Phase portraits on the Poincaré disc of a SIS model

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Introduction

There is a long time that scientists have been curious about the interaction between portions of a population with particular characteristics, e.g. prey and predator interaction. Moreover, the means of how diseases spread has stimulated the interest mainly within researchers of the biological field.

In addition, the application of mathematical tools in different areas of sciences has been frequent and has had a great impact on the scientific and/or experimental conclusions. For example, Lotka (1925) and Volterra (1926) described independently the dynamics present in biological interactions between two species using ordinary differential equations (ODE).

Besides, continuous models composed by ODE have formed a large part of the traditional mathematical epidemiology literature, mainly because mathematicians have been attracted by applying the ODE's tools, such as their qualitative theory, to the study of infectious diseases in the attempt of using mathematics to contribute positively to the science field and because the mathematical models become indispensable to inform decision-making.

In the present poster, we consider the system of first-order ODE

$$\begin{aligned}\dot{x} &= -bxy - mx + cy + mk,\\ \dot{y} &= bxy - (m+c)y,\end{aligned} \qquad (1)$$

with $bm \neq 0$, where x and y represent, respectively, the portion of the population that has been susceptible to the infection and those who have already been infected.

System (1) is a particular case of the class of classical systems known as **susceptible-infected-susceptible (SIS) models**, introduced by Kermack and McKendrick [2] and studied by Brauer [1], who has assumed that **recovery from the nonfatal infective disease does not yield immunity**. In system (1),

- k is the population size (susceptible people plus infected ones);
- mk is the constant number of births;
- m is the proportional death rate;
- b is the infectivity coefficient of the typical Lotka-Volterra interaction term; and
- c is the recovery coefficient.

As system (1) is assumed to be nonfatal, the standard term which removes dead infected people $-ay$ in [1] is omitted. As usual in the literature, **all the critical points of system (1) will henceforth be called (endemic) steady states** (e.g. see [8]).

Much has been studied on SIS models.

- Studies on the local and global stability: C. Vargas-De-Léon in [6,7] (construction of Lyapunov functions);
- Studies on the integrability: Nucci and Leach [5] (using the Painlevé test) and Llibre and Valls [3] (they have shown the explicit expression of its first integral using the Darboux integrability).

It is clear that the above studies contribute to understand the behavior of the solutions of system (1). However, another approach would be the possibility of drawing its global phase portraits.

Motivated by this idea, **the purpose here was to classify all the topological classes of the global phase portraits of system (1) using tools from the qualitative theory of the ODE**.

The main result is the following:

Theorem 1: *The phase portrait on the Poincaré disc of system (1) is topologically equivalent to one of the two phase portraits shown in Figure 1, modulo reversibility.*

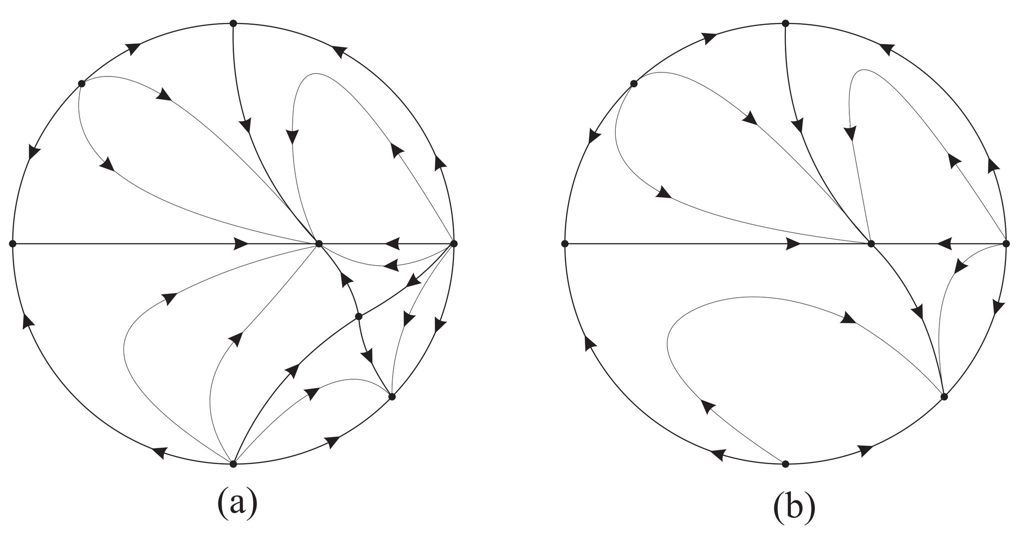

Figure 1: Phase portraits in the Poincaré disc of system (1)

1. Analysis of the system

Finite singular points

Provided that $b \neq 0$, system (1) has only two finite singular points: $p = ((c+m)/b, (-c+bk-m)/b)$, usually known as endemic steady state, and $q = (k, 0)$, usually known as disease-free steady state. In addition, both finite singular points p and q are the same if $bk = c$.

First, we start with the analysis of the endemic steady state p. Translating the singular point p to the origin in system (1), we obtain

$$\begin{aligned}\dot{x} &= -bkx + cx - my - bxy,\\ \dot{y} &= (-c + bk - m)x + bxy,\end{aligned} \qquad (2)$$

which is equivalent to (1). The Jacobian matrix of (2) is given by

$$\mathrm{J}(x,y) = \begin{pmatrix} c - bk - by & -m - bx \\ -c + bk - m + by & bx \end{pmatrix},$$

which implies that

$$\delta = \delta(0,0) = (bk - c - m)m \quad \text{and} \quad \tau = \tau(0,0) = -bk + c.$$

If $(bk - c - m)m < 0$, then p is a saddle point. On the other hand, if $(bk - c - m)m > 0$, then p is a node point, because $\tau^2 - 4\delta = (c - bk + 2m)^2 \geq 0$.

