# Global dynamics of the Kummer-Schwarz differential equation

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## Introduction

The Schwarzian derivative

$$\{x,t\} = \frac{\dddot{x}(t)}{\dot{x}(t)} - \frac{3}{2}\left(\frac{\ddot{x}(t)}{\dot{x}(t)}\right)^2, \tag{1}$$

plays an important role in the treatment of univalent functions, see details in [5] and references therein. Here the dot denotes derivative with respect to the independent variable $t$. When the right hand in equation (1) is taken at zero, the resulting equation is the Kummer-Schwarz equation which is given by

$$2\dot{x}\,\dddot{x} - 3\ddot{x}^2 = 0, \tag{2}$$

and is of special interest due to its relationship to the Schwarzian derivative and its exceptional algebraic properties. This equation is also encountered in the study of geodesic curves in spaces of constant curvature, Lie lists the characteristic functions for its contact symmetries, see more results on this differential equation in [1], [4], [5] and [6]. But up to now nobody has described its global dynamics. This will be the objective of this paper.

The Kummer-Schwarz equation of third order (2) can be written as the following rational differential system of first order

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = \frac{3z^2}{2y}, \tag{3}$$

in $\mathbb{R}^3$. Rescaling the time according to $d\tau/dt = 2y$, we obtain the equivalent polynomial differential system (outside the plane $y = 0$)

$$x' = 2y^2, \qquad y' = 2yz, \qquad z' = 3z^2, \tag{4}$$

here the prime denotes derivative with respect to the new independent variable $\tau$. This differential system is called the *Kummer-Schwarz differential system in* $\mathbb{R}^3$.

We will study the flow of the polynomial differential system (4) in the phase space $\mathbb{R}^3$, of course, in order to describe the flow of the differential system (3) we must omit the plane $y = 0$.

## 1. Symmetries and reduction of the flow to the quadrant $y \geq 0$ and $z \geq 0$

The differential system (4) is invariant under the following two symmetries

$$S_1(x,y,z) = (x,-y,z), \quad \text{and} \quad S_2(x,y,z,\tau) = (-x,y,-z,-\tau).$$

The symmetry $S_1$ says that the flow of system (4) is symmetric with respect to the plane $y = 0$. Therefore, if $(x(\tau), y(\tau), z(\tau))$ is a solution of (4), then $(x(\tau), -y(\tau), z(\tau))$ is also solution of (4). The symmetry $S_2$ says that the flow of system (4) is symmetric with respect to the $y$–axis reversing the sense of the orbits. Therefore, if $(x(\tau), y(\tau), z(\tau))$ is a solution of (4), then $(-x(-\tau), y(\tau), -z(-\tau))$ is also solution of (4). Using both symmetries in order to describe the flow of system (4) in $\mathbb{R}^3$ it is enough to describe the flow of system (4) on the quadrant

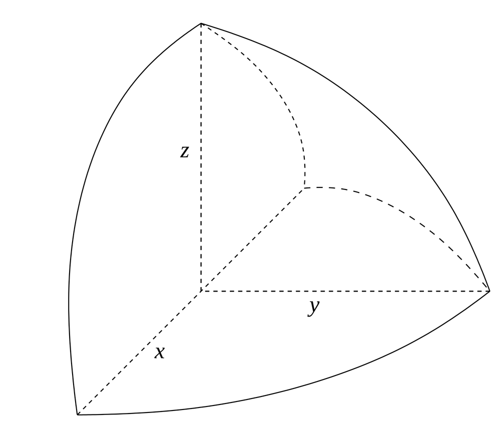
$$Q = \{(x,y,z) \in \mathbb{R}^3 : y \geq 0,\ z \geq 0\}.$$

## 2. The Poincaré compactification

It is know that a polynomial differential system in $\mathbb{R}^3$ can be extended to a unique analytic differential system on the closed ball $B$ of radius $1$ centered at the origin of $\mathbb{R}^3$, called the *Poincaré ball*. More precisely, the whole space $\mathbb{R}^3$ is identified with the interior of $B$, and the infinity of $\mathbb{R}^3$ is identified with the boundary of $B$, i.e. with the 2–dimensional sphere $S^2$. For more details see [2] and [7], and the appendix of this paper. The know technique for making such an extension is called the Poincaré compactification and it allows to study the dynamics of a polynomial differential system in a neighborhood of infinity. Poincaré introduced this compactification for polynomial differential systems in $\mathbb{R}^2$.

