Fractal detection of the first nonzero Lyapunov quantity

Renato Huzak (Hasselt University, Belgium) 17 October 2022 (FRABDYN HRZZ PZS 3055)

Joint work with P. De Maesschalck, A. Janssens and G. Radunović

A simple question in a planar slow-fast setting: we consider a Hopf point

$$\begin{cases} \dot{x} &= y - x^2 + x^3 h_1(x, \lambda) \\ \dot{y} &= \epsilon\big(b(\lambda) - x + x^2 h_2(x, \epsilon, \lambda) + y h_3(x, y, \epsilon, \lambda)\big), \end{cases}$$

where $b(\lambda_0) = 0$.

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Can we intrinsically define the notion of codimension of the Hopf point $(x,y) = (0,0)$? Yes

1. Traditional definition of codimension

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The Hopf point has codimension $j+1 \geq 1$ if

$$h_1(x,\lambda_0) + h_1(-x,\lambda_0) = \alpha x^{2j} + O(x^{2j+2}), \quad \alpha \neq 0.$$

ANDRONOV–HOPF (OR CODIM. 1 HOPF)

- $\begin{cases} \dot{x} = -\omega y + p(x,y,\mu) \\ \dot{y} = \omega x + q(x,y,\mu) \end{cases}$

- $\lambda_{\pm}(\mu) = \alpha(\mu) \pm i\beta(\mu)$

- $\frac{d}{d\mu}\alpha(\mu)\big|_{\mu=0} \neq 0$ (<u>transversality</u>)

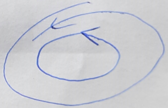
- $l_1 := \frac{1}{16}\left(p_{xxx} + p_{xyy} + q_{xxy} + q_{yyy}\right)$

$$- \frac{1}{16\omega}\left(q_{xy}(q_{xx} + q_{yy}) - p_{xy}(p_{xx} + p_{yy}) + p_{xx}q_{xx} - p_{yy}q_{yy}\right) \neq 0$$

(nondegeneracy)

the first Lyap. coeff.

$\Longrightarrow$ $\exists$ a <u>limit cycle</u>



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- DeMaesschalck, Doan, Wynen, 2021->the criticality of the Hopf bifurcation without normal forms
- Use a fractal approach instead of the differential approach to find the codimension!

2. Our goal is to define the notion of fractal codimension of a Hopf point

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1. Differential interface —> 2. Fractal interface

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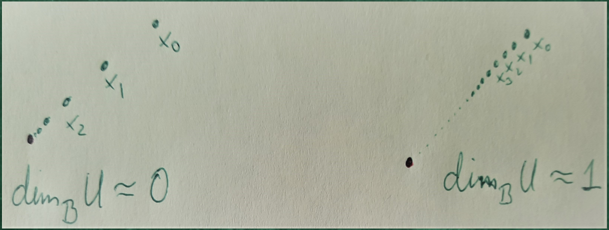

Figure: The box dimension of U.

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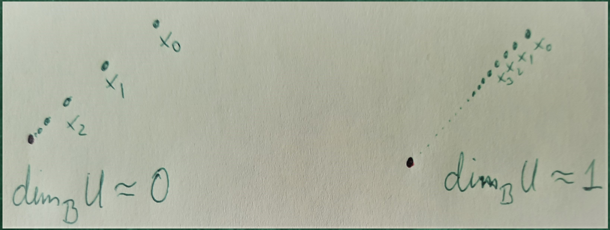
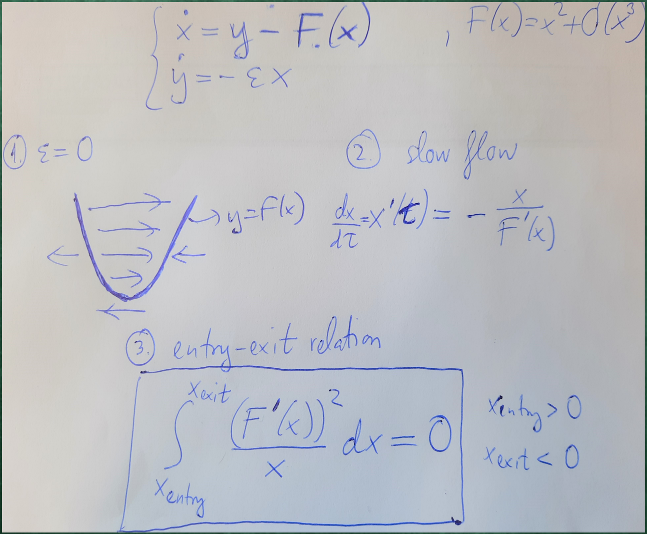
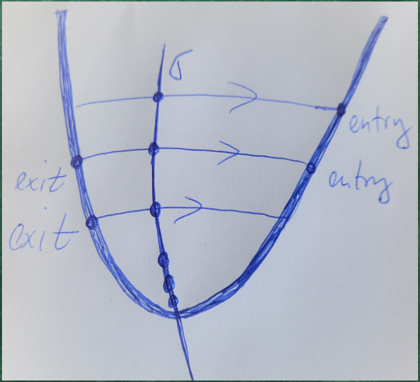


Figure: The box dimension of U.

