Oscillatory motions in the Restricted Circular Planar Three Body Problem

New trends in Dynamical Systems

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The (circular) restricted planar three body problem (RCP3BP)

We consider the motion of a particle q with zero mass under the effects of the Newtonian gravitational force exerted by two primaries q_1 and q_2 of masses μ and $1-\mu$ evolving in circular orbits around their center of mass.

This is a particular case of the elliptic one, where the primaries move in elliptic orbits.

Typical models in the elliptic case with eccentricity e_0:

- Sun–Jupiter–asteroid or comet: $e_0 = 0.048$
- Sun–Earth–Moon systems: $e_0 = 0.016$

In this work we will consider $e_0 = 0$

The equations of the RCP3BP

The motion of the particle q is described by

$$\frac{d^2q}{dt^2} = \frac{(1-\mu)(q_1(t)-q)}{|q_1(t)-q|^3} + \frac{\mu(q_2(t)-q)}{|q_2(t)-q|^3},$$

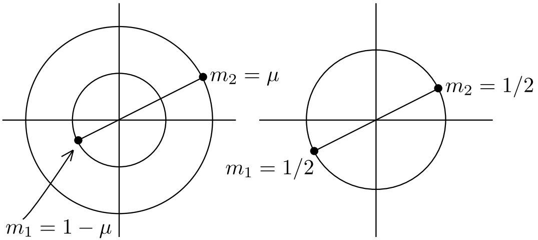
where $q_1(t) = -\mu q_0(t)$, $q_2(t) = (1-\mu)q_0(t)$ and $q_0(t) = (\cos t, \sin t)$ correspond to the circular motion of the primaries.

This is a 2π-periodic in time Hamiltonian system (2 and 1/2 degrees of freedom) with Hamiltonian

$$\mathcal{H}(q,p,t;\mu) = \frac{p^2}{2} - \frac{(1-\mu)}{|q-q_1(t)|} - \frac{\mu}{|q-q_2(t)|}.$$

Parameter: μ, not necessarily small.

Observation: $\mu = 1/2$, The Hamiltonian is π-periodic in time.


$m_2 = \mu$
$m_1 = 1 - \mu$
$m_2 = 1/2$
$m_1 = 1/2$

Oscillatory motions in the RCP3BP

Chazy (1922) gave a classification of all possible states that a three body problem can approach as time tends to infinity.
For the restricted three body problem (either planar or spatial, circular or elliptic) the possible final states are reduced to four:

- $H^{\pm}$ (hyperbolic): $\|q(t)\| \to \infty$ and $\|\dot{q}(t)\| \to c > 0$ as $t \to \pm\infty$.
- $P^{\pm}$ (parabolic): $\|q(t)\| \to \infty$ and $\|\dot{q}(t)\| \to 0$ as $t \to \pm\infty$.
- $B^{\pm}$ (bounded): $\limsup_{t\to\pm\infty} \|q\| < +\infty$.
- $OS^{\pm}$ (oscillatory): $\limsup_{t\to\pm\infty} \|q\| = +\infty$ and $\liminf_{t\to\pm\infty} \|q\| < +\infty$.

Examples of all types of motion except oscillatory were already known by Chazy.

Oscillatory motions in the RCP3BP

Oscillatory motions were proved by

1. Sitnikov, 1960 considered the restricted spatial three body problem with mass ratio $\mu = 1/2$ and the three bodies in a certain symmetric configuration.
2. Moser, 1973 gave a new proof considering the invariant manifolds of infinity and prove that they intersected transversally. Then, one could establish symbolic dynamics close to these invariant manifolds which lead to the existence of oscillatory motions.
3. LLibre and Simó, 1980 (Oscillatory solutions in the planar circular restricted three-body problem, Mathematische Annalen 248) proved their existence for the RCP3BP with μ small enough, using that for $\mu = 0$ the stable and unstable invariant manifolds coincide and Melnikov Theory. They considered the Jacobi constant $\mathcal{J}$ big enough.
   The proof is only valid for μ exponentially small with respect to the Jacobi constant. The orbits that they obtain are far from collisions.

