Invariant algebraic manifolds for ordinary differential equations

Maria V. Demina
maria_dem@mail.ru

National Research University Higher School of Economics, Moscow, Russia

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Invariant manifolds

- Polynomial differential system

$$\dot{x}_j = X_j(x), \quad x = (x_1, \ldots, x_n), \quad X_j(x) \in \mathbb{C}[x], \quad 1 \leq j \leq n$$

- Polynomial vector field

$$\mathcal{X}_{n\mathcal{D}} = X_1(x)\frac{\partial}{\partial x_1} + \ldots + X_n(x)\frac{\partial}{\partial x_n}$$

- Invariant manifold $M \subset \mathbb{C}^n$

$$s_0 \in M \quad \Rightarrow \quad \forall t \in \mathbb{R} \quad x(t; s_0) \in M, \quad \text{where} \quad x(0; s_0) = s_0$$

Invariant algebraic manifolds

- Invariant algebraic manifold $M \subset \mathbb{C}^n$ of codimension k, $1 \leq k \leq n-1$

$$M = \bigcap_{j=1}^{k} \{G_j(x) = 0,\ G_j(x) \in \mathbb{C}[x]\}$$

- Polynomial ordinary differential equation

$$E: \quad E\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^n x}{\mathrm{d}t^n}\right) = 0, \quad E(s_1, \ldots, s_{n+1}) \in \mathbb{C}[s_1, \ldots, s_{n+1}]$$

- A compatible with E polynomial ordinary differential equation of degree $n-k$ defines an invariant algebraic manifold M of codimension k

Invariant algebraic manifolds of codimension $n-1$

- Reduction of order

$$E: \quad E\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^n x}{\mathrm{d}t^n}\right) = 0$$

$$\Big\Downarrow \; \frac{\mathrm{d}x}{\mathrm{d}t} = y(x)$$

$$H: \quad H\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}\right) = 0$$

- Compatible equations: $F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = 0 \;\Rightarrow\; F(x, y) = 0$

- $F(x, y) \in \mathbb{C}[x, y]$ is called an algebraic invariant

  $x(t)$ such that $F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = 0$ is called an algebraically invariant solution of equation (E)

Finding algebraic invariants

The Poincaré problem

For a given polynomial vector field $\mathcal{X}_{2\mathcal{D}}$ find an upper bound on the degrees of its irreducible algebraic invariants: $\mathcal{P}(\mathcal{X}_{2\mathcal{D}})$.

Partial solution 1. (D. Cerveau, A. Lins Neto, 1991)

If all the singularities of irreducible invariant algebraic curves are of nodal type, then the following estimate is valid: $\mathcal{P}(\mathcal{X}_{2\mathcal{D}}) \leq \deg \mathcal{X}_{2\mathcal{D}} + 2$.

Partial solution 2. (M. M. Carnicer, 1994)

If there are no dicritical singularities of the vector field $\mathcal{X}_{2\mathcal{D}}$ on irreducible invariant algebraic curves, then the following estimate is valid: $\mathcal{P}(\mathcal{X}_{2\mathcal{D}}) \leq \deg \mathcal{X}_{2\mathcal{D}} + 2$.

Finding algebraic invariants

The methods of finding algebraic invariants $(2\mathcal{D})$

- The method of undetermined coefficients (the method of Prelle and Singer)
- The Lagutinskii's method (the method of the extactic polynomial)
- Decomposition into weight-homogeneous components: $\mathcal{X}_{2\mathcal{D}}^{(0)} F^{(0)} = \lambda^{(0)}(x,y) F^{(0)}$, $\lambda^{(0)}(x,y) \in \mathbb{C}[x,y]$
- Methods, based on symmetries
- The method of fractional power series (Puiseux series)

Finding algebraic invariants

- Fields of Puiseux series

$$\mathbb{C}_{\infty}\{x\} = \left\{ y(x) = \sum_{k=0}^{+\infty} b_k x^{\frac{l_0}{n} - \frac{k}{n}}, \quad x_0 = \infty \right\},$$

$$\mathbb{C}_{x_0}\{x\} = \left\{ y(x) = \sum_{k=0}^{+\infty} c_k (x - x_0)^{\frac{l_0}{n} + \frac{k}{n}}, \quad x_0 \in \mathbb{C} \right\}$$

- Rings of polynomials over the fields of Puiseux series

$$\mathbb{C}_{\infty}\{x\}[y], \quad \mathbb{C}_{x_0}\{x\}[y]$$

Finding algebraic invariants

Projection operators:

$\{W(x,y)\}_+$ yields the polynomial part of $W(x,y) \in \mathbb{C}_\infty\{x\}[y]$;
$\{W(x,y)\}_-$ yields the non-polynomial part of $W(x,y) \in \mathbb{C}_\infty\{x\}[y]$.

