Global instability in the elliptic restricted three body problem using two scattering maps

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New Trends in Dynamical Systems
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The (planar) elliptic restricted three body problem (ER3BP).

We consider the motion of a particle q with zero mass under the attraction of two particles q_{S} and q_{J}, called *primaries*, which move in elliptic orbits with *eccentricity* e_0 around their center of mass.

Typical models:

- Sun–Jupiter–asteroid or comet: $e_0 = 0.048$
- Sun–Earth–Moon systems: $e_0 = 0.016$

We consider the motion of the particle q (comet) when it moves outside of the orbit of the primaries along nearly parabolic orbits.

The equations

The motion of the particle q (comet) is described by

$$\frac{d^2q}{dt^2} = -(1-\mu)\frac{q - q_{\mathrm{S}}(t,e_0)}{|q - q_{\mathrm{S}}(t,e_0)|^3} - \mu\frac{q - q_{\mathrm{J}}(t,e_0)}{|q - q_{\mathrm{J}}(t,e_0)|^3}.$$

This is a time-periodic Hamiltonian system (2 and 1/2 degrees of freedom) with Hamiltonian

$$H(q,p,t;e_0,\mu) = \frac{p^2}{2} - \frac{(1-\mu)}{|q - q_{\mathrm{S}}(t,e_0)|} - \frac{\mu}{|q - q_{\mathrm{J}}(t,e_0)|}.$$

Parameters: $0 < \mu, e_0 < 1$ small.

The two body problem: Sun-comet

When $\mu = 0$, there is no Jupiter in the equation of motion and the Sun is fixed at the origin: $q_{\mathrm{S}}(t, e_0) = 0$

The Sun q_{S} and the comet q form the two-body problem with the Hamiltonian $H(q, p, t; e_0, 0) = H_0(q, p) = \dfrac{p^2}{2} - \dfrac{1}{|q|}$.

The two–body problem is integrable.

The ER3BP as a perturbation of the 2BP

We shall study the case of $e_0 > 0$ and small $\mu > 0$.

Hamiltonian $H_\mu(q, p, t, e_0)$ is a *small time-periodic perturbation* of the integrable two body problem (Sun-comet).

The perturbation term is

$$\begin{aligned} \Delta H_\mu(q, p, t; e_0) &= H(q, p, t; e_0, \mu) - H_0(q, p) \\ &= (1-\mu)\left(\frac{1}{|q - q_{\mathrm{S}}(t, e_0)|} - \frac{1}{|q|}\right) \\ &+ \mu\left(\frac{1}{|q - q_{\mathrm{J}}(t, e_0)|} - \frac{1}{|q|}\right). \end{aligned}$$

Since *Jupiter* $q_{\mathrm{J}}(t, e_0)$ moves along an ellipse with semi-major axis $1-\mu$, in the case q being uniformly away from the unit ball both terms are of order of μ and tend to zero as $q \to \infty$.

Expression of the primaries in polar coordinates (r, α)

Sun

$$q_{\rm S} = q_{\rm S}(t, e_0) = \mu(r_0 \cos f, r_0 \sin f)$$

Jupiter:

$$q_{\rm J} = q_{\rm J}(t, e_0) = -(1-\mu)(r_0 \cos f, r_0 \sin f)$$

with

$$r_0 = r_0(t; e_0) = \frac{1-e_0^2}{1+e_0 \cos f}, \qquad \frac{df}{dt} = \frac{(1+e_0 \cos f)^2}{(1-e_0^2)^{3/2}},$$

where $f = f(t; e_0)$ is the true anomaly. Also

$$r_0 = r_0(t; e_0) = 1 - e_0 \cos E, \qquad t = E - e_0 \sin E,$$

where E is the eccentric anomaly.

Hamiltonian equations in polar coordinates

Polar coordinates $q = (x, y) = (r\cos\alpha, r\sin\alpha),\ \alpha \in \mathbb{T},\ r \geq 0.$

Hamiltonian

$$H(r, \alpha, P_r, P_\alpha, t; e_0, \mu) = \frac{P_r^2}{2} + \frac{P_\alpha^2}{2r^2} - U(r, \alpha, t; e_0, \mu)$$

where (r, P_r) and (α, P_α) are pairs of conjugate variables,

$$U(r, \alpha, t; e_0, \mu) = \frac{1-\mu}{|q - q_{\mathrm{S}}|} + \frac{\mu}{|q - q_{\mathrm{J}}|},$$

$$\begin{aligned} |q - q_{\mathrm{J}}|^2 &= r^2 - 2(1-\mu) r\, r_0 \cos(\alpha - f) + (1-\mu)^2 r_0^2, \\ |q - q_{\mathrm{S}}|^2 &= r^2 + 2\mu\, r\, r_0 \cos(\alpha - f) + \mu^2 r_0^2. \end{aligned}$$

$$r_0 = r_0(t; e_0) = \frac{1 - e_0^2}{1 + e_0 \cos f}, \qquad \frac{df}{dt} = \frac{(1 + e_0 \cos f)^2}{(1 - e_0^2)^{3/2}}.$$

Hamiltonian equations in polar coordinates

$P_\alpha := G$ is the *angular momentum*.

$$H(r, \alpha, P_r, G, t; e_0, \mu) = \frac{P_r^2}{2} + \frac{G^2}{2r^2} - U(r, \alpha, t; e_0, \mu)$$

The two body problem in polar coordinates

In the polar coordinates: $q = (x, y) = (r\cos\alpha, r\sin\alpha),\ \alpha \in \mathbb{T},\ r \geq 0,$
The Hamiltonian of the two body problem becomes

$$H_0(r, P_r, \alpha, G) = \frac{P_r^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r},$$

$h = H_0$ is the energy.

