Connectivity of Julia sets, Baker domains and weakly repelling fixed points

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Motivation: Newton's method in the complex plane

Given $P(z)$ a complex polynomial, its **Newton's method** is defined as

$$N_P(z) = z - \frac{P(z)}{P'(z)}.$$

N_P is a rational map which acts on the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

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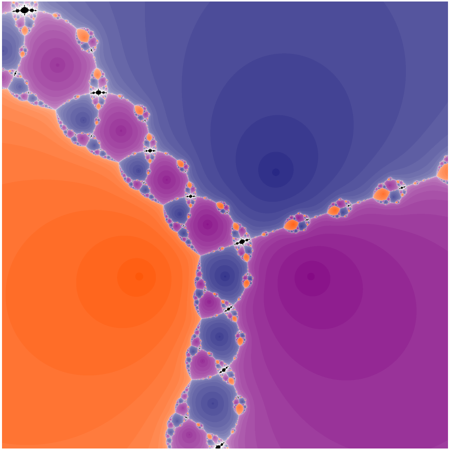
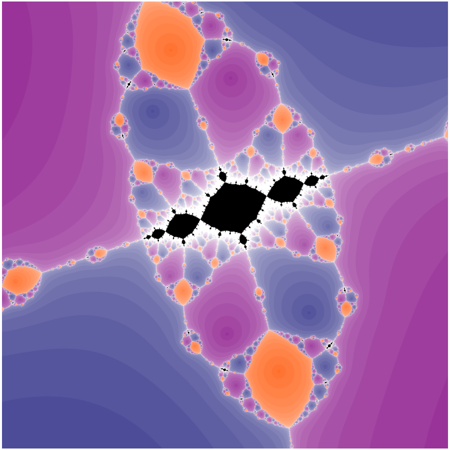
As all complex dynamical systems, its phase space decomposes into two totally invariant sets:

- **The Fatou set (or stable set)**: basins of attraction of attracting or parabolic cycles, Siegel discs (irrational rotation domains) or Herman rings (irrational rotation annuli).
- **The Julia set (or chaotic set)**: the closure of the set of repelling periodic points (boundary between the different stable regions).

Newton's method for $P(z) = z(z-1)(z-a)$.





The study of the distribution and topology of these invariant sets has recently produced efficient algorithms to locate all roots of P. [Hubbard, Schleicher and Sutherland '04 '11].

- An important topological property is the following.

Theorem (Shishikura '90)

For any polynomial P, all Fatou components of N_P are simply connected. (Equivalently, $\mathcal{J}(N_P)$ is connected.)

- In particular, there are no Herman rings: only basins and Siegel disks.

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- Shishikura's theorem had a long history with partial results from Przytycki '86, Meier '89, Tan Lei ..... He actually proved a more general theorem for rational maps, obtaining this as a corollary.

Newton's method of entire transcendental maps

Given $g(z)$ an **entire transcendental map** , i.e., with an essential singularity at infinity, its Newton's method

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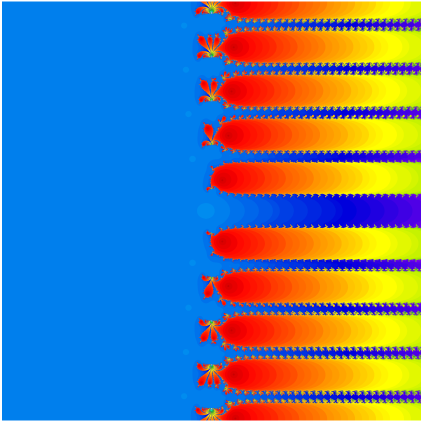
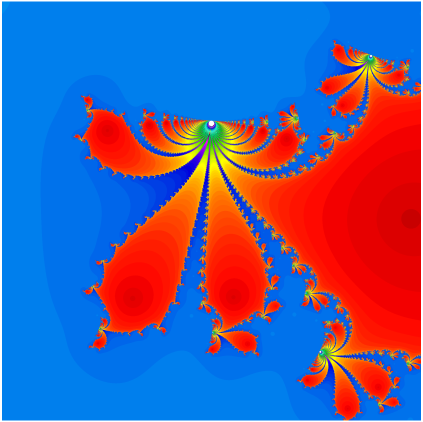
• The Fatou set allows for two extra types of components:

- **Wandering domains** $f^m(U) \cap f^n(U) = \emptyset$ for all n, m.
- **Baker domains** Sets for which all iterates converge uniformly to ∞.