In the case that $(bk - c - m)m = 0$, then p is degenerate. Indeed, it is the case that both finite singular points are the same, i.e. $p = q = (k, 0)$. Here, system (2) becomes

$$\begin{aligned}\dot{x} &= -mx - my - bxy,\\ \dot{y} &= bxy,\end{aligned} \qquad (3)$$

whose Jacobian matrix at $(0,0)$ is

$$\mathrm{J}(0,0) = \begin{pmatrix} -m & -m \\ 0 & 0 \end{pmatrix},$$

so that p is a semi-hyperbolic point. By a linear change of coordinates, we can apply Theorem 2.19 of [4] and conclude that p is a saddle-node point.

The disease-free steady state q can be similarly studied. Thus, we obtain the following:

Proposition 1: *Consider system (1) with $bm \neq 0$ and its two finite steady states p and q. Then:*

1. *If either $m > 0$ and $m > bk - c$, or $m < 0$ and $m < bk - c$, then p is a saddle and q is a node;*
2. *If either $m > 0$ and $m < bk - c$, or $m < 0$ and $m > bk - c$, then p is a node and q is a saddle;*
3. *If $m = bk - c$, then $p = q$ is a semi-hyperbolic saddle-node.*

Infinite singular points

Having classified all the finite singular points, we apply the Poincaré compactification to study the infinite singularities.

In the local chart U_1, where $x = 1/v$ and $y = u/v$, we have:

$$\begin{aligned}\dot{u} &= u(b + bu - cv - cuv - kmv^2),\\ \dot{v} &= v(bu + mv - cuv - kmv^2),\end{aligned} \qquad (4)$$

whose singular points are $(0,0)$ and $(-1,0)$, which are a saddle-node of type SN1 (by Theorem 2.19 of [4]) and a node, respectively.

In the local chart U_2, where $x = u/v$ and $y = 1/v$, the system

$$\begin{aligned}\dot{u} &= -bu - bu^2 + cv + cuv + kmv^2,\\ \dot{v} &= v(-bu + cv + mv)\end{aligned} \qquad (5)$$

has two singular points $(0,0)$ and $(-1,0)$. The latter one is a node and is the same as $(-1,0) \in U_1$, while the former one is a saddle-node of type SN1.

We have just proved the following:

Proposition 2: *The infinite singular points of system (1) are the origin of charts U_1, V_1, U_2 and V_2, which are saddle-node points of type SN1, and $(-1,0)$, belonging to each of the charts U_1 and U_2, which is a node point.*

The existence of invariant straight lines

Knowing the local behavior around each finite and infinite singular points, another useful tool to describe the phase portraits of differential systems is the existence of invariant curves. The next result shows system (1) has at least three invariant straight lines.

Proposition 3: *Let $bm \neq 0$. System (1) has at least three invariant straight lines given by $f_1(x,y) = y$ and $f_2(x,y) = k - x - y$, and additionally $f_3(x,y) = k - x$, if $c = bk$.*

Proof: We can find $K_1(x,y) = bx - m - c$, $K_2(x,y) = -m$ and $K_3(x,y) = -m - by$ as the cofactors of $f_1(x,y)$, $f_2(x,y)$ and $f_3(x,y)$ (if $c = bk$), respectively. ∎

2. Main result

From Propositions 1 and 2 we get all the information about the local behavior of finite and infinite singular points, respectively. Using the continuity of solutions and primary definitions and results of ODE and the existence of invariant straight lines of system (1) stated by Proposition 3, its global phase portraits can be easily drawn.

Essentially, we have only two cases. The finite steady state q is the intersection of the invariant curves $f_1(x,y) = f_2(x,y) = 0$, and the other finite steady state p lies on the curve $f_2(x,y) = 0$. Firstly, we note that the four infinite singular points continue to be the same points no matter what conditions are being considered.

Case 1 According to items (1) and (2) of Proposition 1, p (respectively, q) is a saddle (respectively, a node) the one way and the other a node (respectively, a saddle). The subtle difference here is the position of point p. While q remains on the line $\{y = 0\}$, p is in the lower part of the Poincaré disc the one way and the other in the upper part. The phase portrait of both subcases above is topologically equivalent to the one which is shown in Figure 1(a).

Case 2 Item (3) of Proposition 1 assumes the existence of only one finite singular point, $p = q$. Here, when $m = bk - c$, both p and q become only one degenerate singularity which bifurcates into a saddle-node point. Again, no changes are applied to the infinite singular points. The phase portrait of this case is topologically equivalent to the one which is shown in Figure 1(b).

Finally, Theorem 1 has been proved.

3. Conclusions and discussions

We have proved the existence of only two classes of global phase portraits of the quadratic system (1). In the qualitative theory of ODE it is quite important to know the global behavior of solutions of systems and, in general, this is not an easy task.

In the case represented by Figure 1(a), it is clear that **while the steady state q characterizes the presence of only susceptible individuals, p indicates the mutual presence of susceptible and infected people**. Besides, as q is an asymptotically stable node, the disease seems to be controlled and the whole population tends to be healthy but susceptible to be infected again. As p is an unstable saddle steady state, it suggests that there is no harmony between the number of susceptible people and infected ones, although some of the solutions tend to q, indicating the control of the disease.

In case of Figure 1(b), all the solutions tend to q (regarding that $x, y > 0$), i.e. if $m = bk - c$, the disease is supposed to be controlled and the whole population is inclined to be healthy but susceptible to the reinfection.

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