G and H_0 are both first integrals of motion.

If $h < 0$, motions are elliptic:

Semi-major axis: $a = 1/(-2h)$, eccentricity $e = \sqrt{1 + 2hG^2}$.

If $h = 0$ (which corresponds to $e = 1$) the motion is parabolic.

Diffusion of the angular momentum

In general, Diffusion $\equiv$ Gaining lots of energy by applying small forces.

In the elliptic restricted three body (ERTBP) problem we want to see that the angular momentum of the comet $G(t)$ can have *large changes* when the eccentricity $e_0 > 0$ and $\mu > 0$ are small enough:

Given any $G_1, G_2 \gg 1$, there exist trajectories of the ERTBP whose angular momentum satisfies, for some $T > 0$:

$$G(0) < G_1 \qquad G(T) > G_2$$

Proven for $0 < \mu \ll e_0 \ll 1$ and any $1 \ll G_1, G_2 \leq 1/e_0$.

Likely (need still some work) for any $0 < e_0 < 1$ and $0 < \mu \ll 1$.

Previous results

For oscillatory motions or diffusion close to parabolic orbits:

Llibre-Simó 1980 (oscillatory motions in the CRTBP for $0 < \mu \ll 1$)

Guàrdia-Martín-Seara 2012 (idem for $0 < \mu < 1$)

Xia 1993 (local diffusion in the ERTBP)

Martínez-Pinyol 1994 (Massive computations in the ERTBP)

Other types of oscillatory motions or diffusion:

Llibre-Martínez-Simó 1985 (oscillatory motions close to L_2 in the CRTBP)

Bolotin 2006 (close to collision in the ERTBP)

Capiñski-Zgliczyñski 2011 (close to L_2 in the ERTBP)

Féjoz-Guàrdia-Kaloshin-Roldán 2012 (close to resonances in the ERTBP)

One limit: the two body problem: $\mu = 0$

A priori unstable structure

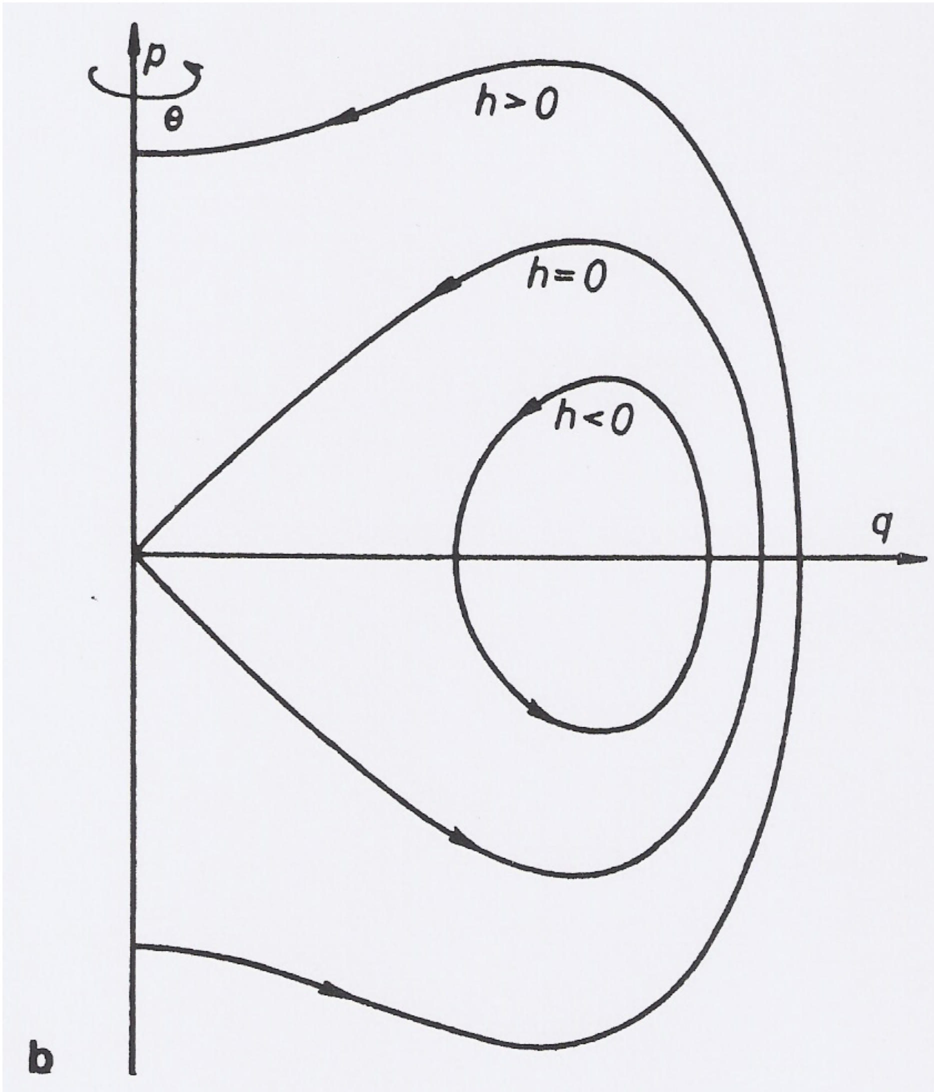
Introducing $x^2 := 1/r$, $y := P_r$, we get new Hamiltonian equations:

$$\begin{aligned} \dot{x} &= -\frac{x^3}{2}\frac{\partial \mathcal{H}_0}{\partial y} & \dot{\alpha} &= \frac{\partial \mathcal{H}_0}{\partial G} & & \\ \dot{y} &= \frac{x^3}{2}\frac{\partial \mathcal{H}_0}{\partial x} & \dot{G} &= -\frac{\partial \mathcal{H}_0}{\partial \alpha} = 0 & \dot{s} &= 1 \end{aligned}$$

with Hamiltonian $\mathcal{H}_0(x, y, G) = \dfrac{y^2}{2} + \dfrac{G^2 x^4}{8} - \dfrac{x^2}{2}$, and Poisson bracket

$$\{f, g\} = -\frac{x^3}{2}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right) + \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial G} - \frac{\partial g}{\partial \alpha}\frac{\partial f}{\partial G}$$

which has the separatrix loop $\gamma_G = \{\mathcal{H}_0(x, y, G) = 0\}$ to the origin.


p
θ
h>0
h=0
h<0
q
b

One limit: the two body problem: $\mu = 0$

A priori unstable structure: An invariant “normally parabolic” cylinder.

Main features we will use:

- The 3 dimensional manifold:

$$\tilde{\Lambda}_\infty = \{x = y = 0,\ (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\}$$

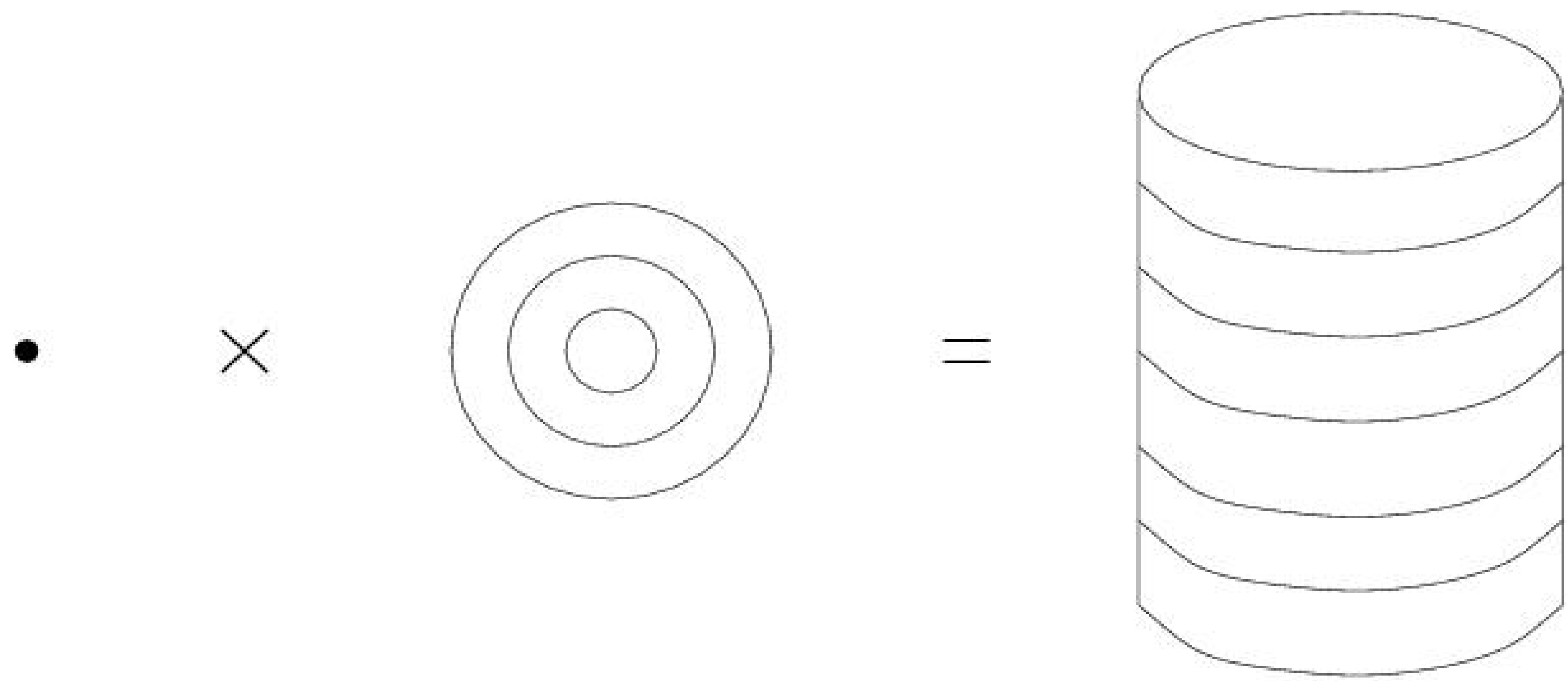
  is invariant.