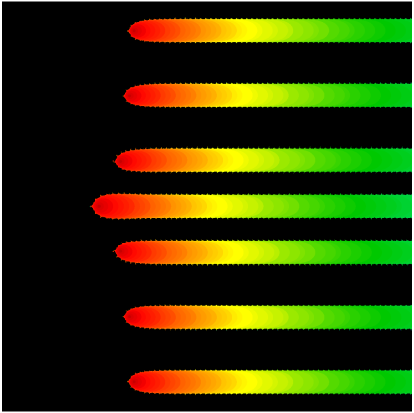
Newton's method for $g(z) = z + e^z$.

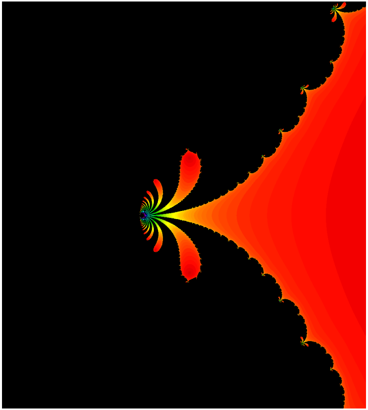




Question: Are all Fatou components simply connected?? Or equivalently, is the Julia set always connected (in $\mathbb{C} \cup \{\infty\}$)?

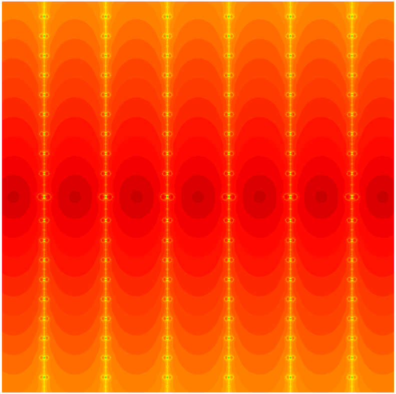
Newton's method for $g(z) = e^z(z + e^z)$.

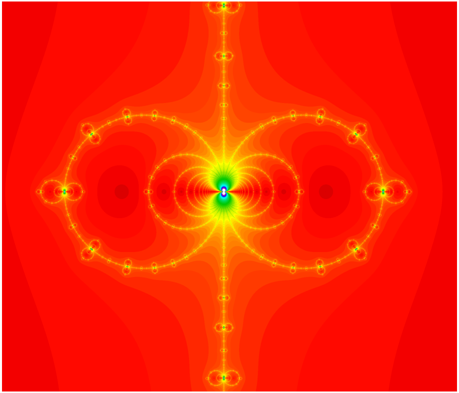




In black, a Baker domain where iterates tend to $-\infty$.

Newton's method for $g(z) = \sin(z)$.





Goal / Main result

Our goal is to complete the proof of the following **more general** theorem. (*Mer* denotes the class of meromorphic transcendental maps).

Main Theorem

$f \in Mer$ $\mathcal{J}(f)$ *is disconnected*	$\Longrightarrow$	*f has at least one weakly repelling fixed point.*

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This theorem can be proved separately for each type of Fatou component.

Previous results

- g entire transcendental; N_g Newton's method ($\in$ Mer).
  - Mayer + Schleicher '06: Basins of attraction and "virtual immediate basins" are simply connected.
- $f \in Mer$.
  - Bergweiler + Terglane '96: case where U is a wandering domain.
  - F + Jarque + Taixés '08: case where U is an attracting basins or a preperiodic comp.
  - F + Jarque + Taixés '11: case where U is a parabolic basin.

Cases left: general Baker domains and Herman rings (for both, Newton's method and for general meromorphic maps).

Main results

Theorem A

Suppose $f \in Mer$. If f has a multiply connected cycle of Baker domains, then f has at least one weakly repelling fixed point.

Theorem B

Suppose $f \in Mer$. If f has a cycle of Herman rings, then f has at least one weakly repelling fixed point.

Absorbing regions

Most of the existing results were based on the existence of **simply connected absorbing regions** for f.

