Dynamics of trace maps motivated by applications in spectral theory of quasicrystals

William N. Yessen

Department of Mathematics

University of California Irvine

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Given two letters, a and b, consider a *substitution* $s : \{a, b\} \mapsto \langle a, b\rangle$.

Extend s to $\langle a, b\rangle$ by concatenation: $s(\alpha_1 \cdots \alpha_k) = s(\alpha_1) \cdots s(\alpha_k)$.

- If there exists k such that $s^k(a)$ and $s^k(b)$ both contain letters a and b, then s is called *primitive*;
- If s extends to an invertible morphism on $\langle a, b\rangle$, s is called *invertible*.

Suppose s is primitive and invertible.

Given a representation $\rho : \langle a, b\rangle \to \mathrm{SL}(2, \mathbb{C})$, there exists $\mathcal{F}_s : \mathbb{C}^3 \to \mathbb{C}^3$ with the following properties.

- $\mathcal{F}_s = (f_s, g_s, h_s)$ with $f_s, h_s, g_s \in \mathbb{Z}[x, y, z]$;
- If $x_1 = \frac{1}{2}\mathrm{Tr}(\rho(a))$, $x_2 = \frac{1}{2}\mathrm{Tr}(\rho(b))$, $x_3 = \frac{1}{2}\mathrm{Tr}(\rho(ab))$, then

$$\frac{1}{2}\mathrm{Tr}(\rho(s^k(a))) = \pi_1 \circ F_s^k(x_1, x_2, x_3) \text{ and } \frac{1}{2}\mathrm{Tr}(\rho(s^k(b))) = \pi_2 \circ F_s^k(x_1, x_2, x_3).$$

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Integrals of Motion

It turns out that all trace maps preserve the so-called *Fricke-Vogt* invariant:

$$I \circ F_s(x, y, z) = I(x, y, z) \text{ where } I(x, y, z) = x^2 + y^2 + z^2 - 2xyz.$$

In particular, F_s preserves the algebraic surfaces

$$S_V \stackrel{\text{def}}{=} \left\{(x, y, z) \in \mathbb{C}^3 : I(x, y, z) = V\right\}.$$

We shall concentrate on $S_V \cap \mathbb{R}^3$ with $V \in \mathbb{R}$. The geometry and topology of S_V depends on V:

- If $V > 0$, then S_V is a smooth connected manifold, which is topologically a four-punctured sphere;
- If $V = 0$, then S_V is connected and smooth everywhere except for four conic singularities;
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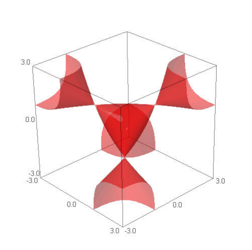
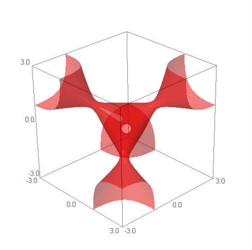
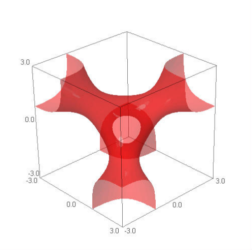


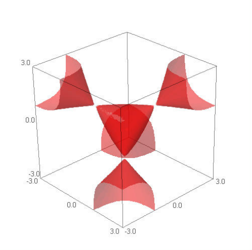
(a) $V = 0.0001$



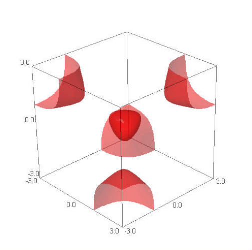
(b) $V = 0.05$



(c) $V = 1$



(d) $V = -0.0001$



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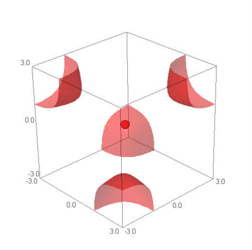

(f) $V = -0.95$

We consider a prominent example of a primitive invertible substitution: *the Fibonacci substitution* $s : a \mapsto ab$, $s : b \mapsto a$.

The corresponding trace map is given by $f(x, y, z) = (2xy - z, x, y)$.

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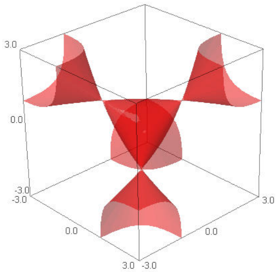
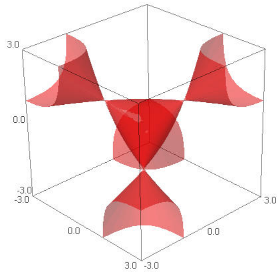


- $S_0 \cap [-1,1]^3$ is invariant under f and is a factor of $(x,y) \mapsto (x+y,y) : \mathbb{T}^2 \hookleftarrow$ under the factor map $(\theta,\phi) \mapsto (\cos 2\pi(\theta+\phi), \cos 2\pi\theta, \cos 2\pi\phi)$;
- The singularities lie on curves of periodic points which are normally hyperbolic;
- Every point on the cones that does not lie on the strong stable (unstable) manifold of a singularity escapes to infinity in forward (backward) time.