Oscillatory motions in the RCP3BP

1. Xia, 1992 (Melnikov method and transversal homoclinic points in the restricted three-body problem, Journal of Differential Equations 96, 1) showed that the invariant manifolds intersect transversally for any mass ratio $\mu \in (0, 1/2]$ except for a finite number of values using analyticity arguments and LLibre-Simó results.
2. Llibre and Simó, 1980 proved the existence of oscillatory motions for the (non necessarily restricted) collinear three body problem (Some homoclinic phenomena in the three-body problem, J. Differential Equations 37,3)
3. J. Galante and V. Kaloshin, 2011 use Aubry-Mather theory and semi-infinite regions of instability to prove the existence of orbits which initially are in the range of our Solar System and become oscillatory as time tends to infinity for the RPC3BP with a realistic mass ratio for the Jupiter-Sun pair. (Destruction of invariant curves in the restricted circular planar three-body problem by using comparison of action, Duke Mathematical Journal, 159)

Oscillatory motions in the RCP3BP

We will show that there exist oscillatory orbits for the RPC3BP, which are orbits that satisfy that

$$\limsup_{t\to\pm\infty} \|q\| = +\infty \quad \text{and} \quad \liminf_{t\to\pm\infty} \|q\| < +\infty.$$

When $\mu = 0$, the motion of the massless body is only influenced by one of the primaries and therefore it satisfies Kepler's laws. This implies that oscillatory motions cannot exist. We will show that oscillatory orbits do exist for any value of mass ratio $\mu \in (0, 1/2]$.

Oscillatory motions in the RCP3BP

The method we will use is similar to the method by LLibre-Simó, where they showed that these orbits come from the existence of chaotic behavior near some parabolic orbits of the two body problem.
This chaotic behaviour comes from the splitting of some invariant manifold near the parabolic orbits of the two body problem.

The orbits they found occur for large values of the Jacobi constant and for μ small enough.

We will use the same fact to see that they exist for any value of the mass ratio μ.
Our result will make the RCTBP an "a priori chaotic" system in the language of Arnold diffusion: existence of a family of periodic orbits with a transversal homoclinic orbit.

Hamiltonian equations in polar coordinates

Polar coordinates: $q = (x, y) = (r\cos\alpha, r\sin\alpha)$, $\alpha \in \mathbb{T}$, $r \geq 0$.
Hamiltonian in polar coordinates:

$$H(r, \alpha - t, P_r, P_\alpha; \mu) = \frac{P_r^2}{2} + \frac{P_\alpha^2}{2r^2} - \widetilde{U}(r, \alpha - t; \mu)$$

where the potential $\widetilde{U}(r, \phi; \mu)$ is given by:

$$\widetilde{U}(r, \phi; \mu) = \frac{(1-\mu)}{(r^2 - 2\mu r\cos\phi + \mu^2)^{1/2}} + \frac{\mu}{(r^2 + 2(1-\mu) r\cos\phi + (1-\mu)^2)^{1/2}}.$$

(r, P_r) and (α, P_α) are pairs of conjugate variables. Notation: $P_\alpha = G$, $P_r = y$

$$H(r, \alpha - t, y, G; \mu) = \frac{y^2}{2} + \frac{G^2}{2r^2} - \widetilde{U}(r, \alpha - t; \mu)$$

Notice that $\widetilde{U}(r, \alpha - t; 0) = \frac{1}{r}$, the two body problem potential.