Definition (Absorbing region)

$f \in Mer$, U invariant Fatou component.
$W \subset U$ is **absorbing in U for f** if $f(W) \subset W$ and for every compact set $K \subset U$ there exists $n = n(K) > 0$, such that $f^n(K) \subset W$.

Examples:

1. Linearization domain (appropriately chosen) around an attracting fixed point.
2. Attracting petals attached to parabolic fixed points

Strategy: Pull back a s.c. absorbing region until it becomes m.c. Then apply surgery.

Absorbing regions in Baker domains

Baker domains, in general, do NOT have **simply connected** absorbing regions (König '99). Existence of any type of absorbing regions was an open problem.
We prove:

Theorem C (Baker domains have absorbing regions)

Let $f \in Mer$ and U be an invariant Baker domain. Then there exists an absorbing region $W \subset U$, which satisfies:

(a) $\overline{W} \subset U$,

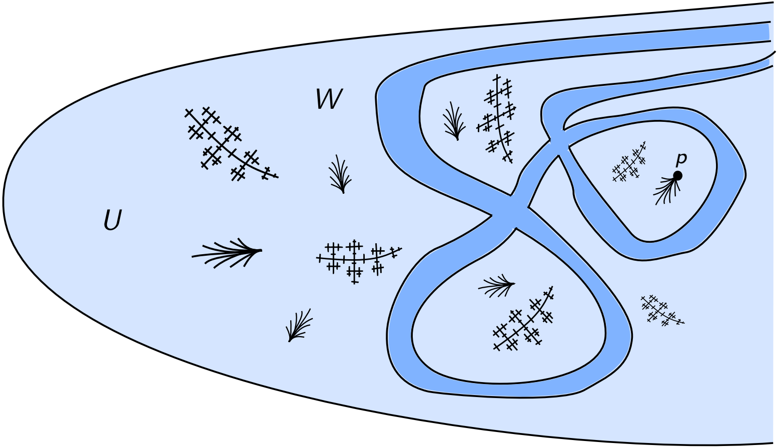
(b) $f^n(\overline{W}) = \overline{f^n(W)} \subset W$ *for every* $n \geq 1$,

(c) $\bigcap_{n=1}^{\infty} f^n(\overline{W}) = \emptyset$,

Moreover, for every point $z \in U$ and every sequence of positive numbers r_n, $n \geq 0$ with $\lim_{n\to\infty} r_n = \infty$, the domain W can be chosen such that

$$W \subset \bigcup_{n=0}^{\infty} \mathcal{D}_U(F^n(z), r_n).$$

Observe that W may be or not be simply connected.



• Theorem C holds for any $p-$**cycle of Baker domains**, just taking f^p.

• In fact we prove Theorem C in much **greater generality**, for f defined only on a hyperbolic domain (in the sense of having $\mathbb{D}$ as the universal covering space).

• We moreover show that on the absorbing region W, the map f is semiconjugate to one of the **Möbius transformations** $\omega + 1$, $a\omega$ with $a > 1$ or $w \pm i$.

Plan

- Main Tools.
- Finding a wrfp in a multiply connected Baker domain: sketch of proof of Theorem A (fixed case).
- (If there is time) Construction of the absorbing region (sketch of proof of Theorem C)

Proof of Theorem A (fixed case). Main tools.

We suppose f has an invariant Baker domain U.
To show that f has a weakly repelling fixed point we use the following ingredients, (which concentrate a good part of the work).

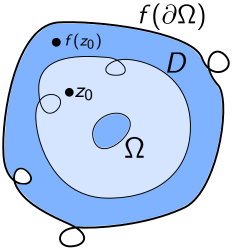
1. **Theorem C** (existence of an absorbing region W, not necessarily s.c.)
2. Results on existence of wrfp:
   - **Lemma 1** (Open set maps over itself)
   - **Corollary 2** (Continum surrounds a pole and maps out)
   - **Proposition 3** (Boundary maps in)
3. **Proposition 4** (Poles in holes)

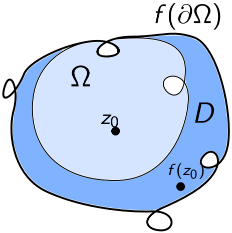
Standing assumption: f holomorphic on a neighborhood of the domain being considered.