The Fibonacci Trace Map - Pseudo Anosov on S_0



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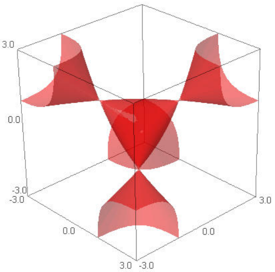

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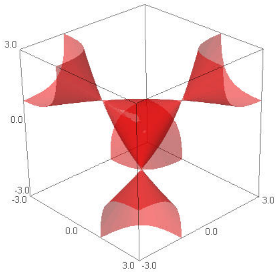

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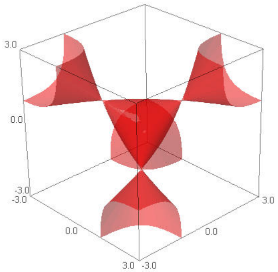

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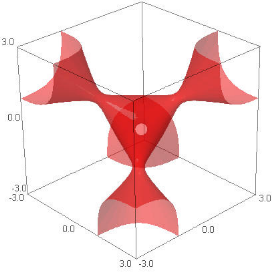

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- A point has a bounded forward (backward) orbit if and only if it belongs to the stable (unstable) lamination;
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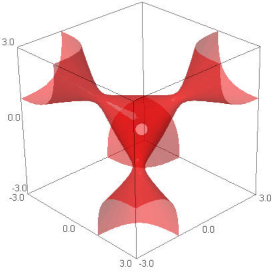

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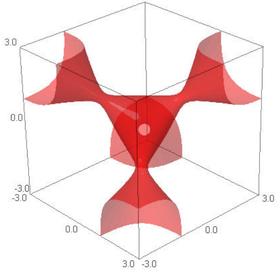

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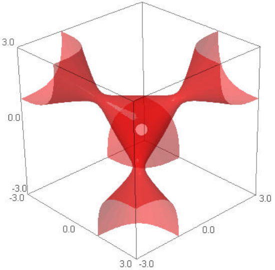

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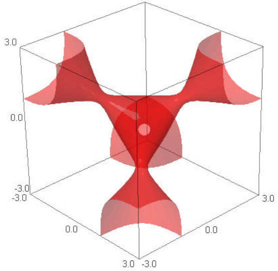

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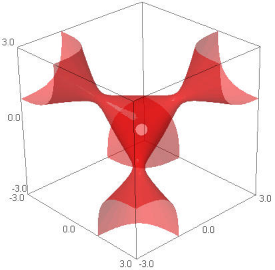

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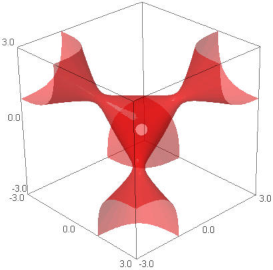

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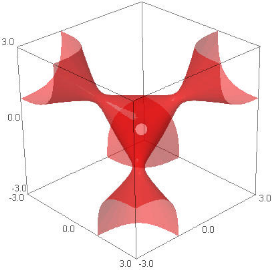

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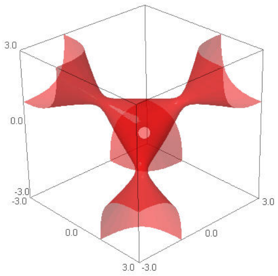

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Given a compact analytic curve $\gamma \subset \bigcup_{V>0} S_V$, what can be said about $B_\infty(\gamma)$?

$B_\infty(\gamma)$ is the intersection of γ with the center stable manifolds;

Theorem

Either $B_\infty(\gamma)$ is finite, it is all of γ, or it is a Cantor set of zero measure together with, possibly, finitely many isolated points; in this case the Hausdorff dimension of $B_\infty(\gamma)$ is strictly between zero and one.

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Connections to Quasicrystals

Consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ obtained by

$$a \mapsto ab \mapsto aba \mapsto abaab \mapsto abaababa \mapsto abaababaabaab \mapsto \cdots .$$

Extend $\{u_n\}$ to the left arbitrarily, and call the resulting sequence $\{\hat{u}_n\}_{n\in\mathbb{Z}}$;

Let Ω be the set of limit points of $\left\{T^k(\hat{u})\right\}_{k\in\mathbb{N}}$, where T is the left shift. The dynamical system (T, Ω) is strictly ergodic.

With $K(a) = 1$, $K(b) \neq 0$, $V(a) = 0$ and $V(b) \in \mathbb{R}$, to each $\omega \in \Omega$ associate a self-adjoint bounded linear operator $H_{\omega,K,V} : \ell^2(\mathbb{Z}, \mathbb{C}) \hookleftarrow$:

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Quantum dynamics of an electron wavepacket in a *quasicrystal* is modeled by

$$\phi(n,t) = e^{-itH}\phi(n,t_0).$$

While it isn't *sufficient*, it is *necessary* to study the spectrum of H in order to understand the quantum dynamics.