Example


ẋ = y − F(x) , F(x) = x² + O(x³)
ẏ = − εx
1. ε = 0
2. slow flow
y = F(x)
dx/dτ = x'(τ) = − x / F'(x)
3. entry-exit relation
∫ from x_entry to x_exit of (F'(x))² / x dx = 0
x_entry > 0
x_exit < 0

Define a fractal sequence $U_0 = \{y_0, y_1, y_2, \dots\} \to 0$!



Compute the Minkowski (or box) dimension of U_0!

$$\dim_B U_0 = \lim_{k\to\infty} \frac{\ln k}{-\ln(y_k - y_{k+1})} \text{ (Cahen-type formula)}$$

$$\dim_B U_0 = \lim_{k\to\infty} \frac{1}{1 - \frac{\ln y_k}{\ln k}} \text{ (Borel rarefaction index of } U_0\text{)}$$

or

$$\dim_B U_0 = \lim_{k\to\infty} \left(1 - \frac{\ln\left(k(y_k - y_{k+1}) + y_k\right)}{\ln\left(\frac{y_k - y_{k+1}}{2}\right)}\right) \text{ (tail and nucleus)}$$

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–Zubrinic,Zupanovic, 2007,2008

Fractal codimension:
If $\dim_B U_0 < 1$, we say that the Hopf point has finite fractal codimension $j + 1 \geq 1$ where

$$j = \frac{3 \dim_B U_0 - 1}{2(1 - \dim_B U_0)} \in \mathbb{N}_0.$$

If $\dim_B U_0 = 1$, then we say that the fractal codimension is infinite.

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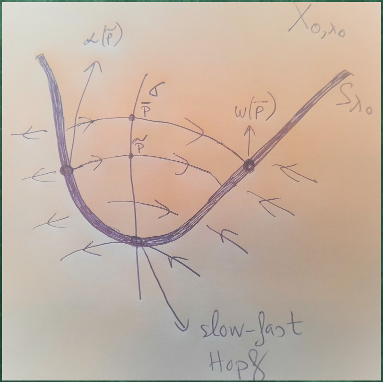

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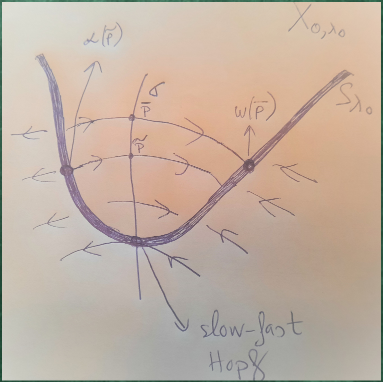
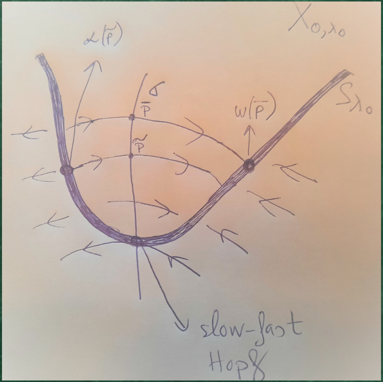


A slow fast Hopf point is intrinsically defined!! (see [De Maesschalck,Dumortier,Roussarie,2021])

The slow divergence integral

$$I(\tilde{p},\bar{p}) := \int_{\alpha(\tilde{p})}^{\omega(\bar{p})} \frac{\operatorname{div} X_{0,\lambda_0}\, dx}{f(x,\lambda_0)} = 0, \quad x' = f(x,\lambda_0)$$



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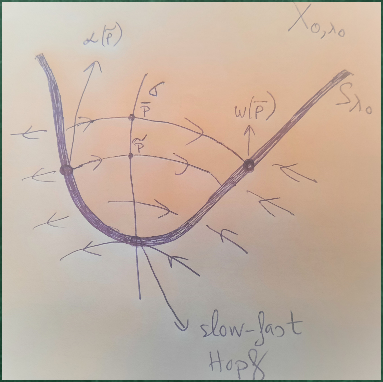

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$\mathcal{S} = \{p_k \mid k \geq 0\}$

Theorem

Consider a smooth slow-fast system $X_{\epsilon,\lambda}$. Let $\mathcal{S}$ be a fractal sequence defined above. Then $\dim_B \mathcal{S}$ *exists and*

$$\dim_B \mathcal{S} \in \{\frac{2j+1}{2j+3} \mid j \in \mathbb{N}_0\} \cup \{1\}.$$

Furthermore, the Minkowski dimension of $\mathcal{S}$ is a coordinate free notion which does not depend on the choice of the section σ, the first element p_0 of the sequence $(p_k)_{k\geq 0}$ from $\mathcal{S}$, and the metric on M.