Lemma 1 (Open set maps over itself)

- Ω bounded domain with finite Euler characteristic,
- D component of $\widehat{\mathbb{C}} \setminus f(\partial\Omega)$,
- $\exists z_0 \in \Omega$ such that $f(z_0) \in D$,
- (left) $\overline{\Omega} \subset D$
- (right) $\Omega \subset D$, Ω s.c., $\partial\Omega$ locally connected and f has no fixed points in $\partial\Omega$.

Then, f has a wrfp in Ω.





Corollary 2: Continuum surounds a pole and maps out

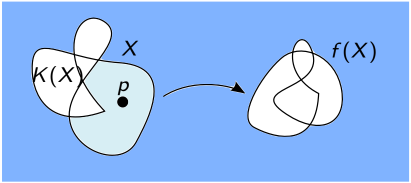
$X \subset \mathbb{C}$ compact. Set:

$\mathrm{ext}(X)$: unbounded component of $\widehat{\mathbb{C}} \setminus X$.

$K(X) = \widehat{\mathbb{C}} \setminus \mathrm{ext}(x)$

- $X \subset \mathbb{C}$ continuum
- f has no poles in X,
- $K(X)$ contains a pole of f,
- $K(X) \subset \mathrm{ext}(f(X))$.

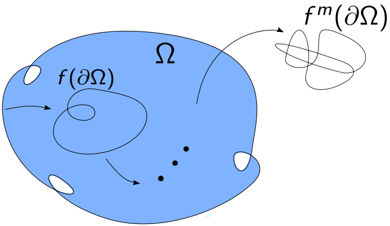
Then, f has a weakly repelling fixed point in the interior of $K(X)$.



Lemma 3: Boundary maps in

- $\Omega \subset \mathbb{C}$ bounded, simply connected domain;
- there exists $m \geq 2$, such that f^m is defined on $\partial\Omega$,
- $f^j(\partial\Omega) \subset \overline{\Omega}$ for $j = 1, \ldots, m-1$,
- $f^m(\partial\Omega) \cap \overline{\Omega} = \emptyset$.

Then, f has a weakly repelling fixed point in Ω.



(Note that $f^m(\partial\Omega)$ could also surround Ω)

Proposition 4: Poles in holes

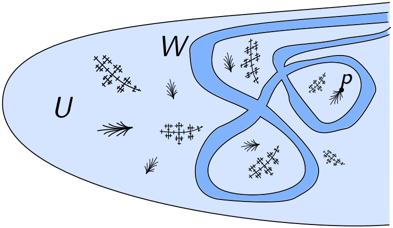
- $V \subset \mathcal{F}(f)$ open and invariant by $f \in Mer$.
- $\gamma \subset V$ closed curve such that $K(\gamma) \cap J(f) \neq \emptyset$.

Then, $\exists\ n \geq 0 \mid K(f^n(\gamma))$ contains a pole of f.

Corollary

- *$V \subset \mathcal{F}(f)$ open and invariant by $f \in Mer$.*
- *V is multiply connected*

Then, there exists at least one bounded connected component of $\widehat{\mathbb{C}} \setminus V$ containing a pole.



Proof of Theorem A (invariant case)

* U invariant Baker domain, multiply connected:

  $f^n \to \infty$ unif. on cpct. subsets of U.

* By Theorem C, $\exists\ \overline{W} \subset U$ absorbing region:

  For all compact $K \subset U$, $\exists\ n_0 \mid f^{n_0}(K) \subset W$.

We split into two cases:

1. W is multiply connected;
2. W is simply connected.

Case 1: W is multiply connected.

By Proposition 4 (Poles in holes):

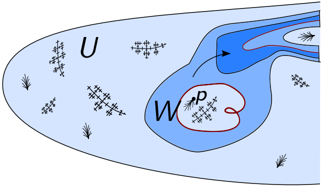
* $\exists$ a component of $\mathbb{C} \setminus \overline{W}$ which contains a pole P.

Consider $W, f(W), f^2(W), \cdots$ and remember $f^n(\overline{W}) \subset f^{n-1}(W)$.

(a) There exists a maximal $k > 0$ such that P is contained in a bdd component Ω of $\mathbb{C} \setminus f^k(W)$.