Hamiltonian equations in polar coordinates

The equations of motion, which are 2π-periodic in t, are:

$$\begin{aligned}
\dot{r} &= y \\
\dot{y} &= \frac{G^2}{r^3} + \partial_r \widetilde{U}(r, \alpha - t; \mu) \\
\dot{\alpha} &= \frac{G}{r^2} \\
\dot{G} &= \partial_\alpha \widetilde{U}(r, \alpha - t; \mu)
\end{aligned}$$

As $\widetilde{U}(\phi, r; \mu) = O(\frac{1}{r})$, for any value of G_0, the "infinity": $(\infty, 0, \alpha_0, G_0)$ is a periodic solution.
We will study the invariant manifolds of this "infinity".
In fact, "infinity" is foliated by periodic orbits which can be parameterized by the angular momentum G_0

The Jacobi constant

The RCTBP has a first integral: the so called Jabobi constant.
In our case, in polar coordinates, it is given by

$$\mathcal{J}(r, \alpha - t, y, G; \mu) = H(r, \alpha - t, y, G; \mu) - G$$

therefore orbits of the RCTBP stay in the hypersurfaces

$$\mathcal{J} = \mathcal{J}_0$$

The periodic orbit of infinity $(\infty, 0, \alpha_0, G_0)$ belongs to the surface

$$\mathcal{J} = -G_0$$

The oscillatory orbits will be generated by the intersection of the invariant manifolds of this periodic orbit, which lie in the same surface.
We will see that they exist for large values of G_0.

The autonomous model: rotating coordinates

With the change $\phi = \alpha - t$ the periodic Hamiltonian system becomes a two degrees of freedom autonomous Hamiltonian, of Hamiltonian

$$\mathcal{J}(r, \phi, y, G, \mu) = H(r, \phi, y, G; \mu) - G$$

Then the Hamiltonian $\mathcal{J}(r, \phi, y, G; \mu)$ is a first integral.
The equations of motion are:

$$\begin{aligned} \dot{r} &= y \\ \dot{y} &= \frac{G^2}{r^3} + \partial_r \widetilde{U}(r, \phi; \mu) \\ \dot{\phi} &= \frac{G}{r^2} - 1 \\ \dot{G} &= \partial_\phi \widetilde{U}(r, \phi; \mu) \end{aligned}$$

These equations are the polar version of the equations in synodic coordinates.

The autonomous model: rotating coordinates

Facts:

- The periodic orbits at infinity contained in $\mathcal{J} = -G_0$ are:

$$\Lambda_{\infty,G_0} = \{(r,\phi,y,G) : r = \infty, y = 0, \phi \in \mathbb{T}, G = G_0\}.$$

  Therefore "infinity" is a two dimensional invariant manifold foliated by periodic orbits.

$$\Lambda_\infty = \cup \Lambda_{\infty,G_0}$$

- The stable and unstable manifolds of each Λ_{∞,G_0}: $\mathcal{W}^{u,s}_{\infty,G_0}$ are two dimensional and are contained in $\mathcal{J} = -G_0$.

The two body problem as the classical limit for $\mu \to 0$

The classical way to see the RCTBP as a perturbation of the TBP is to consider μ small, because $\widetilde{U}(r, \phi; 0) = \frac{1}{r}$.
When $\mu = 0$, the only primary is fixed at the origin: $q_1(t) = 0$. The primary and the massless particle q form the two-body problem

$$H(r, \alpha - t, y, G; 0) = H_0(r, y, G) = \frac{y^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r},$$

$h = H_0$ is the energy.
$\mathcal{J}_0(r, y, G) = H_0(r, y, G) - G$ and H_0 are the first integrals. Therefore G is also preserved.
If $h < 0$, motions are elliptic of eccentricity $e = \sqrt{1 + 2hG^2}$.
If $h = 0$ (which corresponds to $e = 1$) the motion is parabolic.

The two body problem: $\mu = 0$

Equations in rotating coordinates:

$$\begin{aligned} \dot r &= y \\ \dot\phi &= \frac{G}{r^2} - 1 \\ \dot y &= \frac{G^2}{r^3} - \frac{1}{r^2} \\ \dot G &= 0 \end{aligned}$$

G is a first integral. For every fixed value of G, the variables (r, y) form a Hamiltonian system with one degree of freedom

The "infinity" $(r = \infty, y = 0)$ is a critical point! $H_0(r, y, G) = 0$ is a homoclinic manifold to infinity.

