic solutions of the unperturbed linear system

$$\mathcal{Z} = \ker(I_d - M_\omega).$$

$z \in \mathcal{Z}$ if and only if the (forward) solution of

$$x_0 = z, \quad x_{k+1} = A_k x_k, \text{ is (forward) } \omega\text{-periodic.}$$

For differential equations:

$z \in \mathcal{Z}$ if and only if the solution of

$$x_0 = z, \quad x' = A_t x, \text{ is } \omega\text{-periodic.}$$

Periodic solutions of the adjoint system

$$\mathcal{I}^\perp = \ker(I_d - M_\omega^T).$$

$u \in \mathcal{I}^\perp$ if and only if the backward solution $(y_k)_{k\le\omega-1}$ of

$$y_{\omega-1} = u^T, \quad y_{k-1} = y_k A_k, \quad k \le \omega - 1,$$

is backward ω-periodic.

J. M. Cushing, *Periodically forced nonlinear systems of difference equations*, Journal of Difference Equations and Applications, 1998.

For differential equations:

$u \in \mathcal{I}^\perp$ if and only if the solution of

$$y_0 = u^T, \quad y' = -y\,A_t$$

is ω-periodic.

Corollary

$$\mathcal{Z} = \ker(I_d - M_\omega), \quad \mathcal{I} = \mathrm{Im}\,(I_d - M_\omega) \quad \delta = \dim \mathcal{Z} = \dim \mathcal{I}^\perp.$$

We have.

Both the linear system and its adjoint have exactly δ linearly independent ω-periodic solutions.

$$\delta = 0 \iff (I_d - M_\omega) \text{ is invertible } \iff 1 \notin \sigma(M_\omega).$$

$$\delta = d \iff M_\omega = I_d.$$

$$\mathcal{Z} \cap \mathcal{I} = \{0\} \iff \text{the algebraic multiplicity of } 1 \in \sigma(M_\omega) \text{ is } \delta.$$

First results for $B\,z = \varepsilon\,F_\omega(z,\varepsilon)$ with $B = I_d - M_\omega$

Theorem

If $1 \notin \sigma(M_\omega)$ then the trivial solution $\varphi = 0$ of the linear system persists.

A. Halanay, *Solutions périodique et presque-périodique des systèmes d'équations aux difféerences finies*, Arch. Rational Mech. Anal. **12** (1963).

Theorem

If $M_\omega = I_d$ then

$$\mathcal{M}\colon \mathbb{R}^d \to \mathbb{R}^d, \quad \mathcal{M}(z) = F_\omega(z,0)$$

is a Malkin (Melnikov) function.

A. Gasull and C. Valls, *Discrete Melnikov functions*, Journal of Difference Equations and Applications, 2022.

Some linear algebra

Let $B\colon \mathbb{R}^d \to \mathbb{R}^d$ be a linear map, and

$$\mathcal{Z} = \ker B, \quad \mathcal{I} = \operatorname{Im} B, \quad \delta = \dim \mathcal{Z}.$$

There exist the isomorphisms of linear spaces

$$\tilde{\xi}\colon \mathbb{R}^\delta \to \mathcal{Z}, \quad P^T\colon \mathbb{R}^\delta \to \mathcal{I}^\perp, \quad Q^T\colon \mathbb{R}^{d-\delta} \to \mathcal{I}.$$

Then

$$\begin{pmatrix} Q \\ P \end{pmatrix} : \mathbb{R}^d \to \mathbb{R}^d \quad \text{is invertible}$$

and the linear maps

$$PB\colon \mathbb{R}^d \to \mathbb{R}^\delta, \quad QBQ^T\colon \mathbb{R}^{d-\delta} \to \mathbb{R}^{d-\delta}$$

satify

$$PB = 0, \quad \ker QBQ^T \cong \mathcal{Z} \cap \mathcal{I}.$$

Lyapunov-Schmidt reduction for $B\,z = \varepsilon\,F_\omega(z,\varepsilon)$

First we look for

$$\alpha \in \mathbb{R}^\delta \quad \text{and} \quad \beta \in \mathbb{R}^{d-\delta}$$

