Discontinuous piecewise linear differential systems with 2 zones in the plane

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1. Introduction and Main Results

General piecewise linear systems have attracted great attention in the past years specially due to their simplicity. Landmarks in this area are the works of Andronov et al. in [1] and Chua [2]. Here we are interested in piecewise linear systems in the plane with two zones, for short PLSPTZ, that is piecewise linear systems in the plane where the two linearity regions are separated by a straight line $\mathcal{L}$.

An example of a continuous PLSPTZ with one limit cycle can be found in [1]. Lum and Chua in [7] presented other examples of continuous PLSPTZ with one limit cycle and stated the following conjecture: a continuous PLSPTZ has at most one limit cycle.

In [3] was given an affirmative answer to the Lum and Chua conjecture. So, in order to obtain a PLSPTZ with two or more limit cycles it is necessary that the system be discontinuous at points on the line $\mathcal{L}$. In [4] Han and Zhang studied discontinuous PLSPTZ with two limit cycles and stated the following conjecture: a discontinuous PLSPTZ has at most two limit cycles.

The first example of a discontinuous PLSPTZ with three limit cycles was proposed by Huan and Yang in [5] giving a negative answer to the conjecture of Han and Zhang. A rigorous proof of the existence of three limit cycles in the example presented in [5] was given by Llibre and Ponce in [6].

In this poster we consider the following one–parameter family of discontinuous piecewise linear system with two zones [5, 6]

$$X' = \begin{cases} A^- X, & \text{if } x < \varepsilon, \\ A^+ X, & \text{if } x \geq \varepsilon, \end{cases} \tag{1}$$

where ε is a real parameter, the matrices A^- and A^+ are

$$A^- = \begin{pmatrix} \frac{4}{3} & -\frac{20}{3} \\ \frac{377}{750} & -\frac{26}{15} \end{pmatrix}, \quad A^+ = \begin{pmatrix} \frac{19}{50} & -1 \\ 1 & \frac{19}{50} \end{pmatrix} \tag{2}$$

and the prime denotes derivative with respect to the independent variable t (time). System (1) with $\varepsilon = 1$ was studied in [5, 6]. Our main results are the following.

Theorem 1. *The one–parameter family of piecewise linear systems with two zones (1) has:*

(a) One unstable focus at the origin and no limit cycle when $\varepsilon < 0$.

(b) One unstable focus at the origin and no limit cycle when $\varepsilon = 0$.

(c) One stable focus at the origin and three limit cycles surrounding the origin for each $\varepsilon > 0$. One limit cycle is stable and the other two are unstable. See Figure 1. So system (1) presents a triple Hopf bifurcation at $\varepsilon = 0$.

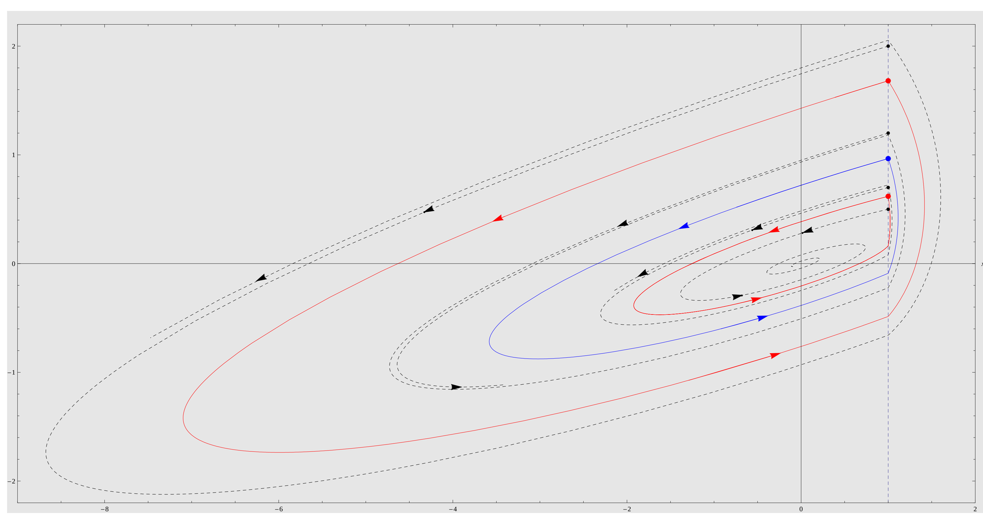

Figure 1: *Three limit cycles of system (1) for $\varepsilon > 0$. In this figure $\varepsilon = 1/2$.*