Then, $X = \partial\Omega$, and $f(X) \subset \mathrm{ext}(X)$ and $f(X)$ does not surround P.

Corollary 2 (Continum surounds a pole and maps out) $\Rightarrow$ $\exists$ wrfp.



Case 1: W is multiply connected.

(b) P is contained in a a bdd component Ω_k of $\mathbb{C} \setminus f^k(W)$ for all $k > 0$.

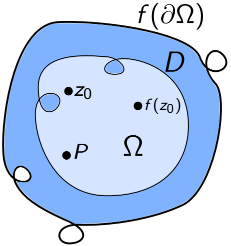
* Choose z_0 and let k be large enough so that

$$\{P, z_0, f(z_0)\} \in \Omega_k$$

Again $f(\partial\Omega_k) \subset \mathbb{C} \setminus \overline{\Omega_k}$ and surounds Ω_k.

* There exists D component of $\mathbb{C} \setminus f(\partial\Omega_k)$ which contains Ω_k. Moreover, $z_0 \in \Omega_k$ and $f(z_0) \in D$.

Lemma 1 (Open set maps over itself)
$\Rightarrow \quad \exists$ wrfp.



Case 2: W is simply connected

By Proposition 4 (Poles in holes):

$*$ $\exists\ \gamma \subset U$ a closed curve which surounds a pole P.

Observe $\gamma \not\subset W$!!!

$*$ Consider

$$\Gamma = \bigcup_{n \geq 0} f^n(\gamma) \text{ closed subset of } \mathbb{C}$$

- $f(\Gamma) \subset \Gamma$
- $\Gamma_j := \bigcup_{n \geq 0} f^{n+j}(\gamma) = f^j(\Gamma) \subset U$
- $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$
- For j large enough, $\Gamma_j \subset W$ (i.e. does not suround P).

Case 2: W is simply connected

* Let $k \geq 0$ be maximal such that

$$\Gamma_k \text{ surounds } P \text{ but } \Gamma_{k+1} \text{ does not.}$$

* Let E be the bdd component of $\mathbb{C} \setminus \Gamma_k$ containing P. Let

$$\Omega = \text{filled}(E) \ \text{ simply connected}$$

* $\partial\Omega$ locally connected, with no fixed points.

* $f(\partial\Omega)$ does NOT suround P. Hence,

$$\begin{cases} \Omega \subset \text{ext}(f(\partial\Omega)) & \textit{or} \\ f(\partial\Omega) \subset \Omega \end{cases}$$

Case 2: W is simply connected

(a) $\Omega \subset \mathrm{ext}(f(\partial\Omega))$

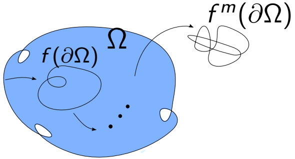
Corollary 2 (Cont. surounds a pole and maps out) $\Rightarrow \exists$ wrfp

(b) $f(\partial\Omega) \subset \Omega$

Since $f^n(\partial\Omega) \to \infty$, there exists $m > 1$ minimal such that

$$f(\partial\Omega), f^2(\partial\Omega), \cdots f^{m-1}(\partial\Omega) \text{ and } f^m(\partial\Omega) \cap \overline{\Omega} = \emptyset.$$

Lemma 3 (Boundary maps in)
$\Rightarrow \quad \exists$ wrfp.



q.e.d.

Observations

- This method is entirely different from the former proofs: there are **NO PULLBACKS**. This means asymptotic values present no danger!

- Work in progress: We believe this proof with very few modifications could offer a **UNIFIED** proof of **ALL** cases at once (rational or transcendental, attracting or parabolic or Baker). Right now, the proof of the main result splits into 4 papers, each with a different method!

Thank you for your attention!!
and

gràcies i feliç aniversari Jaume!!