such that

$$z = \tilde{\xi}\,\alpha + \varepsilon\,Q^T\beta \quad \text{is a solution} \quad (\tilde{\xi}\,\alpha \in \mathcal{Z},\ Q^T\beta \in \mathcal{I}). \text{ Then}$$

$$B\tilde{\xi}\,\alpha + \varepsilon\,B\,Q^T\beta = \varepsilon F_\omega(\tilde{\xi}\,\alpha + \varepsilon\,Q^T\beta,\varepsilon) \implies B\,Q^T\beta = F_\omega(\tilde{\xi}\,\alpha + \varepsilon\,Q^T\beta,\varepsilon)$$

Now we multiply with the invertible map $\begin{pmatrix} Q \\ P \end{pmatrix} : \mathbb{R}^d \to \mathbb{R}^d$ and obtain

$$Q\,B\,Q^T\beta = Q\,F_\omega(\tilde{\xi}\,\alpha + \varepsilon\,Q^T\beta,\varepsilon)\,, \quad P\,F_\omega(\tilde{\xi}\,\alpha + \varepsilon\,Q^T\beta,\varepsilon) = 0\,.$$

Recall that

$$\ker QBQ^T \cong \mathcal{Z} \cap \mathcal{I}.$$

The non degeneracy condition $\mathcal{Z} \cap \mathcal{I} = \{0\}$

Let

$$G_1(\beta, \varepsilon, \alpha) = Q\,B\,Q^T\beta - Q\,F_\omega(\tilde{\xi}\,\alpha + \varepsilon\,Q^T\beta, \varepsilon).$$

For the map $G_1 \colon \mathbb{R}^{d-\delta} \times \mathbb{R} \times \mathbb{R}^{\delta} \to \mathbb{R}^{d-\delta}$ and

$$\alpha^* \text{ fixed and } \beta^* = [Q\,B\,Q^T]^{-1}Q\,F_\omega(\tilde{\xi}\,\alpha^*, 0)$$

we have

$$G_1(\beta^*, 0, \alpha^*) = 0, \quad D_\beta G_1(\beta^*, 0, \alpha^*) = Q\,B\,Q^T.$$

The IFT assures the existence of a smooth function $\beta(\alpha, \varepsilon)$ with $\lim_{\varepsilon\to 0,\ \alpha\to\alpha^*} \beta(\alpha, \varepsilon) = \beta^*$ such that $G_1(\beta(\alpha,\varepsilon), \varepsilon, \alpha) = 0$ for $\varepsilon \in (-\rho, \rho)$ and $\alpha \in V$, a small neighborhood of α^*.

The Malkin function

Let

$$G_2(\alpha, \varepsilon) = P\, F_\omega(\tilde{\xi}\, \alpha + \varepsilon\, Q^T \beta(\alpha, \varepsilon), \varepsilon)$$

and note that $G_2(\alpha, 0) = P\, F_\omega(\tilde{\xi}\, \alpha, 0)$.

Let

$$\mathcal{M} \colon \mathbb{R}^\delta \to \mathbb{R}^\delta, \quad \mathcal{M}(\alpha) = P\, F_\omega(\tilde{\xi}\, \alpha, 0)$$

and assume that α^* is a simple zero of $\mathcal{M}$.

The IFT assures the existence of a smooth map $\alpha(\varepsilon)$ with $\lim_{\varepsilon \to 0} \alpha(\varepsilon) = \alpha^*$ such that, finally, $z(\varepsilon) = \tilde{\xi}\alpha(\varepsilon) + \varepsilon Q^T \beta(\alpha(\varepsilon), \varepsilon)$ is a solution of $B\, z = \varepsilon F_\omega(z, \varepsilon)$ with $\lim_{\varepsilon \to 0} z(\varepsilon) = \tilde{\xi}\alpha^* \in \mathcal{Z}$.