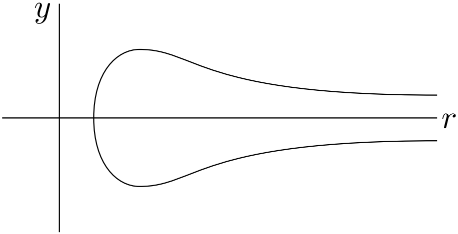
The autonomous model: rotating coordinates

Facts:

- One can consider in the level of energy $\mathcal{J}(r, \phi, y, G) = -G_0$ a Poincaré map

$$\begin{array}{rccc} \mathcal{P}_{\phi_0} & : & \{\phi = \phi_0\} \longrightarrow \{\phi = \phi_0\} \\ & & (r, y) \quad \mapsto \quad \mathcal{P}_{\phi_0}(r, y) \end{array}$$

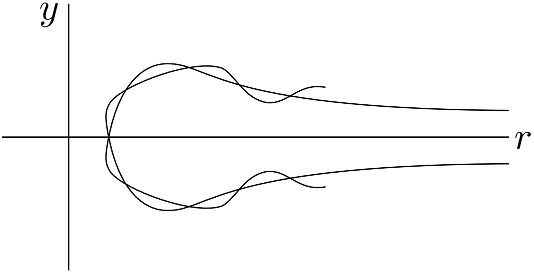
  This Poincaré map is area preserving (it preserves the symplectic form $\Omega = dr \wedge dy$). The point $(\infty, 0)$ is fixed and their invariant manifolds are curves. A natural way to quantify the splitting of separatrices is to measure the angle at their intersection or the area of the lobes that are formed between them.



Theorem

Fix $\mu \in (0, 1/2]$. Then, there exists $G^* > 0$ such that for any $G_0 > G^*$, the invariant manifolds of infinity $\mathcal{W}^s_{\infty,G_0}$ and $\mathcal{W}^u_{\infty,G_0}$ intersect transversally in $\mathcal{J}(r, \phi, y, G) = -G_0$. Moreover, the area of the lobes between the corresponding invariant curves $\gamma^{u,s}$ of the Poincaré map $\mathcal{P}_{\phi_0}$ is given by

$$A = \mu(1-\mu)\sqrt{\pi}\left[\frac{1-2\mu}{\sqrt{2}} G_0^{-3/2} e^{-\frac{G_0^3}{3}} + 8 G_0^{1/2} e^{-\frac{2G_0^3}{3}}\right]\left(1 + O\left(G_0^{-1/2}\right)\right).$$



Theorem

There exist G_0^* and a curve η in the parameter region

$$(\mu, G_0) \in \left(0, \frac{1}{2}\right] \times (G_0^*, +\infty),$$

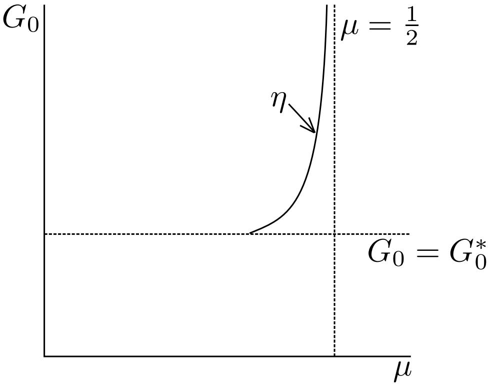
of the form

$$\mu = \mu^*(G_0) = \frac{1}{2} - 16\sqrt{2} G_0^2 e^{-\frac{G_0^3}{3}} \left(1 + O\left(G_0^{-1/2}\right)\right),$$

such that, for $(\mu, G_0) \in \eta$,

- *the invariant curves $\gamma^{u,s}$ of the Poincaré map $\mathcal{P}_{G_0,\phi_0}$ have a cubic homoclinic tangency and a transversal homoclinic point and*
- *the area of the lobes between the invariant curves $\gamma^{u,s}$ between the homoclinic tangency and a consecutive transversal homoclinic point is given by*

$$A = 10\sqrt{\pi} G_0^{1/2} e^{-2\frac{G_0^3}{3}} \left(1 + O\left(G_0^{-1/2}\right)\right).$$



Bifurcation curve η in the parameter space where the homoclinic tangency is undergone.

