Workshop "Limit cycles on Chemistry" Bellaterra, 9-12 December, 2025

Abstracts of the Talks

Limit cycles for ODE with few complex monomials with emphasis on quadratic systems

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Any planar polynomial system can be written in complex form as $z' = \sum_{j=1}^{N} A_j M_j(z,\bar{z})$ where $M_j(z,\bar{z}) = z^{k_j}\bar{z}^{m_j}$ are different monic monomials and A_j are complex numbers. It is known that when $N = 1$ these equations do not have limit cycles and when $N = 2$ they have at most one limit cycle. On the other hand, it is also known that for $N \geq 3$ there is no upper bound for their number of limit cycles, and it is natural to believe that the upper bound (if exists) should depend on the degree of the differential equation.

In this talk we denote as $H_N(n)$ the maximum number limit cycles that equations with N monomials and degree n can have. In this notation the above results read as $H_1(n) = 0$ and $H_2(n) \leq 1$. More concretely, $H_2(2n) = 0$ and $H_2(2n+1) = 1$. As a first result we will show that for each $N \geq 4$, $H_N(n) \geq n - 3$.

The rest of the talk focuses on the quadratic systems (QS). It is known that if they have limit cycles then they can be written as $z' = Az + Bz^2 + Cz\bar{z} + D\bar{z}^2$. Hence the known QS with 4 limit cycles prove that $H_4(2) \geq 4$. Firstly, we fix our attention on QS with 3 complex monomials. There are 20 such subcases. For 6 of them we prove that the QS have 0 limit cycles; for 3 of them they have at most 1 limit cycle; for other 3, they have at most 2 limit cycles; for other 7 families they have at least 1 limit cycle, while for 1 family they have at least 2 limit cycles. In short, we know that $H_3(2) \geq 2$ and we suspect that $H_3(2) = 2$.

Finally, if time permits, we will discuss about the number of limit cycles of QS with other restrictions.

References

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Introduction to the mathematical theory of (chemical) reaction systems-I&II

Gheorghe Craciun

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We give a general introduction to reaction systems, which are a class of polynomial dynamical systems on the positive quadrant of $\mathbb{R}^2$, or in general, the positive orthant of $\mathbb{R}^d$. We will explain how, essentially, all polynomial dynamical systems of degree n on the positive quadrant of $\mathbb{R}^2$ can be regarded as reaction systems of degree $n + 2$. We will describe the classical notions of detailed balanced systems, complex balanced systems, Horn–Jackson Lyapunov functions (inspired by Boltzmann entropy), and elements of deficiency theory. We will also discuss some modern developments: representation of $2D$ reaction systems via "embedded graphs" (E-graphs) in the plane, (weakly) reversible reaction systems, sufficient conditions for existence and/or uniqueness of positive equilibria, persistence and permanence of reaction systems, reversible reaction systems with an "oval" (i.e., smooth closed algebraic curve) of steady states, and a simple way to obtain a reversible reaction system with an algebraic limit cycle.

Application of polynomial and topological invariants to the study of phase portraits of quadratic systems. The birth of an Encyclopedia on quadratic phase portraits

Joan Carles Artés
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In this talk, I will introduce the most powerful tools we have for the study of phase portraits of quadratic systems. These are the polynomial invariants, which allow us to obtain in a simple way the algebraic bifurcation diagram of a given normal form with several parameters. Up to now, several normal forms with 4 parameters have already been studied using this method, another with 5 parameters, and even a special one with 9 parameters. The non-algebraic bifurcations can later be added to this bifurcation diagram with the use of continuity arguments and the help of the program P4.

Secondly, I will present the topological invariants, which allow us to identify similar or topologically equivalent phase portraits, or to distinguish them, whether within the same paper with many portraits or among different papers.

And finally, I will present, for the first time, the birth of an encyclopedia of phase portraits of quadratic systems, which aims to become a complete one within the next few years.

