CENTRE DE RECERCA MATEMÀTICA

EXCELENCIA MARÍA DE MAEZTU

Jet transport and applications

Sem-GSDUAB

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Variational Equations

Automatic differentiation

Jet transport

On power expansions of Poincaré maps

The parameterization method

Computing a splitting

Section 1

Variational Equations

Variational equations

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Under general conditions:

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- If we write the Taylor Expansion of φ_t (w.r.t. x_0):

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- Then $k!\varphi_t^{(k)} = D^k_{x_0^k}\varphi_t(x_0)$ can be regarded as the solution of a differential equation: $VE_k(f)$

First order variational equations

- Given a trajectory $\varphi_t^{(0)}(x_0)$ of the original system, the first order variational equation ($VE_1(f)$) is the following linear system

$$\begin{cases} \frac{d}{dt}\varphi_t^{(1)} = Df(\varphi_t^{(0)})\varphi_t^{(1)}, \\ \varphi^{(1)}(0) = I_n. \end{cases}$$

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- Classically, $VE_1(f)$ are computed by hand and integrated numerically together with the original differential equation. The whole system is of dimension $n + n^2$.

Example: van der Pool

```
extern MY_FLOAT mu;

diff(x,t) = y;
diff(y,t) = mu * (1 - x*x) * y - x;


a11=0;
a12=1;
a21=-2*mu*x*y-1;
a22=mu*(1-x*x);

diff(v11,t)= a11 * v11 + a12 * v21;
diff(v12,t)= a11 * v12 + a12 * v22;
diff(v21,t)= a21 * v11 + a22 * v21;
diff(v22,t)= a21 * v12 + a22 * v22;
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- In general: $VE_k(f) = VE_1(VE_{k-1}(f))$.

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```
b11=0;
b12=0;
b22=0;

c11=-2*mu*y;
c12=-2*mu*x;
c22=0;

diff(u111,t)=(b11*v11*v11 + 2*b12*v11*v21 + b22*v21*v21)*.5 + 1*(a11*u111 + a12*u211);
diff(u211,t)=(c11*v11*v11 + 2*c12*v11*v21 + c22*v21*v21)*.5 + 1*(a21*u111 + a22*u211);
diff(u112,t)=(b11*v11*v12 + b12*(v21*v12 + v11*v22) + b22*v21*v22)*1 + 1*(a11*u112 + a12*u212);
diff(u212,t)=(c11*v11*v12 + c12*(v21*v12 + v11*v22) + c22*v21*v22)*1 + 1*(a21*u112 + a22*u212);
diff(u122,t)=(b11*v12*v12 + 2*b12*v22*v12 + b22*v22*v22)*.5 + 1*(a11*u122 + a12*u222);
diff(u222,t)=(c11*v12*v12 + 2*c12*v22*v12 + c22*v22*v22)*.5 + 1*(a21*u122 + a22*u222);


diff(u121,t)=(b11*v11*v12 + b12*(v21*v12 + v11*v22) + b22*v21*v22) + (a11*u121 + a12*u221);
diff(u221,t)=(c11*v11*v12 + c12*(v21*v12 + v11*v22) + c22*v21*v22) + (a21*u121 + a22*u221);
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Section 2

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- EXAMPLE: Let us consider $f(x) = x^2 + x$ and the extended arithmetic $(x + y\delta)$ where x and y are real numbers and $\delta^2 = 0$.

- Then,

$$\begin{aligned} f(1+\delta) &= (1+\delta)(1+\delta) + (1+\delta) = 1 + 2\delta + \delta^2 + 1 + \delta \\ &= 2 + 3\delta = f(1) + f'(1)\delta. \end{aligned}$$

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- Given an algorithm, we have to replace each operation by the corresponding one for formal series.