## 3. The global dynamics

Our main result is the description of the global dynamics of the Kummer-Schwarz differential system (4) on the compactified quadrat $\emptyset Q$ of $Q$ inside the Poincaré ball $B$, see Figure below. More precisely, we describe all the $\alpha$– and $\omega$–limit sets of all the orbits of the Kummer-Schwarz differential system (4). For the standard definitions of orbit, $\alpha$– and $\omega$–limit sets of an orbit, and of the Poincaré compactification, see for instance [3].




**Theorem 1.** *The following statements hold for the Kummer-Schwarz differential system* (4)*.*

*(a) On the quadrant $\overline{Q}$ the equilibrium points are all the points of the $x$–axis, including its endpoints at infinity ($X_-$ at the end of the negative $x$–axis and $X_+$ at the end of the positive $x$–axis), and additionally the endpoint $Z_+$ at infinity of the positive $z$–axis, see Figure first below.*

*(b) On the invariant boundary $y = 0$ of the quadrant $\emptyset Q$ the orbits are the half–straight lines parallel to the $z$–axis having $\alpha$–limit an equilibrium point of the $x$–axis and $\omega$–limit the equilibrium point $Z_+$, see Figure below.*

*(c) On the invariant boundary $z = 0$ of the quadrant $\emptyset Q$ the orbits are the straight lines parallel to the $x$-axis having $\alpha$-limit the equilibrium point $X_-$ and $\omega$-limit the equilibrium point $X_+$, see Figure first figure below.*

*(d) On the infinity $\S^2 \cap \overline{Q}$ the flow is qualitatively the one described in Figure (the second figure below), i.e., without taken into account the three equilibrium points $Z_+$, $X_-$ and $X_+$ at the infinity of the quadrant $\overline{Q}$ all the other orbits have $\omega$-limit at the equilibrium point $Z_+$, and $\alpha$–limit either at $X_-$, or at $X_+$.*

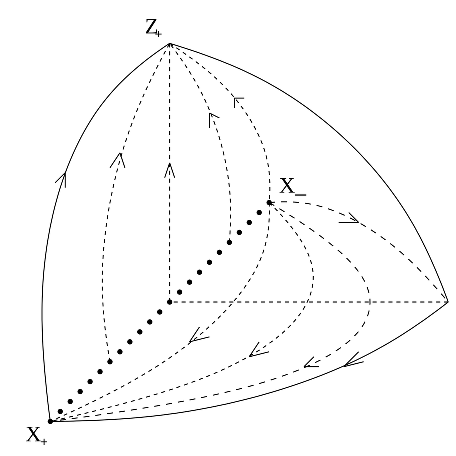
*(e) The explicit solution $(x(\tau), y(\tau), z(\tau))$ of the differential system* (4) *such that $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ is*

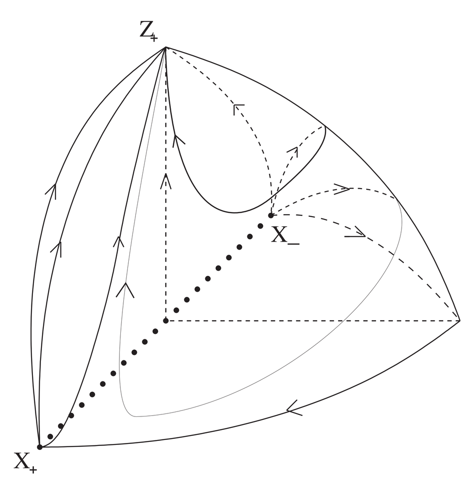
$$\begin{aligned} x(\tau) &= x_0 + \frac{2y_0^2}{z_0}\left(\frac{1}{(1-3z_0\tau)^{1/3}} - 1\right), \\ y(\tau) &= \frac{y_0}{(1-3z_0\tau)^{2/3}}, \\ z(\tau) &= \frac{z_0}{1-3z_0\tau}. \end{aligned} \tag{5}$$