The two body problem as the singular limit for $G_0 \to \infty$

As we want to study the manifolds of the periodic orbits Λ_{∞, G_0}, we should restrict ourselves to $\mathcal{J} = -G_0$.

The existence of exponentially small phenomena usually arise when

- The system possesses two different time scales.
- The system has combined fast elliptic behavior and hyperbolic (or parabolic) behavior.

In $\mathcal{J} = -G_0$ we perform the following changes of variables

$$r = G_0^2 \widetilde{r}, \quad y = G_0^{-1} \widetilde{y}, \quad \alpha = \widetilde{\alpha} \quad \text{and} \quad G = G_0 \widetilde{G}$$

and we rescale time as

$$t = G_0^3 s.$$

The rescaled system is Hamiltonian with respect

$$\widetilde{H}(\widetilde{r}, \alpha - G_0^3 s, \widetilde{y}, \widetilde{G}; \mu, G_0) = \frac{\widetilde{y}^2}{2} + \frac{\widetilde{G}^2}{2\widetilde{r}^2} - \widetilde{V}(\widetilde{r}, \alpha - G_0^3 s; \mu, G_0),$$

The equations in scaled variables for big G_0

where

$$\widetilde{V}(\widetilde{r},\phi;\mu,G_0)=\frac{1-\mu}{\left(\widetilde{r}^2-2(\frac{\mu}{G_0^2})\widetilde{r}\cos\phi+(\frac{\mu}{G_0^2})^2\right)^{1/2}}+\frac{\mu}{\left(\widetilde{r}^2+2(\frac{1-\mu}{G_0^2})\widetilde{r}\cos\phi+(\frac{1-\mu}{G_0^2})^2\right)^{1/2}}.$$

The equations of motion are

$$\begin{aligned}
\frac{d}{ds}\widetilde{r} &= \widetilde{y}\\
\frac{d}{ds}\widetilde{y} &= \frac{\widetilde{G}^2}{\widetilde{r}^3}+\partial_{\widetilde{r}}\widetilde{V}(\widetilde{r},\alpha-G_0^3 s;\mu,G_0)\\
\frac{d}{ds}\alpha &= \frac{\widetilde{G}}{\widetilde{r}^2}\\
\frac{d}{ds}\widetilde{G} &= \partial_\alpha\widetilde{V}(\widetilde{r},\alpha-G_0^3 s;\mu,G_0).
\end{aligned}$$

Now the two time scales become clear. Indeed, now we have that $\dot{\widetilde{y}}\sim\dot{\widetilde{r}}\sim 1$, which are the variables that will define the separatrix, whereas the perturbation depends fast in time.

Note that, for any μ, $\widetilde{V} = \frac{1}{\widetilde{r}} + O(G_0^{-2})$ and its dependence on time is through $\phi = \alpha - G_0^3 s$, thus, for $G_0 \gg 1$, we are dealing with a *fast oscillating small* perturbation of the two body problem.

The Jacobi constant is now $\mathcal{J} = G_0^{-2}\tilde{H} - G_0\widetilde{G}$ and the periodic orbit at infinity is given by $(\widetilde{r}, \alpha, \widetilde{y}, \widetilde{G}) = (\infty, \alpha, 0, 1)$, which belongs to the surface of Jacobi constant $\widetilde{J} = -G_0$.

The two body problem is at the same time the regular (when $\mu \to 0$) and singular (when $G_0 \to \infty$) limit of our system.