- Generally, the output of each operation can be written as a recursion. For instance,

$$A^{\alpha} = \sum_{k \geq 0} c_k \delta^k, \quad \alpha \neq 0, 1$$

with

$$c_k = \frac{1}{k a_0} \sum_{j=0}^{k-1} [\alpha k - (\alpha + 1) j] a_{k-j} c_j, \quad c_0 = a_0^{\alpha},$$

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- Then (1) encodes the jet of partial derivatives of f at 0.

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- A^α contains terms of order higher than M that we are neglecting.
- The complexity of all standard operations is similar to the cost of the product.
- The efficiency of the operations depends on the efficiency of the product of homogeneous polynomials.

Section 3

Jet transport

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- QUESTIONS:
  a) How the error behaves?
  b) Does depend on the degree of the jet?
  c) How do we choose the step-size?

Example

Consider $(\dot{x} = f(x), VE_1(f))$, for $x \in \mathbb{R}$.

$$\begin{cases} \dot{x} = f(x), & x(0) = x_0, \\ \dot{\zeta} = df(x)\zeta, & \zeta(0) = 1. \end{cases}$$

A step of the Euler method is:

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- Notice that $f(x_n + \zeta_n\delta) = f(x_n) + df(x_n)\zeta_n\delta + \mathcal{O}(\delta^2)$

Equivalency theorem

Theorem (Explicit 1-step integrators)

Let us consider the Cauchy problem

$$\begin{cases} \dot{x} = f(t,x), \\ x(t_0) = x_0, \end{cases}$$

and a stepper

$$x_{n+1} = x_n + h\phi_f(t_n, x_n; h), \tag{2}$$

such that

$$D_x\phi_f(t_n, x_n; h) = \phi_{D_x f}(t_n, x_n; h).$$

Then, applying jet transport of order m to (2) is equivalent to apply (2) to the ODE (with suitable initial conditions):

$$\left(f, VE_1(f), \overline{VE}_2(f), \ldots, \overline{VE}_m(f)\right),$$

where

$$\overline{VE}_k = \frac{1}{k!} VE_k.$$

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- If the error of the method is $\mathcal{O}(h^p)$ on the flow, then the error on all the derivatives behaves as $\mathcal{O}(h^p)$.

- The step-size has to be computed using all the coefficients of the jet as they were the coefficients a large ODE.

Section 4

On power expansions of Poincaré maps

Poincaré maps

- A standard tool: The Poincaré map. Reduce dimensionality of things.

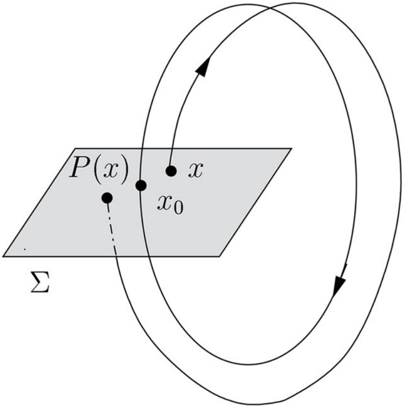

Figure: Source:

Power expansions of Poincaré maps

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  5. HINT: Do not use Euler.

Section 5

The parameterization method

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  That is, there exist an analytic map $x : I \mapsto \mathcal{U}$ defined for some interval I such that

$$F(x(s)) = x(\lambda s). \tag{3}$$

  Equation (3) is called Invariance equation.

The parameterization method

Our goal is to compute a semi-analytic approximation of this parametrization. Let us name:

$$x(s) = \sum_{j=0}^{\infty} a_j s^j.$$

We solve the invariance equation order by order.

- Order 0 is given by the coordinates of the fixed point.

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- Order 0 is given by the coordinates of the fixed point.
- Order 1 is given by the eigenvector related to λ.