Recall that $\tilde{\xi} \colon \mathbb{R}^\delta \to \mathcal{Z}$ and $P^T \colon \mathbb{R}^\delta \to \mathcal{I}^\perp$ are isomorphisms.

The equation in $\mathbb{R}^d$ in both cases

Find $z = z(\varepsilon) \in \mathbb{R}^d$ solution of the equation

$$\boxed{\psi(z,\varepsilon)_\omega = z}$$

where $\psi(z,\varepsilon)_\omega = \varphi(z)_\omega + \varepsilon F_\omega(z,\varepsilon)$.
Thus, we have to solve the equation

$$z - \varphi_\omega(z) = \varepsilon\, F_\omega(z,\varepsilon)$$

Here

$$\varphi_\omega \text{ is the Poincaré translation map at time } \omega$$

of the unperturbed system, which, in the discrete case has the form

$$\varphi_\omega = f_{\omega-1} \circ f_{\omega-2} \circ \cdots \circ f_1 \circ f_0.$$

Since our analysis is at first order in ε, we will need the expression of

$$F_\omega(z,0).$$

$F_\omega(z,0)$

$$A_k(z) = Df_k(\varphi_k(z)), \quad M_k(z) = D\varphi_k(z).$$

If we assume that the matrices A's are invertible, we can write

$$F_\omega(z,0) = \sum_{i=0}^{\omega-1} M_\omega(z) M_{i+1}^{-1}(z) g_i(\varphi(z)_i, 0).$$

For differential systems we have

$$A_t(z) = Df_t(\varphi_t(z)), \quad M_t(z) = D\varphi_t(z).$$

$$F_\omega(z,0) = \int_0^\omega M_\omega(z) M_\tau^{-1}(z) g_\tau(\varphi(z)_\tau, 0) d\tau$$

$z - \varphi_\omega(z) = \varepsilon F_\omega(z, \varepsilon)$

Here φ_ω is the Poincaré translation map of the unperturbed system. Let

$$\tilde{Z} = \{z \in \mathbb{R}^d \ : \ \varphi_\omega(z) = z\}, \quad M_\omega(z) = D\varphi_\omega(z).$$

Theorem

Assume that there exists $z^* \in \mathbb{R}^d$ such that

$$z^* \in \tilde{Z} \ \ \textit{and} \ \ 1 \notin \sigma(M_\omega(z^*)).$$

Then the ω-periodic solution $\varphi(z^*)$ of the unperturbed system persists.

Theorem

If there exists an open $V \subset \mathbb{R}^d$ such that $V \subset \tilde{\mathcal{Z}}$ then

$$\mathcal{M} \colon V \to \mathbb{R}^d, \quad \mathcal{M}(z) = F_\omega(z, 0)$$

is a Malkin (Melnikov) function.

On the unperturbed system

$$\tilde{Z} = \{z \in \mathbb{R}^d \ : \ \varphi_\omega(z) = z\}.$$

Assume that there exist a nonempty open set $V \subset \mathbb{R}^{\tilde{\delta}}$ and $\xi \colon V \to \mathbb{R}^d$

$$\xi(V) \subset \tilde{Z} \ \text{ and the rank of } \ D\xi(\alpha) \ \text{ is } \ \tilde{\delta}.$$

Consider $M_\omega(\alpha) = D\varphi_\omega(\xi(\alpha)), \quad B(\alpha) = I_d - M_\omega(\alpha),$

$$\mathcal{Z}(\alpha) = \ker B(\alpha), \quad \mathcal{I}(\alpha) = \mathrm{Im}\ B(\alpha),$$

$$\delta = \dim \mathcal{Z}(\alpha).$$

We have $\varphi_\omega(\xi(\alpha)) = \xi(\alpha) \implies D\xi(\alpha) \in \mathcal{Z}(\alpha)$, thus

$$\delta \geq \tilde{\delta}.$$

The nondegeneracy condition

Assume that, for each $\alpha \in V \subset \mathbb{R}^{\tilde{\delta}}$, and for $\delta = \dim \mathcal{Z}(\alpha)$ we have

$$\tilde{\delta} = \delta, \quad \mathcal{Z}(\alpha) \cap \mathcal{I}(\alpha) = \{0\}.$$