Proof, sort of

In rotating coordinates, with $\phi = \alpha - G_0^3 s$, the Hamiltonian becomes

$$\mathcal{H}(\widetilde{r}, \phi, \widetilde{y}, \widetilde{G}; \mu, G_0) = \frac{\widetilde{y}^2}{2} - G_0^3 \widetilde{G} + \frac{\widetilde{G}^2}{2\widetilde{r}^2} - \widetilde{V}(\widetilde{r}, \phi; \mu, G_0).$$

We study the invariant manifolds of infinity without using McGehee Coordinates.
To study the invariant manifolds of infinity, we will follow the approach of Lochak-Marco-Sauzin:
The invariant manifolds are Lagrangian and therefore they can be locally parameterized as graphs of a generating function $S(\widetilde{r}, \phi)$:

$$(\widetilde{y}, \widetilde{G}) = (\partial_{\widetilde{r}} S, \partial_\phi S)$$

And S is a solution of the Hamilton-Jacobi equation:

$$\mathcal{H}(\widetilde{r}, \phi, \partial_{\widetilde{r}} S, \partial_\phi S; \mu; G_0) = -G_0^3.$$

When $G_0 = \infty$ ($\mu = 0$)

- The solution of H-J equation, S_0, is explicit.
- The homoclinic of the periodic orbits at infinity is known:

$$(\widetilde{r}, \widetilde{\alpha}, \widetilde{y}, \widetilde{G}) = (\widetilde{r}_h(s), \alpha_0 + \widetilde{\alpha}_h(s), \widetilde{y}_h(s), 1).$$

Singularities at $\pm i/3$.

Then, for $G_0 >> 1$,

- $S = S_0 + S_1$.
- Change of variables: $T_1(v, \xi) = S_1(\widetilde{r}_h(v), \xi + \widetilde{\alpha}_h(v))$. We look for $T^{u,s}$ such that satisfies H-J and
    - $T^{u,s}(v, \xi)$ are 2π-periodic respect to ξ
    - $T^{s,u}(v, \xi) \to 0$ as $v \to \pm\infty$
- A fixed point scheme and

$$T^u - T^s \sim \int_{-\infty}^{+\infty} V(\widetilde{r}_{\mathrm{h}}(v+s), \xi - G_0^3 s + \widetilde{\alpha}_{\mathrm{h}}(v+s); \mu, G_0) ds$$

where $V(r, \phi; \mu, G_0) = \widetilde{V}(r, \phi; \mu, G_0) - 1/r$.

Computation of the modified Melnikov potential

Recipe to compute

$$\begin{aligned} L(v,\xi;\mu,G_0) &= \int_{-\infty}^{+\infty} V(\widetilde{r}_{\mathrm{h}}(v+s),\xi - G_0^3 s + \widetilde{\alpha}_{\mathrm{h}}(v+s);\mu,G_0)ds \\ &= \int_{-\infty}^{+\infty} V(\widetilde{r}_{\mathrm{h}}(t),\xi - G_0^3 t + G_0^3 v + \widetilde{\alpha}_{\mathrm{h}}(t);\mu,G_0)dt, \end{aligned}$$

- Write $V(\widetilde{r},\phi;\mu,G_0) = \sum_{\ell\in\mathbb{Z}} V_\ell(\widetilde{r};\mu,G_0)e^{i\ell\phi}$
- $L(v,\xi;\mu,G_0) = \sum_{\ell\in\mathbb{Z}} L_\ell(v;\mu,G_0)e^{i\ell\xi}$
- $L_\ell(v;\mu,G_0) = \bar{L}_\ell(\mu;G_0)e^{i\ell G_0^3 v}$
- $\bar{L}_\ell(\mu,G_0) = \int_{-\infty}^{+\infty} V_\ell(\widetilde{r}_{\mathrm{h}}(t);\mu,G_0)e^{i\ell\widetilde{\alpha}_{\mathrm{h}}(t)}e^{-i\ell G_0^3 t}\ dt.$