- For $k > 0$, order $k+1$ is given by the solution of the linear system:

  $$(DF(0) - \lambda^{k+1})x = -b_{k+1}.$$

  Here, b_{k+1} is the $k+1$-th term of the evaluation of manifold up to degree k by the map F, that is:

  $$F^{k+1}(x^k(s)) = \sum_{j=0}^{k} b_j s^j + b_{k+1} s^{k+1}.$$

Using jet transport

Notice that, the operation

$$F^{k+1}(x^k(s)) \tag{4}$$

requires the composition of two polynomials of degrees $k+1$ and k. This is an extremely expensive operation.

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- $F(x(s)$ is obtained from the composition of the Taylor expansion of the flow with the manifold.
- The derivatives of $F(x(s))$ verify some subset of variational equations.
- We can regard operation (4) as an integration of a jet of one symbol. This can be done efficiently.

Expanding the manifold

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- Sometimes, the points in the fundamental domain are close to the fixed points, so one needs a large number of iterates to draw the manifold.

- A higher order approximation of the manifold, allows us to start the iterations far away from the fixed point.

Stopping criterion

At each step k we have to:

- Integrate a jet with 1 symbol and order k.
- Solve a linear system of dimensions $n \times n$.

Moreover:

- We can scale the parameterization to have radius of convergence 1.

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- Integrate a jet with 1 symbol and order k.
- Solve a linear system of dimensions $n \times n$.

Moreover:

- We can scale the parameterization to have radius of convergence 1.
- We stop the computation when the gain of radius of convergence is less than $1/100$.

Example: Henon-Heiles at $h = 0.1$

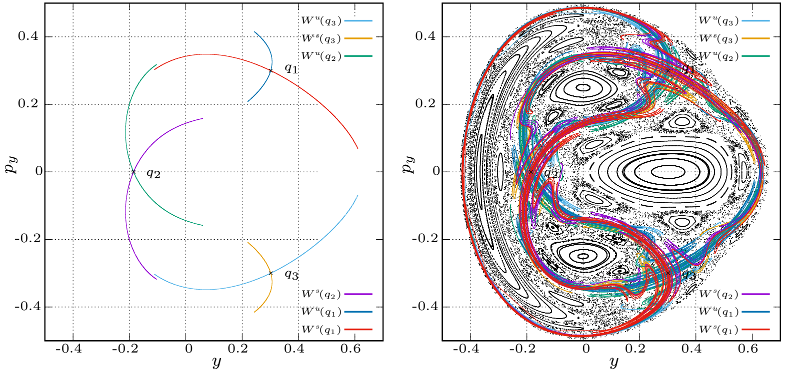

Figure: Each of the globalizations took around 40 seconds, and 7 or 8 iterations of the fundamental interval with 10^4 equispaced points in it.

Section 6

Computing a splitting

Splitting

$$\dot{x} = y,$$
$$\dot{y} = -\sin x + \mu \sin \frac{t}{\varepsilon}.$$

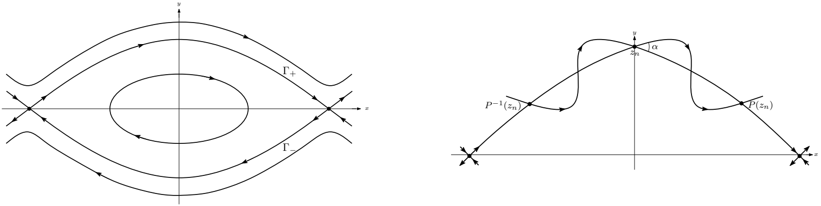

Figure: Sketch of the pendulum phase space; in the unperturbed case (left)the (un)stable manifolds coincide while in the perturbed one (right) intersect transversally.

Computing the angle

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5. Using the tangent vector, we compute α^*.
6. The splitting angle is $\alpha = 2\alpha^*$

Comparison

Order	TM (s)	It	OAF	TS(s)	Total(s)
1		384	32	29	29
8	<1	84	7	21	<22
12	<1	59	5	16	<17
16	1	45	3	12	13
20	3	37	3	10	13
32	11	24	2	7	18
50	40	16	1	4	44
78	146	11	0	3	149

Table: Metrics for the computation of the splitting using different orders and a mantissa of 65 digits.

Note: $\varepsilon = 1/32$, $\mu = 1/1024$, $\lambda \approx 6/5$, $\alpha = \mathcal{O}(10^{-23})$.

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Numerical integration of high-order variational equations of ODEs

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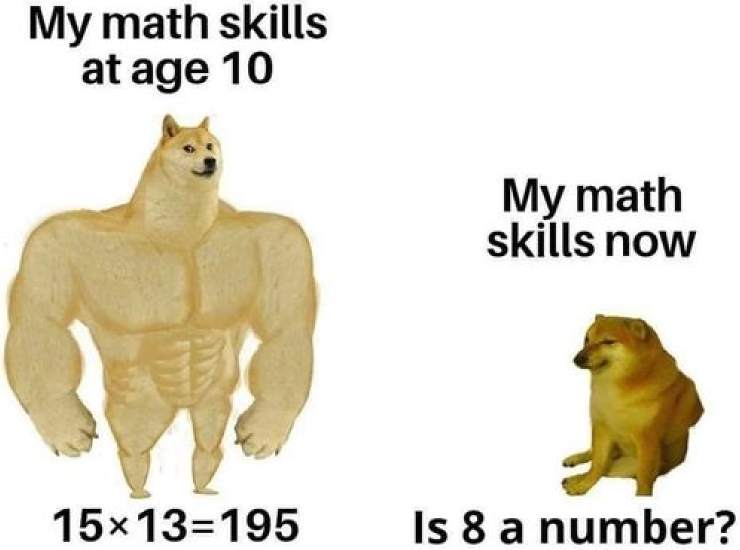
Questions?


My math skills
at age 10
My math
skills now
15×13=195
Is 8 a number?