2. Proof of Theorem 1

Define $\mathcal{L}_\varepsilon = \{(x, y) \in \mathbb{R}^2 : x = \varepsilon\}$ for each $\varepsilon \in \mathbb{R}$. For any initial condition $(\varepsilon, y_0) \in \mathcal{L}_\varepsilon$, the solutions of $X' = A^+X$ will be denoted by $X^+(t, \varepsilon, y_0) = (x^+(t, \varepsilon, y_0), y^+(t, \varepsilon, y_0))$ while the solutions of $X' = A^-X$ will be denoted by $X^-(t, \varepsilon, y_0) = (x^-(t, \varepsilon, y_0), y^-(t, \varepsilon, y_0))$, where

$$\begin{aligned} x^+(t, \varepsilon, y_0) &= e^{\frac{19t}{50}}(\varepsilon \cos(t) - y_0 \sin(t)), \\ y^+(t, \varepsilon, y_0) &= e^{\frac{19t}{50}}(y_0 \cos(t) + \varepsilon \sin(t)), \end{aligned} \tag{3}$$

$$\begin{aligned} x^-(t, \varepsilon, y_0) &= \frac{1}{15} e^{-\frac{t}{5}}(15\varepsilon \cos(t) + (23\varepsilon - 100 y_0)\sin(t)), \\ y^-(t, \varepsilon, y_0) &= \frac{1}{750} e^{-\frac{t}{5}}(750 y_0 \cos(t) + (377\varepsilon - 1150 y_0)\sin(t)). \end{aligned} \tag{4}$$

The origin is a real unstable focus for $X' = A^+X$ and a virtual stable focus for $X' = A^-X$ when $\varepsilon < 0$ and it is a virtual unstable focus for $X' = A^+X$ and a real stable focus for $X' = A^+X$ when $\varepsilon > 0$.

2.1 System (1) for $\varepsilon = 0$

Proposition 1. *Consider the one–parameter family of piecewise linear systems with two zones (1). Assume that $\varepsilon = 0$. Then the origin is an unstable focus.*

Proof. Consider an initial condition $(0, y_0) \in \mathcal{L}_0$ with $y_0 \neq 0$. By (3) and (4) we have

$$X^+(t, 0, y_0) = \left(-y_0\, e^{\frac{19t}{50}} \sin(t), y_0\, e^{\frac{19t}{50}} \cos(t)\right)$$

and

$$X^-(t, 0, y_0) = \left(-\frac{20}{3} y_0\, e^{-\frac{t}{5}} \sin(t), -\frac{20}{3} y_0\, e^{-\frac{t}{5}} \left(\frac{23}{100}\sin(t) - \frac{3}{20}\cos(t)\right)\right).$$

Consider $\Sigma_0 = \{(0, y) \in \mathcal{L}_0 : y > 0\}$ and define the Poincaré map (first return map) $P : \Sigma_0 \to \Sigma_0$ by $P(0, y_0) = X^+(\pi, X^-(\pi, 0, y_0))$. From the above expressions of $X^+(t, 0, y_0)$ and $X^-(t, 0, y_0)$ it follows that $P(0, y_0) = \left(0, y_0\, e^{\frac{9\pi}{50}}\right)$. As $y_0\, e^{\frac{9\pi}{50}} > y_0$ the Poincaré map is an expansive map. □