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The spectrum of $H_{\omega,K,V}$ is independent of $\omega \in \Omega$. Moreover, the spectrum consists of those $\lambda \in [-2, 2]$, for which $\gamma(\lambda) \in B_\infty(\gamma)$, where

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In the case $K(a) = K(b) = 1$:

- The problem was introduced by physicists Kohmoto et. al. and Ostlund et. al. in 1983;
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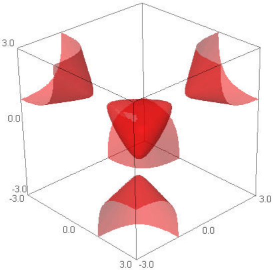

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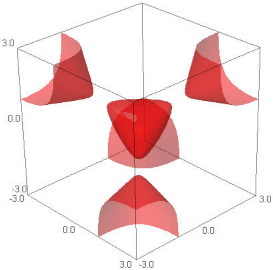
Denote the sphere by $\mathbb{S}_V$...



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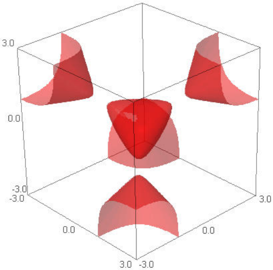

- Every point on the discs escapes to infinity;
- The spheres in the center are invariant under f;
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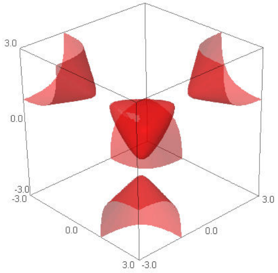

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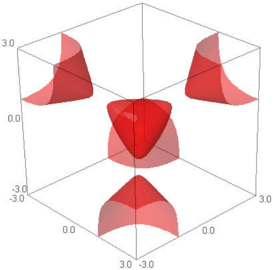

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A. Romanelli, *The Fibonacci Quantum Walk and its Classical Trace Map*

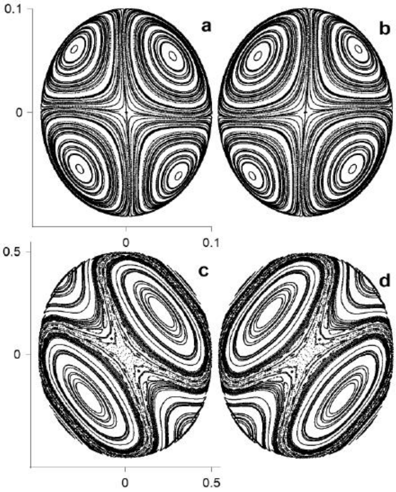

FIG. 1: Poincare sections for the two hemispheres of the trace map. Arbitrary units are used. For C=-0.99 a) back and b) front hemispheres. For C=-0.7 c) back and d) front hemispheres.

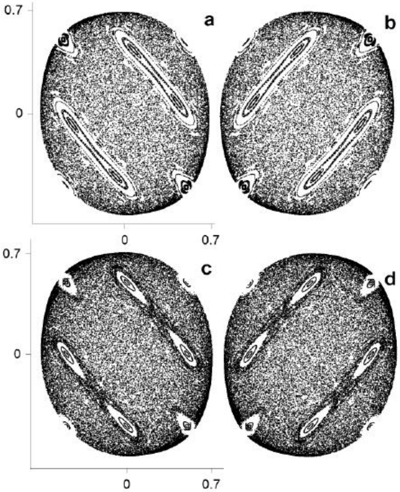

FIG. 2: Poincare sections for the two hemispheres of the trace map. Arbitrary units are used. For C=-0.53 a) back and b) front hemispheres. For C=-0.5 c) back and d) front hemispheres.

Theorem (Joint with A. Gorodetski)

There exists $V_0 \in (-1, 0)$ such that for all $V \in (V_0, 0)$, the map f_V has a locally maximal compact hyperbolic set Λ_V in $\mathbb{S}_V$ with the following properties.

1. *The sequence $\{\Lambda_V\}$ is dynamically monotone; that is, for $V_2 > V_1$, Λ_{V_2} contains the continuation of Λ_{V_1};*
2. *The Hausdorff dimension of Λ_V tends to two as V tends to zero;*
3. *Λ_V exhibits persistent generically unfolding quadratic homoclinic tangencies;*
4. *There exists a residual set $\mathcal{R} \subset (V_0, 0)$ such that for all $V \in \mathcal{R}$, $f|_{\mathbb{S}_V}$ has a nested sequence of hyperbolic sets $\Lambda_V^{(0)} \subseteq \Lambda_V^{(1)} \subseteq \cdots$, with $\Lambda_V^{(0)} = \Lambda_V$, and the Hausdorff dimension of $\Lambda_V^{(n)}$ tends to two as n tends to infinity;*
5. *The set $\Omega_V = \overline{\bigcup_{n \in \mathbb{N}} \Lambda_V^{(n)}}$ is a transitive invariant set of $f|_{\mathbb{S}_V}$ whose Hausdorff dimension is equal to two;*
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Question: Is Ω_V of positive Lebesgue measure?

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