- $\tilde{\Lambda}_\infty = \bigcup_{\alpha,G} \tilde{\Lambda}_{\alpha,G}$

- The inner dynamics on $\tilde{\Lambda}_\infty$ is trivial:

$$(\alpha, G, s) \to (\alpha, G, s + t)$$

- $\tilde{\Lambda}_\infty$ has stable and unstable manifolds.



Λ

One limit: the two body problem: $\mu = 0$

A priori unstable structure: An invariant homoclinic manifold to $\tilde{\Lambda}_\infty$.

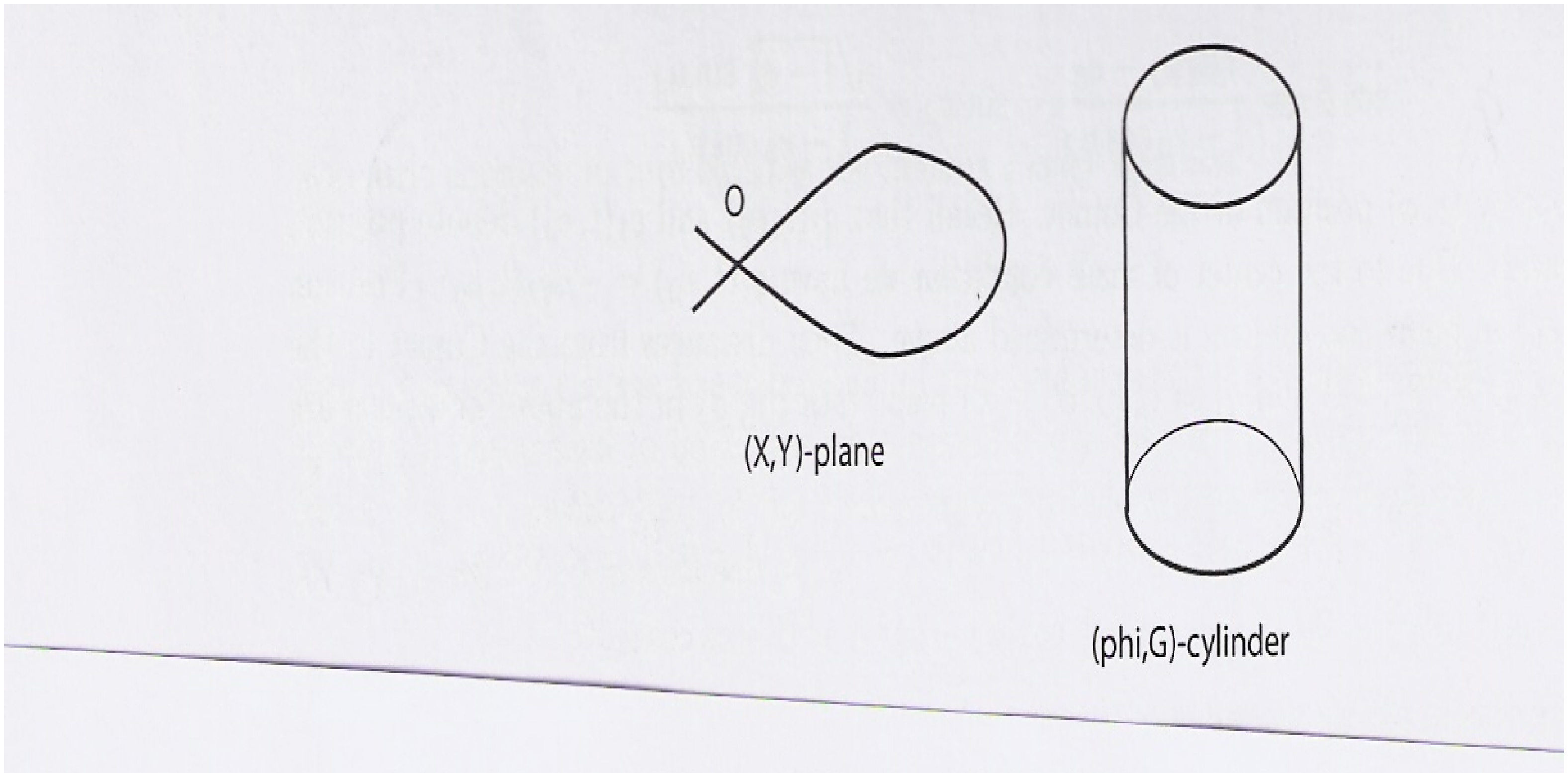
$$\begin{aligned} \tilde{\gamma} &= W_0^s(\tilde{\Lambda}_\infty) = W_0^u(\tilde{\Lambda}_\infty) \\ &= \{\mathcal{H}_0(x, y, G) = 0,\ (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\} \end{aligned}$$

that can be seen as a union of homoclinic orbits to $\tilde{\Lambda}_\infty$ (homoclinic manifold).

$$\tilde{\gamma} = \bigcup_{(\alpha,G)} \tilde{\gamma}_{\alpha,G}$$

We can parameterize the 4-dimensional homoclinic manifold as:

$$\tilde{\gamma} = \{\tilde{z}_0 := (x_G(\tau), y_G(\tau), \alpha_G(\tau) + \alpha, G, s), \tau \in \mathbb{R}, G \in \mathbb{R}_+, (\alpha, s) \in \mathbb{T}^2\}$$