2.2 System (1) for $\varepsilon > 0$

Consider $\varepsilon > 0$ fixed and define $\mathcal{H}_\varepsilon^+ = \{(x, y) \in \mathbb{R}^2 : x > \varepsilon\}$ and $\mathcal{H}_\varepsilon^- = \{(x, y) \in \mathbb{R}^2 : x < \varepsilon\}$. The flow of (1) enters to the half plane $\mathcal{H}_\varepsilon^+$ through the set $\mathcal{E}_\varepsilon^+ = \{(\varepsilon, y) \in \mathcal{L}_\varepsilon : y < \varepsilon/5\}$ and exits it through the set $\mathcal{E}_\varepsilon^- = \{(\varepsilon, y) \in \mathcal{L}_\varepsilon : y > 19\varepsilon/50\}$. The segment $\mathcal{S}_\varepsilon = \{(\varepsilon, y) \in \mathcal{L}_\varepsilon : \varepsilon/5 < y < 19\varepsilon/50\}$ is the sliding segment on $\mathcal{L}_\varepsilon$. Thus if system (1) for $\varepsilon > 0$ has non–sliding periodic orbits these must surround the segment $\mathcal{S}_\varepsilon$.

Suppose that for $y_0 > 19\varepsilon/50$ there is a periodic orbit in the phase portrait of (1), denoted here by Γ. If $t^- > 0$ is the smallest time such that $X^-(t^-, \varepsilon, y_0) \in \mathcal{L}_\varepsilon$ and $t^+ > 0$ is the smallest time such that $X^+(-t^+, \varepsilon, y_0) \in \mathcal{L}_\varepsilon$, then the point (ε, y_0) of the set $\Gamma \cap \mathcal{L}_\varepsilon$ is associated with a solution of the form $(t^+, t^-, y_0, \varepsilon)$ of

$$\begin{cases} F_1(t^+, t^-, y_0, \varepsilon) = x^-(t^-, \varepsilon, y_0) - \varepsilon = 0, \\ F_2(t^+, t^-, y_0, \varepsilon) = x^+(-t^+, \varepsilon, y_0) - \varepsilon = 0, \\ F_3(t^+, t^-, y_0, \varepsilon) = y^-(t^-, \varepsilon, y_0) - y^+(-t^+, \varepsilon, y_0) = 0, \end{cases} \tag{5}$$

where $y^-(t^-, \varepsilon, y_0) = y^+(-t^+, \varepsilon, y_0) < \varepsilon/5$.

According to [6], for $\varepsilon = 1$, we have that

$$\left(t_1^+, t_1^-, y_1^0\right) = (1.48, 3.45, 1.68), \tag{6}$$

$$\left(t_2^+, t_2^-, y_2^0\right) = (0.85, 3.78, 0.96), \tag{7}$$

$$\left(t_3^+, t_3^-, y_3^0\right) = (0.39, 4.46, 0.61), \tag{8}$$

are solutions of (5) and, therefore, are points of intersection of three periodic orbits with the set $\mathcal{L}_1$. In other words, from (5), (6), (7) and (8) we have $F_i(t_k^+, t_k^-, y_k^0, 1) = 0$, $i = 1, 2, 3$, $k = 1, 2, 3$.

We will prove that, for each $\varepsilon > 0$ fixed, system (5) has at least three solutions of the form

$$p_1(\varepsilon) = (t_1^+, t_1^-, y_1^0\varepsilon, \varepsilon),\ p_2(\varepsilon) = (t_2^+, t_2^-, y_2^0\varepsilon, \varepsilon),\ p_3(\varepsilon) = (t_3^+, t_3^-, y_3^0\varepsilon, \varepsilon), \tag{9}$$

where t_k^+, t_k^- and y_k^0, for $k = 1, 2, 3$, are given by (6), (7) and (8). In other words, the values of y_0 associated with periodic solutions vary linearly with $\varepsilon > 0$ and so for every $\varepsilon > 0$ there are three isolated periodic orbits in the phase portrait of (1).