Mapping dynamical systems into chemical reactions

Tomislav Plesa
University of Cambridge

The dynamics of chemical reaction networks under mass-action kinetics is described with a special subset of polynomial dynamical systems (DSs) called chemical dynamical systems (CDSs). A fundamental problem of great interest is to map general polynomial DSs into dynamically similar CDSs. I will introduce the quasi-chemical map (QCM) that can systematically solve this problem. The QCM introduces suitable state-dependent perturbations into any given polynomial DS which then becomes a CDS under sufficiently large translations of variables. This map preserves robust structures, such as hyperbolic equilibria and limit cycles. Furthermore, the resulting CDSs are

at most one degree higher than the original DSs. I will discuss how the QCM can be applied to address Hilbert's 16th problem for CDSs. In this context, I will prove a certain lower bound for the "chemical" Hilbert number, discuss how the QCM can be adapted for two-dimensional quadratic DSs, and put forward some open problems about two-nest limit cycles in quadratic and cubic CDSs.

Planar and non-planar chemical systems with algebraic and non-algebraic limit cycles

Radek Erban
University of Oxford

The dynamics of chemical reaction systems involving N chemical species can be described (under mass-action kinetics) by a system of N autonomous ordinary differential equations (ODEs) with polynomial right-hand sides. These polynomials have degrees of at most n, where n represents the maximum order of the chemical reactions within the system. I will discuss chemical systems exhibiting limit cycles for both planar ($N = 2$) and non-planar ($N > 2$) cases. The question of the number of limit cycles will be examined for various classes of chemical reaction networks, including: (i) chemical systems with reactions up to the n-th order; (ii) systems with up to n-molecular chemical reactions; and (iii) weakly reversible chemical reaction networks. Additionally, lower bounds on the corresponding Hilbert numbers will be provided for both algebraic and non-algebraic limit cycles.

Lower bounds for the Hilbert numbers

Joan Torregrosa
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Centre de Recerca Matemàtica

We will see the main techniques used to obtain the best lower bounds, known up to now, for the Hilbert numbers of polynomial vector fields of degree N, $H(N)$, for small values of N. Following [1], these limit cycles appear bifurcating from symmetric Darboux reversible centers with very high simultaneous cyclicity. The considered systems have at least three centers: one on the reversibility straight line and two symmetric with respect to it. More concretely, the limit cycles appear in a three-nest configuration, and the total number of limit cycles is at least $2n+m$, for some values of n and m. The new lower bounds are obtained using simultaneous degenerate Hopf bifurcations. The first-order approach used in [1] was improved in [3], using higher-order developments as in [2], to prove that $H(4) \geq 28$ and $H(5) \geq 37$. In particular, for $N \geq 5$, new families of Darboux centers needed to be obtained.

References

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[2] L. F. Gouveia and J. Torregrosa. Lower bounds for the local cyclicity of centers using high order developments and parallelization. J. Differential Equations 271 (2021), 447–479.

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Oscillations in quadratic mass-action differential equations

Balázs Boros

University of Szeged

We give an overview of the recent developments on the smallest quadratic mass-action reaction networks that admit Andronov–Hopf, Bautin, or Bogdanov–Takens bifurcations. Then, we discuss the analogous questions in the class of bimolecular mass-action reaction networks. Finally, we briefly discuss the inheritance theory of mass-action reaction networks, which allows us to lift nondegenerate behaviors from a network to its enlargement.

Limit cycles in Lotka–Volterra systems

Josef Hofbauer
University Vienna

Lotka–Volterra systems

$$\dot{x}_i = x_i \left(r_i + \sum_j a_{ij} x_j \right), \quad i = 1, \ldots, n,$$

are a special case of quadratic (some even bimolecular) mass-action systems. I will review some results about limit cycles in such systems and mention some open problems.

How period functions behave: three criteria and examples from planar centres

David Rojas
Universitat Autònoma de Barcelona

We introduce the concept of the period function associated with planar centres and present three analytical criteria: one for isochronicity, one for monotonicity, and one for convexity. Each criterion is illustrated with a representative example that highlights its applicability.

Global stability of planar vector fields

Luis Fernando Mello
Universidade Federal de Itajubá

We will review the definition of global stability of a vector field in the plane, as well as the main results on the subject. We will analyze the global stability of vector fields associated with a chemical model known as the Lengyel–Epstein model, sometimes called the Belousov–Zhabotinsky model. Such vector fields are topologically equivalent to a two-parameter family of cubic vector fields in the plane. We will show that, for each pair of admissible parameters, the unique equilibrium point is not globally stable. On the other hand, we will find explicit conditions for this unique equilibrium point to be asymptotically stable and study its basin of attraction. We will also study generic and degenerate Hopf bifurcations and identify a subset of the set of admissible parameters for which the phase portraits have two limit cycles.