Then, for each $\alpha \in V$, $D\xi(\alpha) \colon \mathbb{R}^{\delta} \to \mathcal{Z}(\alpha)$ is an isomorphism

and there exist two isomorphisms

$$\mathbb{Q}(\alpha)^T \colon \mathbb{R}^{d-\delta} \to \mathcal{I}(\alpha), \quad P(\alpha)^T \colon \mathbb{R}^{\delta} \to \mathcal{I}(\alpha)^{\perp}$$

with smooth dependence on α.

Then the linear maps

$$P(\alpha)B(\alpha) \colon \mathbb{R}^d \to \mathbb{R}^{\delta}, \quad Q(\alpha)B(\alpha)Q(\alpha)^T \colon \mathbb{R}^{d-\delta} \to \mathbb{R}^{d-\delta}$$

satisfy for any $\alpha \in V$,

$$P(\alpha)B(\alpha) = 0, \quad \ker Q(\alpha)B(\alpha)Q^T(\alpha) \cong \mathcal{Z}(\alpha) \cap \mathcal{I}(\alpha).$$

Lyapunov-Schmidt reduction for $z - \varphi_\omega(z) = \varepsilon F_\omega(z, \varepsilon)$

First we look for

$$\alpha \in V \subset \mathbb{R}^\delta \quad \text{and} \quad \beta \in \mathbb{R}^{d-\delta}$$

such that

$$z = \xi(\alpha) + \varepsilon\, Q(\alpha)^T \beta \quad \text{is a solution.}$$

Then we expand around $\varepsilon = 0$ and obtain for $B(\alpha) = I_d - D\varphi_\omega(\xi(\alpha))$

$$B(\alpha)Q(\alpha)^T\beta + O(\varepsilon) = F_\omega(\xi(\alpha) + \varepsilon\, Q(\alpha)^T\beta, \varepsilon).$$

we multiply with the invertible map $\begin{pmatrix} Q(\alpha) \\ P(\alpha) \end{pmatrix} : \mathbb{R}^d \to \mathbb{R}^d$ and obtain

$$Q(\alpha)B(\alpha)Q^T(\alpha)\beta + O(\varepsilon) = Q(\alpha)F_\omega(\xi(\alpha) + \varepsilon\, Q(\alpha)^T\beta, \varepsilon)$$

$$O(\varepsilon) = P(\alpha)F_\omega(\xi(\alpha) + \varepsilon\, Q(\alpha)^T\beta, \varepsilon).$$

The Malkin function

$$\mathcal{M}\colon V \to \mathbb{R}^{\delta}$$

$$\mathcal{M}(\alpha) = P(\alpha)\, F_{\omega}(\xi(\alpha), 0), \quad \alpha \in V \subset \mathbb{R}^{\delta}$$

The symmetric Euler top

R. Abraham and J. E. Marsden, *Foundations of Mechanics*, 1978.

V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, 1989.

$$\dot{x} = -yz, \;\; \dot{y} = xz, \;\; \dot{z} = 0$$

R. Hirota and K. Kimura, *Discretization of the Euler top*, J. Phys. Soc. Japan, 2000.

S. Saito and N. Saitoh, *Invariant varieties of periodic points for the discrete Euler top*, SIGMA, 2006.

$$\begin{aligned} x_{k+1} - x_k &= -\frac{y_{k+1}z_k + y_k z_{k+1}}{2} \\ y_{k+1} - y_k &= \frac{x_{k+1}z_k + x_k z_{k+1}}{2} \\ z_{k+1} - z_k &= 0. \end{aligned}$$

The symmetric Euler top

$$\dot{x} = -yz, \;\; \dot{y} = xz, \;\; \dot{z} = 0$$

The first discrete symmetric Euler top :

$$\begin{aligned} x_{k+1} &= x_k \cos\Omega(z_k) - y_k \sin\Omega(z_k) \\ y_{k+1} &= x_k \sin\Omega(z_k) + y_k \cos\Omega(z_k) \\ z_{k+1} &= z_k \end{aligned}$$

where $\cos\Omega(z) = \frac{4-z^2}{4+z^2}$, thus $\sin\Omega(z) = \frac{4z}{4+z^2}$.