Proposition 2. *Consider (1) with $\varepsilon > 0$. Then there are at least three limit cycles surrounding the origin.*

Proof. The proof is immediate since for each $i = 1, 2, 3$, the function F_i in (5) is positively homogeneous in the variable ε, that is, for each $i = 1, 2, 3$, $F_i(p_k(\varepsilon)) = \varepsilon F_i(p_k(1)) = 0$, $k = 1, 2, 3$, for all $\varepsilon > 0$. □

2.3 System (1) for $\varepsilon < 0$

Consider $\varepsilon < 0$ fixed and define $\mathcal{H}_\varepsilon^+ = \{(x, y) \in \mathbb{R}^2 : x > \varepsilon\}$ and $\mathcal{H}_\varepsilon^- = \{(x, y) \in \mathbb{R}^2 : x < \varepsilon\}$. The flow of (1) enters to the half plane $\mathcal{H}_\varepsilon^+$ through the set $\mathcal{E}_\varepsilon^+ = \{(\varepsilon, y) \in \mathcal{L}_\varepsilon : y < 19\varepsilon/50\}$ and exits it through the set $\mathcal{E}_\varepsilon^- = \{(\varepsilon, y) \in \mathcal{L}_\varepsilon : y > \varepsilon/5\}$. The segment $\mathcal{S}_\varepsilon = \{(\varepsilon, y) \in \mathcal{L}_\varepsilon : 19\varepsilon/50 < y < \varepsilon/5\}$ is the sliding segment on $\mathcal{L}_\varepsilon$. Figure 2 illustrates $\mathcal{S}_\varepsilon$ and the nullclines of system (1) in the whole plane: continuous lines for the vector field defined by A^- and dashed ones for the vector field defined by A^+.

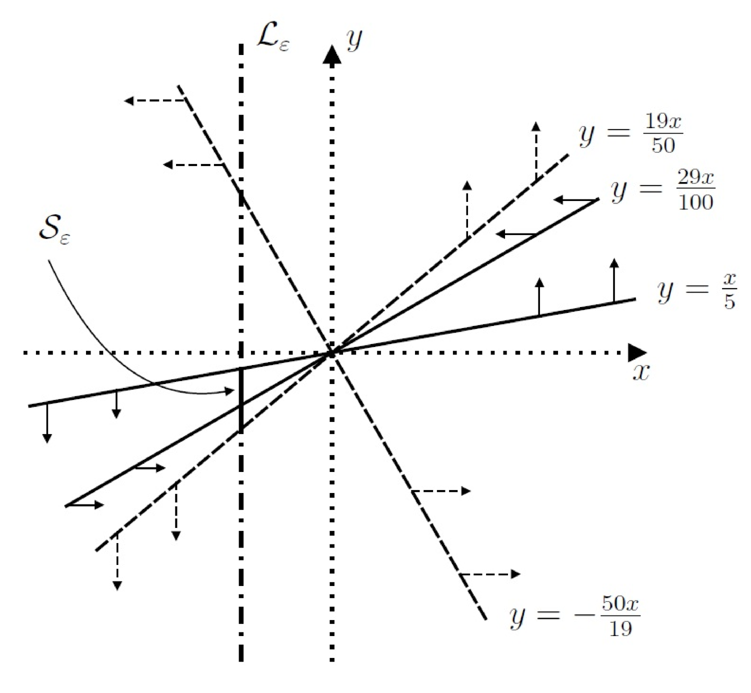

Figure 2: *Nullclines of system (1) in the whole plane.*

Let $Q_0 = (\varepsilon, 19\varepsilon/50)$ be an endpoint of the sliding segment $\mathcal{S}_\varepsilon$. Let Q_2 be the first intersection of the solution $X^+(t, Q_0)$ with $\mathcal{L}_\varepsilon$ for $t > 0$. Denote by Q_1 the intersection of $\mathcal{L}_\varepsilon$ with the line $y = -50x/19$, that is $Q_1 = (\varepsilon, -50\varepsilon/19)$. See Figure 3.