O
(X,Y)-plane
(phi,G)-cylinder

One limit: the two body problem: $\mu = 0$

Outer dynamics: the scattering map (D-Llave-Seara 2000) in $\tilde{\Lambda}_\infty$. We can define a map in $\tilde{\Lambda}_\infty$ associated to the homoclinic manifold $\tilde{\gamma}$

$$S_0 : \tilde{\Lambda}_\infty \to \tilde{\Lambda}_\infty$$

by $\tilde{z}_+ = S_0(\tilde{z}_-)$ iff $\exists \tilde{z} \in \tilde{\gamma}$ such that

$$\mathrm{d}(\varphi(t;\tilde{z}), \varphi(t;\tilde{z}_\pm)) \to 0 \text{ as } t \to \pm\infty.$$

The orbit through $\tilde{z}$ is a heteroclinic connection between the orbits through $\tilde{z}_\pm$.

Using the point of $\tilde{z} = \tilde{z}_0 = (x_G(\tau), y_G(\tau), \alpha_G(\tau) + \alpha, G, s)$, one can compute S_0 in coordinates:

$$S_0(\alpha, G, s) = (\alpha, G, s)$$

One limit: the two body problem: $\mu = 0$

Outer dynamics: the *scattering map* in $\tilde{\Lambda}_\infty$.

As $S_0 = Id$,

The unperturbed periodic orbits $\tilde{\Lambda}_{\alpha,G}$ only have homoclinic connections.

Main goal:

For $\mu > 0$ we want to see that we can define a scattering map such that the image of one periodic orbit intersects other periodic orbits with larger angular momentum G. Then we will have heteroclinic orbits between periodic orbits

Arnold diffusion: $e_0 > 0$, $\mu > 0$

In variables (x, y), the Hamiltonian is:

$$H(x, y, \alpha, G, s; e_0, \mu) = \frac{y^2}{2} + \frac{G^2 x^4}{2} - U(x, \alpha, G, s; e_0, \mu)$$

with $U(x, \alpha, G, s; e_0) = x^2 \tilde{U}(x, \alpha, G, s; e_0, \mu)$

Implications:

- $\tilde{\Lambda}_\infty = \{x = y = 0,\ (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\}$ is still invariant.
- The periodic orbits $\tilde{\Lambda}_{\alpha,G}$ persist.
- The inner dynamics on $\tilde{\Lambda}_\infty$ is trivial:

$$(\alpha, G, s) \to (\alpha, G, s + t)$$

Arnold diffusion: $e_0 > 0$, $\mu > 0$

For $\mu > 0$, $e_0 > 0$, the manifolds $W^s_\mu(\tilde{\Lambda}_\infty)$ and $W^u_\mu(\tilde{\Lambda}_\infty)$ intersect transversally along TWO homoclinic manifolds.

This result is based on a Melnikov type computation.

Melnikov potential:

$$\mathcal{L}(\alpha, G, s; e_0) = \int_{\mathbb{R}} \overline{\Delta U}(x_G(t), \alpha_G(t) + \alpha, s + t; e_0)\, dt.$$

where $U(x, \alpha, s; e_0, \mu) = x^2 + \mu\overline{\Delta U}(x, \alpha, s; e_0) + O(\mu^2)$

Intersection property: If the function

$$\tau \mapsto \mathcal{L}(\alpha, G, s - \tau; e_0)$$

has a *non-degenerate critical point* $\tau^*(\alpha, G, s; e_0)$, then there is a transversal intersection between $W^u(\tilde{\Lambda}_\infty)$ and $W^s(\tilde{\Lambda}_\infty)$ close to $\tilde{z}_0 = (x_G(\tau), y_G(\tau), \alpha_G(\tau) + \alpha, G, s)$.

Arnold diffusion: $e_0 > 0$, $\mu > 0$

For any fixed (α, G, e_0), we just need to find a critical point $s^*(\alpha, G; e_0)$ of $s \mapsto \mathcal{L}(\alpha, G, s; e_0)$, that is, a solution $s^*(\alpha, G; e_0)$ of the equation

$$\frac{\partial \mathcal{L}}{\partial s}(\alpha, G, s; e_0) = 0$$

and we recover $\tau^*(\alpha, G, s; e_0) = s - s^*(\alpha, G; e_0)$

Once we have $\tau^*(\alpha, G, s; e_0)$ we can consider the *Poincaré reduced function*

$$\mathcal{L}^*(\alpha, G; e_0) = \mathcal{L}(\alpha, G, -\tau^*(\alpha, G, 0; e_0); e_0) = \mathcal{L}(\alpha, G, s^*(\alpha, G; e_0); e_0)$$

Arnold diffusion: $e_0 > 0$, $\mu > 0$

The scattering map S given by the homoclinic intersection associated to the critical point s^* is given as:

$$(\alpha, G, s) \mapsto (\alpha - \mu \frac{\partial \mathcal{L}^*}{\partial G} + O(\mu^2), G + \mu \frac{\partial \mathcal{L}^*}{\partial \alpha} + O(\mu^2), s)$$

S is given, up to first order in μ, as the time $-\mu$ Hamiltonian flow of the autonomous Hamiltonian $\mathcal{L}^*(\alpha, G)$!