Definition

If $\dim_B \mathcal{S} < 1$, we say that the contact point p for $\lambda = \lambda_0$ has finite fractal codimension $j + 1 \geq 1$ where

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Consider a smooth slow-fast family $X_{\epsilon,\lambda} = X_{0,\lambda} + \epsilon Q_\lambda + O(\epsilon^2)$ *that has a slow-fast Hopf point* p *at* λ_0.

1. *If the fractal codimension of* p *is equal to* 1*, then* $\text{Cycl}(X_{\epsilon,\lambda}, p) \leq 1$.
2. *If* p *has finite fractal codimension* $j+1 \geq 1$ *and of Liénard type, then* $\text{Cycl}(X_{\epsilon,\lambda}, p)$ *is finite and bounded by* $j+1$.
3. *If* $X_{\epsilon,\lambda}$ *is analytic on an analytic surface* M*, then* $\text{Cycl}(X_{\epsilon,\lambda}, p)$ *is finite. Moreover, if* p *has finite fractal codimension* $j+1 \geq 1$*, then* $\text{Cycl}(X_{\epsilon,\lambda}, p) \leq j+1$.

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a generalization of [Dumortier,Roussarie,2009]

The notion of fractal codimension can be defined for any contact point when the contact order $c_{\lambda_0}(p)$ of p is even, the singularity order $s_{\lambda_0}(p)$ of p is odd and p has finite slow divergence, i.e. $s_{\lambda_0}(p) \leq 2(n_{\lambda_0}(p) - 1)$.

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(Huzak,2017), (Huzak,Vlah,2018), (Crnkovic,Huzak,Vlah,2021), (Dimitrovic, Huzak, Vlah, Zupanovic, 2021), (Huzak, Vlah, Zubrinic,Zupanovic,2022)

Calculating the Minkowski dimension in a normal form for C^∞-equivalence

$$\begin{cases} \dot{x} = y - f(x,\lambda) \\ \dot{y} = \epsilon\left(g(x,\epsilon,\lambda) + (y - f(x,\lambda))\, h(x,y,\epsilon,\lambda)\right), \end{cases}$$

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The singularity order $m \geq 0$ is the order at $x = 0$ of $g(x,0,\lambda_0)$.
We suppose that n and m are finite and write

$$f(x,\lambda_0) = x^n \tilde{f}(x)$$

Calculating the Minkowski dimension in a normal form

If $\tilde{f}(0) > 0$ (resp. $\tilde{f}(0) < 0$), then the smooth diffeomorphism

$$(x, y) \to (x\tilde{f}(x)^{\frac{1}{n}}, y) \left(\text{resp. } (x, y) \to (-x(-\tilde{f}(x))^{\frac{1}{n}}, -y)\right)$$

brings the system into

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upon multiplication by a smooth strictly positive function

$$g(x, 0) = g_m x^m + x^{m+1}\tilde{g}(x)$$

where $g_m = \pm 1$ and $\tilde{g}$ is a smooth function.

Calculating the Minkowski dimension in a normal form

Definition

We say that the contact point $p = (0, 0)$ has finite (fractal) codimension $j + 1 \geq 1$ if

$$\tilde{g}(x) + \tilde{g}(-x) = \alpha x^{2j} + O(x^{2j+2}),\ \alpha \neq 0.$$

If j with the above property does not exist, we say that the codimension is infinite.