*(f) Let $\gamma$ be an orbit contained in the interior of the quadrant $\overline{Q}$. Then the $\alpha$–limit of $\gamma$ is the equilibrium point $X_-$ and its $\omega$–limit is the equilibrium point $Z_+$.*

*(g) The differential system* (4) *has two independent first integrals $H_1 = z^2/y^3$ and $H_2 = x - 2y^2/z$. The set $\{H_1 = h_1\} \cap \{H_2 = h_2\} \cap Q$ is an orbit $\gamma$ with endpoints at $X_-$ and $Z_+$.*

The two independent first integrals $H_1$ and $H_2$ of statement (g) of Theorem 1 are due to Goviender and Leach [4].




## 4. Proof of Theorem 1

From the equations of the differential system (4) it follows immediately that the $x$–axis is filled with equilibrium points, because $x' = y' = z' = 0$ when $y = z = 0$. Now we shall study the equilibriums points at the infinity of the quadrant $\overline{Q}$ using the Poincaré compactification of $\mathbb{R}^3$ described in the appendix.

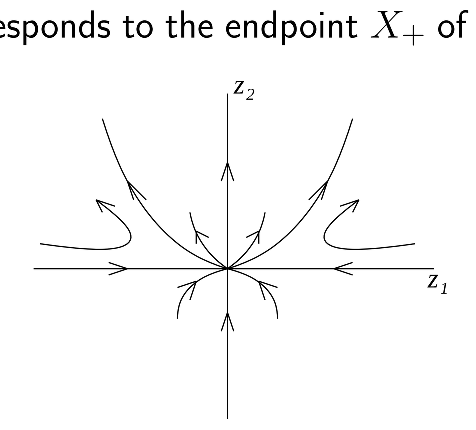
We start studying the equilibrium points at the infinity located on the local chart $U_1$, i.e. in $x > 0$ and its boundary at infinity. Thus the compactified differential system (4) in the local chart $U_1$ is given by

$$\dot{z}_1 = -2z_1^3 + 2z_1z_2, \quad \dot{z}_2 = -2z_1^2z_2 + 3z_2^2, \quad \dot{z}_3 = -2z_1^2z_3. \tag{6}$$

At the infinity $z_3 = 0$ of $U_1$, i.e. in the points of the sphere $S^2$ system (6) reduces to

$$\dot{z}_1 = 2z_1(-z_1^2 + z_2), \qquad \dot{z}_2 = z_2(-2z_1^2 + 3z_2). \tag{7}$$

So, the unique equilibrium point at the infinity of $U_1$ is the origin $(0,0,0)$ of $U_1$. Its linear part has all its eigenvalues equal to zero. Therefore we need to study it using the technique of blow ups, see for more details the Chapter 3 of [3]. Then, we obtain that the local phase portrait of the equilibrium point $(0,0)$ of the differential system (7) is qualitatively the one of Figure below. The equilibrium point $(0,0,0)$ of $U_1$ corresponds to the endpoint $X_+$ of the positive $x$–axis.




## 4. Proof of Theorem 1

The local phase portrait at the equilibrium point $(0,0,0)$ of $V_1$ which corresponds to the endpoint $X_-$ of the negative $x$–axis, is obtained doing symmetry with respect to the center of the sphere $S^2$ and reversing the orientation of the orbits because the degree of the polynomial differential system (4) is $2$.

The flow of system (4) in the local chart $U_2$ is given by the differential system

$$\dot{z}_1 = -2z_1z_2 + 2, \quad \dot{z}_2 = -2z_1^2z_2 + 3z_2^2, \quad \dot{z}_3 = -2z_2z_3. \tag{8}$$

So there are no equilibrium points at infinity in this local chart.

In the local chart $U_3$ the system (4) becomes

$$\dot{z}_1 = -3z_1 + 2z_2^2, \quad \dot{z}_2 = -z_2, \quad \dot{z}_3 = -3z_3. \tag{9}$$

At the infinity of $U_3$ the point $(0,0,0)$ is the unique equilibrium point, and its linear part has the eigenvalues $-1$ and $-3$ with multiplicity two. Therefore, by the Hartman Theorem this equilibrium point is a local attractor, and it corresponds to the endpoint $Z_+$ of the positive $z$–axis.