Then, looking at the level curves of $\mathcal{L}^*(\alpha, G)$ we get the images under the scattering map.

Arnold diffusion: $e_0 > 0$, $\mu > 0$

The inner dynamics in $\tilde{\Lambda}_\infty$ is trivial:

$$(\alpha, G, s) \mapsto (\alpha, G, s + t)$$

The classical geometric mechanism to obtain diffusion does not work: there is no possibility of combining the inner and the outer dynamics to obtain large changes of G.

The Poincaré map $P(\alpha, G, s) = (\alpha, G, s)$, therefore $S \circ P = S$

Only with one scattering map we cannot get large changes in G.

Arnold diffusion: $e_0 > 0$, $\mu > 0$

The function $\mathcal{L}(\alpha, G, s; e_0)$ has two non-degenerate critical points s_+^*, s_-^* which give rise to two different perturbed scattering maps S_+, S_-.

The foliations of their level curves are transversal.

We can construct heteroclinic chains of periodic orbits with increasing angular momentum choosing the right scattering map any time

Computation of the Melnikov potential $\mathcal{L}$ for $e_0 G \ll 1$

Fourier expanding in the angle s (and α), we get

$$\begin{aligned} \mathcal{L}(\alpha, G, s; e_0) &= \qquad \mathcal{L}_0(\alpha, G; e_0) + \mathcal{L}_1(\alpha, G, s; e_0) \\ &\quad + F(\alpha, G; e_0) + E(\alpha, G, s; e_0) \end{aligned}$$

$$\begin{aligned} \mathcal{L}_0(\alpha, G; e_0) &= -\frac{\pi}{G^3} - \frac{15\pi e_0}{8G^5} \cos\alpha, \\ \mathcal{L}_1(\alpha, G, s; e_0) &= \sqrt{\frac{\pi}{8}} \frac{e^{-G^3/3}}{G^{1/2}} \left(\cos(s-\alpha) + p\cos(s-2\alpha)\right), \end{aligned}$$

where $p = 10ee_0G^2$, F is small: $F = O\left(e_0^2 G^{-7}\right)$, and E is exponentially small: $E = e^{-G^3/3} O(G^{-3/2}, e_0 G^{1/2}, e_0^2 G^{5/2})$.

- $\mathcal{L}_0$ contains no harmonics in s and one first order harmonic in α.
- $\mathcal{L}_0$ contains two first order harmonics in s.
- $e_0 G \leq 1$ needed for the convergence of the expansions.

Computation of the term $\mathcal{L}_1$ for $e_0 G \ll 1$

$s \mapsto \mathcal{L}_1(\alpha, G, s; e_0)$ is indeed a cosine function:

$$\mathcal{L}_1(\alpha, G, s; e_0) = \sqrt{\frac{\pi}{8}} \frac{e^{-G^3/3}}{G^{1/2}} \sqrt{1 + 2p\cos\alpha + p^2}\cos(s - \alpha - \alpha*),$$

where $\alpha* = \alpha*(p, \alpha) = 2\arctan\frac{p\sin\alpha}{1+p\cos\alpha}$ ($p = 10ee_0G^2$), with a unique non-degenerate maximum (minimum) for $s = \alpha + \alpha^*$ ($s = \alpha + \alpha^* + \pi$), where $\mathcal{L}_1$ takes the values

$$\pm\mathcal{L}_1^*(\alpha, G; e_0) = \pm\sqrt{\frac{\pi}{8}} \frac{e^{-G^3/3}}{G^{1/2}} \sqrt{1 + 2p\cos\alpha + p^2}.$$

Note that for $e_0 = 0$, $\mathcal{L}_1^*(\alpha, G; 0) = \pm\sqrt{\frac{\pi}{8}}\frac{e^{-G^3/3}}{G^{1/2}}$ does not depend on α.

Computation of the reduced Poincaré functions $\mathcal{L}_1^*$

Since $\left|\dfrac{\partial E}{\partial s}\right| \ll \left|\dfrac{\partial \mathcal{L}_1}{\partial s}\right|$, the function $s \mapsto \mathcal{L}(\alpha, G, s; e_0)$ is a "cosine-like" function, with unique non-degenerate maximum and minimum at $s_\pm^*$. We can define the *Poincaré reduced functions*

$$\mathcal{L}_\pm^*(\alpha, G; e_0) = \mathcal{L}(\alpha, G, s_\pm^*; e_0) = \mathcal{L}_0 \pm \mathcal{L}_1^* + F + E_\pm^*$$