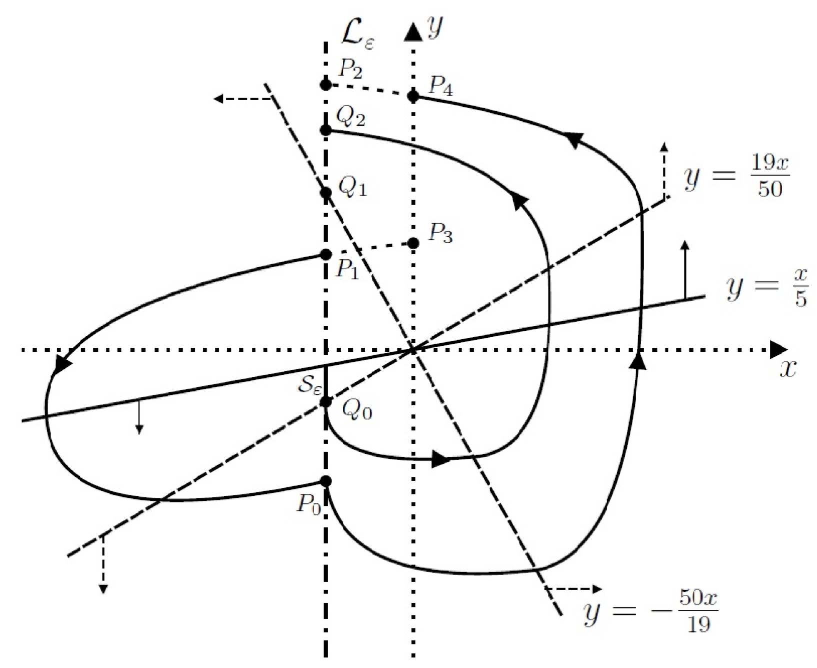

Figure 3: *Points Q_0, Q_1 and Q_2.*

Lemma 1. *Let $Q_1 = (\varepsilon, -50\varepsilon/19)$ and $Q_2 = X^+(t_0, Q_0) = (\varepsilon, y^+(t_0, Q_0)) \in \mathcal{L}_\varepsilon$, $t_0 > 0$. Then $y^+(t_0, Q_0) > -50\varepsilon/19$. See Figure 3.*

Consider a point $P_0 = (\varepsilon, y_0) \in \mathcal{E}_\varepsilon^+$ (see Figure 3). Denote by $\tau_1 > 0$ the smallest time such that

$$P_1 = X^-(-\tau_1, P_0) = \left(x^-(-\tau_1, P_0), y^-(-\tau_1, P_0)\right) \in \mathcal{L}_\varepsilon$$

and by $\tau_2 > 0$ the smallest time such that

$$P_2 = X^+(\tau_2, P_0) = \left(x^+(\tau_2, P_0), y^+(\tau_2, P_0)\right) \in \mathcal{L}_\varepsilon.$$

In analogous way, denote by $\tau_3 > 0$ the smallest time such that

$$P_3 = X^-(-\tau_3, P_0) = \left(x^-(-\tau_3, P_0), y^-(-\tau_3, P_0)\right) \in \{x = 0, y > 0\}$$

and by $\tau_4 > 0$ the smallest time such that

$$P_4 = X^+(\tau_4, P_0) = \left(x^+(\tau_4, P_0), y^+(\tau_4, P_0)\right) \in \{x = 0, y > 0\}$$

(see Figure 3). In order to study the nonexistence of non–sliding limit cycles of system (1) when $\varepsilon < 0$ we will prove that the function

$$d(\varepsilon, y_0) = y^+(\tau_2, P_0) - y^-(-\tau_1, P_0) \tag{10}$$

is positive.

From Lemma 1 it is sufficient to prove that the function

$$\tilde{d}(\varepsilon, y_0) = y^+(\tau_4, P_0) - y^-(-\tau_3, P_0) \tag{11}$$

is positive, for all $\varepsilon < 0$. In fact,

$$d(\varepsilon, y_0) = y^+(\tau_2, P_0) - y^-(-\tau_1, P_0) > y^+(\tau_4, P_0) - y^-(-\tau_3, P_0) = \tilde{d}(\varepsilon, y_0),$$

since $y^+(\tau_2, P_0) > y^+(\tau_4, P_0)$ and $y^-(-\tau_3, P_0) > y^-(-\tau_1, P_0)$. See Figures 2 and 3. We have the following proposition.

Proposition 3. *The function $\tilde{d}(\varepsilon, y_0)$ is positive for all $\varepsilon < 0$.*

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