*Proof of statement $(b)$ of Theorem 1.* From the differential system (4) it follows that $x' = 0$ and $y' = 0$ when $y = 0$, so the plane $y = 0$ and the straight lines $\{y = 0\} \cap \{x = \text{constant}\}$ are invariant by the flow of system (4).

In short, on the invariant boundary $y = 0$ of the quadrant $\overline{Q}$ the orbits are the half–straight lines parallel to the $z$–axis having $\alpha$–limit an equilibrium point of the $x$–axis and $\omega$–limit the equilibrium point $Z_+$, see for more details the proof of statement (a). □

*Proof of statement $(c)$ of Theorem 1.* From the differential system (4) we have that $y' = 0$ and $z' = 0$ when $z = 0$, so the plane $z = 0$ and the straight lines $\{z = 0\} \cap \{y = \text{constant}\}$ are invariant by the flow of system (4).

In summary, on the invariant boundary $z = 0$ of the quadrant $\overline{Q}$ the orbits are the straight lines parallel to the $x$–axis having $\alpha$–limit the equilibrium point $X_-$ and $\omega$–limit the equilibrium point $X_+$, see again for more details the proof of statement (a). □

*Proof of statement $(d)$ of Theorem 1.* From the fact that the infinity $S^2$ is invariant by the compactified flow of the polynomial differential system (4), and the local phase portraits at the three equilibrium points $Z_+$, $X_-$ and $X_+$ at the infinity of the quadrant $\overline{Q}$ studied in the proof of statement (a), it follows that on the infinity $S^2 \cap \overline{Q}$ the flow is qualitatively the one described in Figure below. □

*Proof of statement $(e)$ of Theorem 1.* Let $(x(\tau), y(\tau), z(\tau))$ be the solution of the differential system (4) such that $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$. From the differential equation $z' = 3z^2$ it follows easily that $z(\tau) = z_0/(1 - 3z_0\tau)$. Substituting we have $y(\tau) = y_0/(1-3z_0\tau)^{2/3}$ and $x(\tau) = x_0 + \dfrac{2y_0^2}{z_0}\left(\dfrac{1}{(1-3z_0\tau)^{1/3}} - 1\right)$. □

*Proof of statement $(f)$ of Theorem 1.* Since $x' = 2y^2 > 0$ and $z' = 3z^2 > 0$ in the interior of the quadrant $\emptyset Q$, for every orbit $\gamma$ contained in the interior of the quadrant $\overline{Q}$ we have that its $\alpha$–limit has its $x$–coordinate equal to $-\infty$ and its $z$–coordinate equal to $0$, and its $\omega$–limit has its $x$–coordinate equal to $+\infty$ and its $z$–coordinate equal to $+\infty$. Taking into account either the solution of statement (e), or the phase portrait on the boundary of the quadrant $\overline{Q}$ described in the statements (b), (c) and (d) we get that the $\alpha$–limit of $\gamma$ is the equilibrium point $X_-$ and its $\omega$–limit is the equilibrium point $Z_+$. □

*Proof of statement $(g)$ of Theorem 1.* Let $H_1 = z^2/y^3$ and $H_2 = x - 2y^2/z$. Then, since

$$\frac{dH_k}{d\tau} = \frac{\partial H_k}{\partial x}x' + \frac{\partial H_k}{\partial y}y' + \frac{\partial H_k}{\partial z}z' = 0,$$

for $k = 1, 2$, we obtain that $H_1$ and $H_2$ are first integrals of the differential system (4). Since there gradients are linearly independent except at the points of $z = 0$ and $y = 0$ which have zero Lebesgue measure, these two first integrals are independent.

It is not difficult to show that the set $\{H_1 = h_1\} \cap \{H_2 = h_2\} \cap Q$ has a unique component. Then, from either statement (e), or statement (f) it follows that this set is formed by an orbit $\gamma$ with endpoints at $X_-$ and $Z_+$. □

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