so that the associated *scattering maps* $S_\pm$ are given by

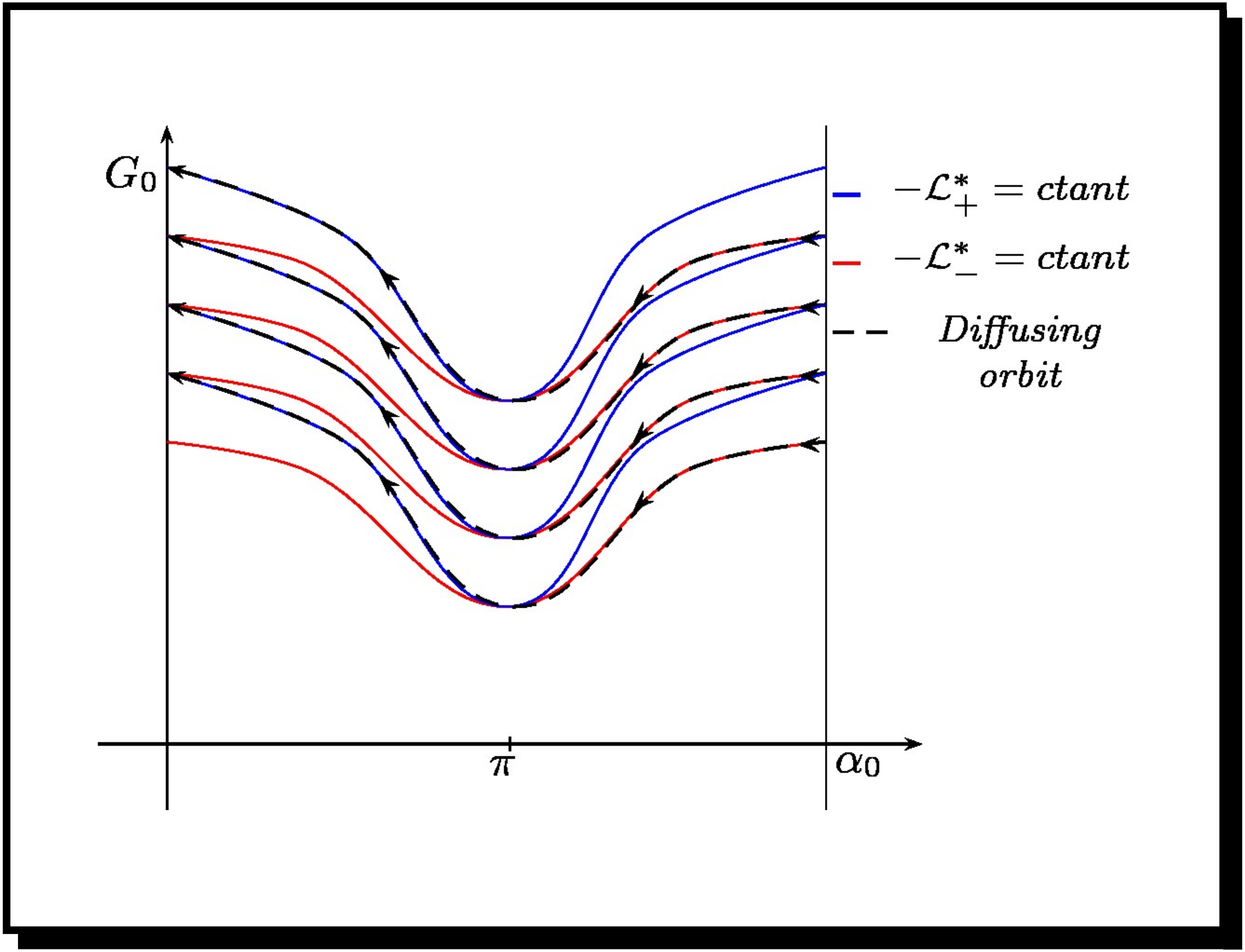
$$(\alpha, G) \mapsto \left(\alpha - \mu \frac{\partial \mathcal{L}_\pm^*}{\partial G} + O(\mu^2), G + \mu \frac{\partial \mathcal{L}_\pm^*}{\partial \alpha} + O(\mu^2)\right).$$

Functionally independent Scattering maps $S_\pm$

The *scattering maps* $S_\pm$ are given by

$$(\alpha, G) \mapsto \left(\alpha - \mu \frac{\partial \mathcal{L}^*_\pm}{\partial G} + O(\mu^2), G + \mu \frac{\partial \mathcal{L}^*_\pm}{\partial \alpha} + O(\mu^2) \right).$$

- $S_\pm$ are given, except for $O(\mu^2)$, as the time μ Hamiltonian flow of the autonomous Hamiltonians $-\mathcal{L}^*_\pm(\alpha, G)$.
- The iterates under $S_\pm$ follow the level curves of $\mathcal{L}^*_\pm$.
- Since $\{\mathcal{L}^*_+, \mathcal{L}^*_-\} = -2\{\mathcal{L}_0, \mathcal{L}^*_1\} + \cdots$ only vanishes on $\alpha = 0, \pi$, we can choose alternatively $S_\pm$ to get diffusing pseudo-orbits and get diffusion along $1 \ll G \leq 1/e_0$.



G_0
π
α_0
$-\mathcal{L}^*_+ = ctant$
$-\mathcal{L}^*_- = ctant$
Diffusing orbit

Computation of the term $\mathcal{L}_1$ for general e_0G

$$\mathcal{L}_1(\alpha, G, s; e_0) = \sqrt{\frac{\pi}{8}}\frac{e^{-G^3/3}}{G^{1/2}} \times$$
$$\times \left(\cos(s-\alpha) + 8eG\sum_{m=1}^{\infty} a_m R^m \cos\left(s-(m+1)\alpha\right)\right)$$

where $a_m = \dfrac{16}{m!}\dfrac{(2m+3)!!}{(2m+6)!!}$, $R = 2e_0G$, which can be also written as

$$\mathcal{L}_1(\alpha, G, s; e_0) = \sqrt{\frac{\pi}{8}}\frac{e^{-G^3/3}}{G^{1/2}}\mathrm{Re}\left\{e^{i(s-\alpha)}\left(1+8eGM\left(Re^{-i\alpha}\right)\right)\right\}$$

where $M(z) := M(5/2, 4, z) = \sum_{m=1}^{\infty} a_m z^m = 1 + 5z/8 + O(z^2)$ is the *confluent hypergeometric Kummer function*, solution of the Kummer equation $zM'' + (4-z)M' - 5M/2 = 0$.