The flow of the symmetric Euler top

$$\dot{x} = -yz,\ \ \dot{y} = xz,\ \ \dot{z} = 0$$

$$\varphi_t(x, y, z) = \left(\begin{array}{c} x\cos(z\,t) - y\sin(z\,t) \\ x\sin(z\,t) + y\cos(z\,t) \\ z \end{array} \right).$$

$$\begin{aligned} x_{k+1} &= x_k \cos\Omega(z_k) - y_k \sin\Omega(z_k) \\ y_{k+1} &= x_k \sin\Omega(z_k) + y_k \cos\Omega(z_k) \\ z_{k+1} &= z_k \end{aligned}$$

Continuous and discrete systems with the same Poincaré operator

$$\dot{X} = g(X), \qquad X_{k+1} = f(X_k).$$

Let $\omega \in \mathbb{N}^*$, $g\colon \mathbb{R}^d \to \mathbb{R}^d$ be C^1 such that any solution of $\dot{X} = g(X)$ is well-defined on the interval $[0, \omega]$.

Let $\varphi_t\colon \mathbb{R}^d \to \mathbb{R}^d$ be the flow at time t of $\dot{X} = g(X)$ and let

$$f = \varphi_1.$$

Then, the flow property $\varphi_{t+s} = \varphi_s \circ \varphi_t$ assures that the solution of

$$X_{k+1} = f(X_k), \quad k \in \mathbb{N}, \quad X_0 = z$$

is $\varphi_k(z) = (\underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}})(z)$. Then

$$\varphi_\omega\colon \mathbb{R}^d \to \mathbb{R}^d$$

is the Poincaré map at time ω of both systems.

A second discrete symmetric Euler top

$$\dot{x} = -yz, \;\; \dot{y} = xz, \;\; \dot{z} = 0$$

$$\begin{aligned} x_{k+1} &= x_k \cos z_k - y_k \sin z_k \\ y_{k+1} &= x_k \sin z_k + y_k \cos z_k \\ z_{k+1} &= z_k \end{aligned}$$

both have the Poincaré map

$$\varphi_\omega(x,y,z) = \begin{pmatrix} x\cos(z\,\omega) - y\sin(z\,\omega) \\ x\sin(z\,\omega) + y\cos(z\,\omega) \\ z \end{pmatrix}$$

while the Poincaré map of the first discrete system is

$$\tilde{\varphi}_\omega(x,y,z) = \begin{pmatrix} x\cos(\Omega(z)\omega) - y\sin(\Omega(z)\omega) \\ x\sin(\Omega(z)\omega) + y\cos(\Omega(z)\omega) \\ z \end{pmatrix}$$

The symmetric Euler top has many ω-periodic solutions

We have the following examples of functions ξ such that $\xi(\alpha)$ is the initial value of an ω-periodic solution.

$$\xi_0^h(\alpha) = (\alpha, 0) \text{ for all } \alpha \in \mathbb{R}^2 \setminus \{(0,0)\}$$
$$\xi^v(\alpha) = (0,0,\alpha), \text{ for all } \alpha \in (0, \mu_\omega) \subset \mathbb{R}$$
$$\xi_1^h(\alpha) = (\alpha, \mu_\omega) \text{ for all } \alpha \in \mathbb{R}^2 \setminus \{(0,0)\}$$

where $\mu_\omega = 2\pi/\omega$ in the continuous case (and the second discrete system),

while $\mu_\omega = 2\sqrt{(1-\cos\frac{2\pi}{\omega})/(1+\cos\frac{2\pi}{\omega})}$ in the first discrete system.

The non degeneracy conditions are fulfilled in each situation.

A.B. and I. García, Periodic solutions of the perturbed symmetric Euler top, TMNA, 2010.

Thank you very much for your attention!