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Finite slow divergence: $m \leq 2(n-1)$.

$$I(y, \tilde{y}) = -\int_{-y^{1/n}}^{\tilde{y}^{1/n}} \frac{1}{g(x,0)} (nx^{n-1})^2 dx$$

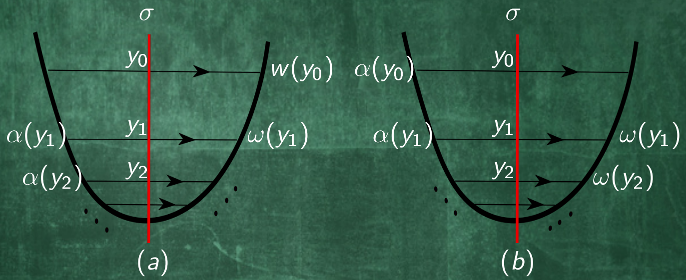


Figure: A fractal sequence starting at $(0, y_0)$ defined near the contact point $(x, y) = (0, 0)$ where $\alpha(y) = \{(-y^{1/n}, y)\}$ is the α-limit of the fast orbit through $y \in \sigma$ and $\omega(y) = \{(y^{1/n}, y)\}$ is the ω-limit of the same orbit. (a) We use $I(y_{k+1}, y_k) = 0$ to generate $(y_k)_{k\geq 0}$. (b) We use $I(y_k, y_{k+1}) = 0$ to generate $(y_k)_{k\geq 0}$.

Theorem

Suppose that the normal form has finite fractal codimension $j+1 \geq 1$. Then $\mathcal{S}$ is Minkowski nondegenerate and

$$\dim_B \mathcal{S} = \frac{2j+1}{n+2j+1} \in]0,1[.$$

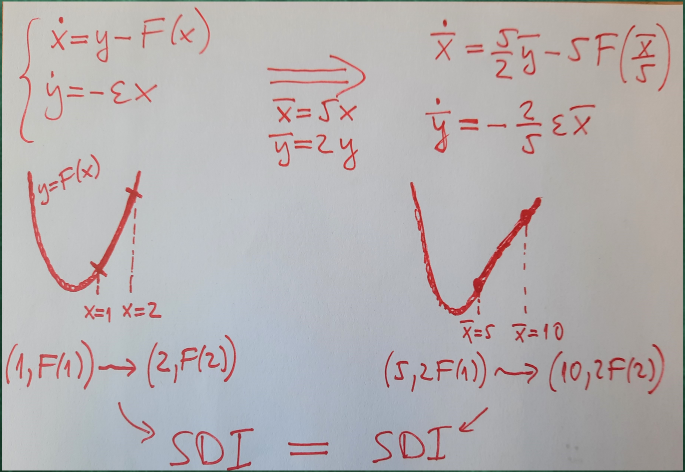
Moreover, when the codimension is infinite, we have $\dim_B \mathcal{S} = 1$. The results do not depend on the choice of the initial point $y_0 \in]0, y^*[$.

The Minkowski dimension is invariant under bi-Lip. maps

- $F: A \subseteq \mathbb{R}^N \to \mathbb{R}^M$ is a bi-Lipschitz map ($\exists\, k_1, k_2 > 0$ such that $k_1 \|x-y\| \leq \|F(x)-F(y)\| \leq k_2 \|x-y\|, \ \forall x, y \in A$)

$\Rightarrow$ $\underline{\dim}_B A = \underline{\dim}_B F(A)$, $\overline{\dim}_B A = \overline{\dim}_B F(A)$

The slow divergence integral is invariant under C^∞-equivalence



A two-stroke oscillator

$$\begin{cases} \dot{x} &= y(\delta - y) \\ \dot{y} &= (-x + \alpha y) \cdot (\delta - y) - \epsilon\big(\beta - \gamma x\big), \end{cases}$$

where $\alpha, \beta, \gamma, \delta > 0$ and $\epsilon \geq 0$ is the singular perturbation parameter.

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Following Wechselberger (2020), we deal with a slow-fast Hopf point (in a non-standard form) at $p = (\alpha\delta, \delta)$, for $\beta = \alpha\gamma\delta$.

$$\dim_B U_0 = \lim_{k\to\infty} \frac{\ln k}{-\ln(y_k - y_{k+1})} \text{ (Cahen-type formula)}$$

$$\dim_B U_0 = \lim_{k\to\infty} \frac{1}{1 - \frac{\ln y_k}{\ln k}} \text{ (Borel rarefaction index of } U_0\text{)}$$

or

$$\dim_B U_0 = \lim_{k\to\infty} \left(1 - \frac{\ln\left(k(y_k - y_{k+1}) + y_k\right)}{\ln\left(\frac{y_k - y_{k+1}}{2}\right)}\right) \text{ (tail and nucleus)}$$

# Iter	$\tilde{y}_0$	α	δ	γ	β	Theo. Value	Results
1000	1.1	1	1	1	1	$\frac{1}{3} = 0.3333...$	0.335137
1000	1.1	1	1	10	10	$\frac{1}{3} = 0.3333...$	0.335137
1000	1.1	2	1	1	2	$\frac{1}{3} = 0.3333...$	0.324280
1000	10.1	5	10	1	50	$\frac{1}{3} = 0.3333...$	0.331570

Table: Numerical results for the two-stroke oscillator.

Thank you!