Cyclicity of a simple focus via the vanishing multiplicity of inverse integrating factors

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We consider planar real analytic differential systems

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

defined in a neighborhood $U \subset \mathbb{R}^2$ of the origin. We assume that $(0,0)$ is a *simple focus*, i.e., one monodromic singularity which after a (generalized) polar blow-up $(x,y) \mapsto (\theta, r)$ is transformed into a periodic orbit. In short, system (1) can be written as

$$\frac{dr}{d\theta} = F(r,\theta) = \sum_{i \geq \ell} F_i(\theta) r^i, \tag{2}$$

where $F(r,\theta)$ is analytic on the cylinder $C = \big\{(r,\theta) \in \mathbb{R} \times S^1 : |r| \text{small}\big\}$ with $S^1 = \mathbb{R}/\mathbb{Z}T$ where $T > 0$ is the minimum constant period associated to the polar change. Here we have $F(0,\theta) = 0$ for all $\theta \in S^1$.

Consider *inverse integrating factors* $V(r,\theta)$ of (2), i.e., a function $V : C \to \mathbb{R}$ non-locally null and solution of

$$\frac{\partial V(r,\theta)}{\partial \theta} + \frac{\partial V(r,\theta)}{\partial r} F(r,\theta) = \frac{\partial F(r,\theta)}{\partial r} V(r,\theta).$$

It is well known (see [1] and [2]) that (2) has a unique (modulo multiplicative constants) inverse integrating factor $V(r,\theta)$ of class C^∞ and non-flat at $r = 0$. Therefore V admits the Taylor series $V(r,\theta) = \sum_{i \geq m} v_i(\theta) r^i$.

In [3] it is proved that $\Pi(r_0) = r_0 + c_m r_0^m + O(r_0^{m+1})$ with $c_m \neq 0$ is the expression of the Poincaré map associated to the origin of (2).

We shall prove that $m \geq \ell \geq 1$. Moreover:

(a) $m = \ell$ if and only if $v_k(\theta)$ are constant for $k = m, \ldots, 2\ell - 1$.

(b) Assume $\ell \geq 2 + k$ with $k \geq 0$ a positive integer. If $\int_0^T F_\ell(\theta)\, d\theta = \int_0^T F_{\ell+1}(\theta)\, d\theta = \cdots = \int_0^T F_{\ell+k-1}(\theta)\, d\theta = 0$, but $\int_0^T F_{\ell+k}(\theta)\, d\theta \neq 0$, then $m = \ell + k$.

This result is next applied to study in some cases the cyclicity of the focus at the origin in the normal form for nilpotent monodromic singularities $(\dot{x}, \dot{y}) = (-y, x^{2n-1} + y b(x))$, with $b(x) = \sum_{j \geq \beta} b_j x^j$.

References

[1] A. Enciso and D. Peralta-Salas, *Existence and vanishing set of inverse integrating factors for analytic vector fields*, Bull. London Math. Soc. **41** (2009), 1112–1124

[2] I.A. García, H. Giacomini and M. Grau, *The inverse integrating factor and the Poincaré map*, Trans. Amer. Math. Soc. **362** (2010), 3591–3612.

[3] I.A. García, H. Giacomini and M. Grau, *Generalized Hopf Bifurcation for Planar Vector Fields via the Inverse Integrating Factor*, J. Dyn. Differ. Equ. **23** (2011), 251–281.