On the period function in a generalized Lotka–Volterra systems with real exponents

Jordi Villadelprat
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In this talk, we will discuss a generalization of the Lotka–Volterra model introduced by Farkas and Noszticzius in their 1985 paper "Generalized Lotka–Volterra schemes and the construction of two-dimensional explodator cores and their Liapunov functions via 'critical' Hopf bifurcations" (J. Chem. Soc. Faraday Trans. II, 81(10):1487–1505). Through a linear transformation and a constant rescaling of time, the system can be rewritten as:

$$\dot{x} = x^p(1 - y^q), \quad \dot{y} = \mu y^q(x^p - 1),$$

where p and q are natural numbers, and μ is a positive real parameter (depending on p and q).

More generally, we consider $p, q \in \mathbb{R}$ with $pq > 0$, and assume that μ is a positive independent constant. Under these assumptions, the system is analytic in the (open) first quadrant, and the singular point at $(x, y) = (1, 1)$ is a center. Our goal is to investigate the qualitative behavior of the associated period function.

It can be shown that when p and q are natural numbers (corresponding to parameter values with physical relevance) the period function is monotonically increasing. However, the general case with real exponents reveals a more intricate dynamical behavior: the system has two isochrones and parameter values with at least one or two critical periods.

Abstracts of the Posters

A result on k-hyperbolicity and period-doubling bifurcation

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The study of periodic solutions in ordinary differential equations is fundamental in dynamical systems theory. The averaging method is a central tool for detecting periodic orbits and analyzing their stability (see [1, 5]). However, in dimensions $n \geq 3$, additional difficulties arise, such as the presence of non-isolated zeros in the averaged functions.

This work presents an approach to identifying *k-hyperbolic* orbits, which emerge precisely from the structure of these non-isolated zeros. We use modified versions of the Implicit Function Theorem (see [2, 3]) to select, within these zero sets, those solutions that preserve the truncated hyperbolicity of the Poincaré map.

We also show that these *k-hyperbolic* orbits may undergo period-doubling (fold) bifurcations, a mechanism that generates more complex dynamics and possible bifurcation cascades (see [4]). The results extend the scope of classical averaging in autonomous systems of dimension $n \geq 3$ and allow the detection of periodic solutions that remain hidden from traditional methods.

This is a joint work with Murilo R. Cândido.

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The palomba economic model within the framework of piecewise-smooth Kolmogorov systems

Yagor R. Carvalho
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In this work, we study isolated crossing periodic orbits, the so-called crossing limit cycles, for a class of piecewise-smooth Kolmogorov systems defined in two zones separated by a straight line. We investigate the number of small-amplitude crossing limit cycles and show the existence of at least one such cycle in the Palomba economic model from a piecewise-smooth perspective.

Global dynamics of a cubic resource-consumer model

Teodoro Mayayo
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We analyze a cubic planar model describing resource-consumer dynamics with facilitation under progressive habitat loss. Our study characterizes the parameter space and enumerates all the phase portraits within the Poincaré disk under ecologically relevant conditions. We show that the system has a unique stable limit cycle and characterize analytically the heteroclinic bifurcation curve involving the collapse of the resource and the consumer, enabling us to determine how the parameter region sustaining coexistence oscillations narrows under habitat destruction. To further explore these dynamics, we construct a piecewise-linear (PWL) approximation that preserves the system's qualitative behavior, allowing us to obtain an explicit expression for the heteroclinic bifurcation.

Limit cycles and invariant algebraic curves

Paulo Santana
Universidade Estadual Paulista

Given a prescribed invariant algebraic curve F, consider the space of planar vector fields of degree n having F invariant. We prove that generically such vector fields have F as unique invariant algebraic curve. We also give lower bounds in terms of n for the number of limit cycles. We then apply these results to the second part of Hilbert's 16th problem, and use the Kolmogorov vector fields as example.