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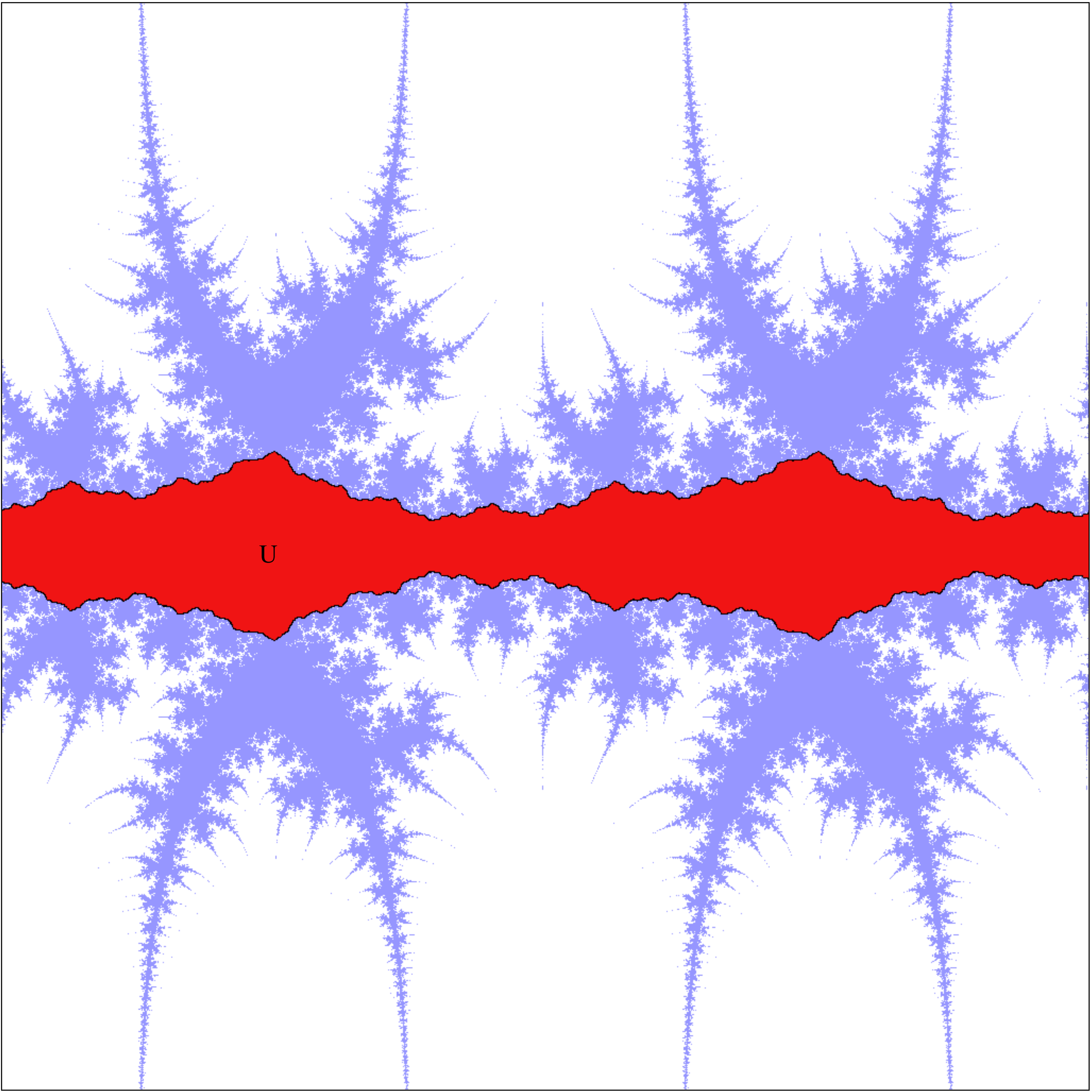
PERIODIC POINTS OF HOLOMORPHIC MAPS VIA LEFSCHETZ NUMBERS

NÚRIA FAGELLA AND JAUME LLIBRE

Abstract. In this paper we study the set of periods of holomorphic maps on compact manifolds, using the periodic Lefschetz numbers introduced by Dold and Llibre, which can be computed from the homology class of the map. We show that these numbers contain information about the existence of periodic points of a given period; and, if we assume the map to be transversal, then they give us the exact number of such periodic orbits. We apply this result to the complex projective space of dimension n and to some special type of Hopf surfaces, partially characterizing their set of periods. In the first case we also show that any holomorphic map of $\mathbb{C}P(n)$ of degree greater than one has infinitely many distinct periodic orbits, hence generalizing a theorem of Fornaess and Sibony. We then characterize the set of periods of a holomorphic map on the Riemann sphere, hence giving an alternative proof of Baker's theorem.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