Computation of the Melnikov potential $\mathcal{L}$ for general $e_0 G$

$$\begin{aligned}\mathcal{L}(\alpha, G, s; e_0) &= \mathcal{L}_0(\alpha, G; e_0) + \mathcal{L}_1(\alpha, G, s; e_0) \\ &\quad + F(\alpha, G; e_0) + E(\alpha, G, s; e_0)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_0(\alpha, G; e_0) &= -\frac{\pi}{G^3} - \frac{15\pi e_0}{8G^5}\cos\alpha, \\ \mathcal{L}_1(\alpha, G, s; e_0) &= \sqrt{\frac{\pi}{8}}\frac{e^{-G^3/3}}{G^{1/2}}\mathrm{Re}\left\{e^{i(s-\alpha)}\left(1 + 8eGM\left(Re^{-i\alpha}\right)\right)\right\},\end{aligned}$$

where F is small and E is exponentially small, which gives rise to two different scattering maps:

- For $e_0 G \leq 1$ coincides with the previous computations.
- For $e_0 G \gg 1$ can be computed as in Martínez-Pinyol 1994.
- For $e_0 G \not\leq 1$ and $e_0 G \not\gg 1$ requires a (numerical and validated) computation.

Arnold diffusion: $e_0 > 0$, $\mu > 0$

- For $e_0 G \leq 1$ analytic proof.
- It remains to check the case $e_0 G > 1$ via analytical, numerical o computer assisted methods.
- All the previous results need μ to be exponentially small with respect to $G \gg 1$.
- A priori stable: Using the same techniques as in Guàrdia, Martín and Seara 2012, prove diffusion for μ small, independent of G and arbitrary $0 < e_0 < 1$.

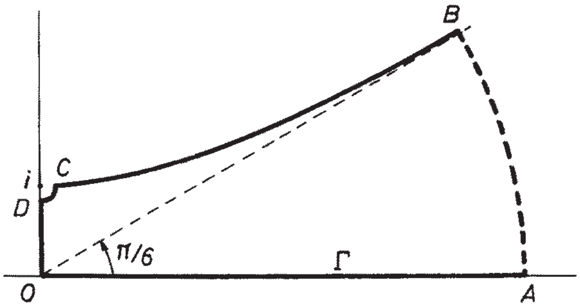


Fig. 7

if we compute the integral in (5.10) with $\bar{\theta}=\pi$. Computing the derivative $\partial c_1/\partial\bar{\theta}$ we state the result so far obtained.

Proposition 5.1. *In order to verify the non-tangentiality condition one has the following relation*

$$I=\frac{\partial}{\partial\bar{\theta}}q_1(\bar{\theta},m)\Big|_{\substack{m=0\\ \bar{\theta}=\pi}}=-(2-64/C^3)$$

$$\cdot\int_0^{\pi/2}\left\{\frac{16\sin z\cos^3 z}{2C^3-64\cos^4 z}\sin\theta[4+2\Delta^{-3/2}-3\Delta^{-5/2}(2+16\cos^2 z\cos\theta/C^2)]\right.$$

$$\left.+\frac{4096\sin z\cos^7 z}{(2C^3-64\cos^4 z)^2}\sin\theta[1-\Delta^{-3/2}]\right\}dz\,, \tag{5.12}$$

where $\theta=-C^3(\mathrm{tg}z+\mathrm{tg}^3 z/3)/16+\pi+2z$ *and* $\Delta=1+16\cos^2 z\cos\theta/C^2+64\cos^4 z/C^4$.

To end the proof of Theorem 5.1 we shall prove that the dominant terms in (5.12), for C large, are not zero.

We introduce $\tau=\mathrm{tg}z$ and $\delta=C^3/16$ in (5.12). The evaluation of the integrals involved will be done through integration in the complex domain along the curve Γ of Fig. 7.

Lemma 5.1. *Let* $I_k=\int_0^\infty\cos(\delta(\tau+\tau^3/3))(1+\tau^2)^{-k}d\tau$, $k\geqq 1$. *For* $\delta>0$ *sufficiently large we have*

$$I_k=\mathrm{Re}\left\{(\pi/2)i\,\mathrm{Res}\left[\frac{\exp(i\delta(\tau+\tau^3/3))}{(1+\tau^2)^k}\,;\,\tau=i\right]\right\}(1+o(1))\,. \tag{5.13}$$

Proof. Let Γ be the curve OABCDO of Fig. 7. Points C and D belong to the circle $\tau=i+\varepsilon\exp(i\phi)$, ε small. The curve CB is defined by $\mathrm{Re}(\tau+\tau^3/3)=0$, i.e., if $\tau=\xi+i\eta$, it is a branch of a hyperbola: $3+\xi^2-3\eta^2=0$ (see [3]). We integrate $h=\exp(i\delta(\tau+\tau^3/3))/(1+\tau^2)^k$ along Γ.