The Newton–Puiseux theorem

Any solution $y(x)$ of the equation $F(x,y)=0$, $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$ can be locally represented by a convergent Puiseux series.

We are interested in Puiseux series satisfying the equation

$$H: \quad H\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}\right) = 0$$

Finding algebraic invariants

Theorem 1 (M. V. Demina, 2018)

Let $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$ be an irreducible algebraic invariant of equation (E). Then $F(x,y)$ takes the form

$$F(x,y) = \left\{ \mu(x) \prod_{j=1}^{N} \{y - y_{j,\infty}(x)\} \right\}_{+}, \quad \mu(x) \in \mathbb{C}[x],$$

where $y_{1,\infty}(x)$, …, $y_{N,\infty}(x)$ are pairwise distinct Puiseux series from the field $\mathbb{C}_{\infty}\{x\}$ that satisfy equation (H).

Finding the polynomial $\mu(x)$

Theorem 2 (M. V. Demina, 2021)

Let $F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}[x]$ be an irreducible algebraic invariant of equation (E). If $x_0 \in \mathbb{C}$ is a zero of the polynomial $\mu(x)$, then the following statements are valid:

- *At least one Puiseux series from the field $\mathbb{C}_{x_0}\{x\}$ that has a negative exponent in the leading-order term solves equation (H).*

Finding the polynomial $\mu(x)$

- If the number of distinct Puiseux series from the field $\mathbb{C}_{x_0}\{x\}$ that solve equation (H) and have negative exponents in leading-order terms

$$y_{j,x_0}(x) = c_0^{(j)}(x-x_0)^{-q_j} + o\left((x-x_0)^{-q_j}\right), \quad c_0^{(j)} \neq 0, \\ q_j \in \mathbb{Q}, \quad q_j > 0, \quad 1 \leq j \leq L \in \mathbb{N} \tag{1}$$

is finite, then the following inequality $m_0 \leq \sum_{j=1}^{L} q_j$ holds, where $m_0 \in \mathbb{N}$ is the multiplicity of the polynomial $\mu(x)$ at its zero x_0.

The uniqueness properties

Theorem 3 (M. V. Demina, 2021)

Suppose for some $x_0 \in \overline{\mathbb{C}}$ *a Puiseux series* $y(x)$ *from the field* $\mathbb{C}_{x_0}\{x\}$ *satisfies equation (H) and possesses uniquely determined exponents and coefficients. Then there exists at most one irreducible algebraic invariant* $F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}[x]$ *of of the related equation (E) such that this series is annihilated by* $F(x, y)$*, i.e. the series* $y(x)$ *solves the equation* $F(x, y) = 0$*.*

The uniqueness properties

Theorem 4 (M. V. Demina, 2021)

If for some $x_0 \in \overline{\mathbb{C}}$ the number of distinct Puiseux series from the field $\mathbb{C}_{x_0}\{x\}$ that satisfy equation (H) is finite, then the related equation (E) possesses a finite number (possibly none) of irreducible algebraic invariants. Moreover, the number of pairwise distinct irreducible algebraic invariants does not exceed the number of distinct Puiseux series from the field $\mathbb{C}_{x_0}\{x\}$ that satisfy equation (H).

The Poincaré problem

The finiteness property $(A_{f,f})$

1. There exists only a finite number of Puiseux series from the field $\mathbb{C}_\infty\{x\}$ that satisfy equation (H).
2. There exists only a finite number of complex numbers $x_0 \in \mathbb{C}$ and a only finite number of Puiseux series belonging to each of the fields $\mathbb{C}_{x_0}\{x\}$ that have negative exponents in the leading-order terms and satisfy equation (H).

Theorem 5 (Partial solution 3, M. V. Demina, 2022)

Let (H) belong to the set $A_{f,f}$, then the Poincaré problem for the related equation (E) has a solution: $\mathcal{P}(E) \leq \deg^* H$.

Finding algebraic invariants

The method of Puiseux series

1. Find all Puiseux series (centered at finite points and infinity) that satisfy equation (H).
2. Consider all possible factorizations of algebraic invariants in the ring $\mathbb{C}_\infty\{x\}[y]$.
3. Construct and solve the algebraic system resulting from the condition

$$\left\{\mu(x)\prod_{j=1}^{N}\{y-y_{\infty,j}(x)\}\right\}_{-}=0.$$

Finding algebraic invariants

Power geometry

1. Newton polygon of equation (H).
2. Dominant balances $U[y(x), x]$ and reduced equations $U[y(x), x] = 0$ related to the vertices and edges of the Newton polygon.
3. Power asymptotics $y(x) = b_0 x^{r_0}$, $b_0 \in \mathbb{C} \setminus \{0\}$, $x \to \infty$ or $x \to 0$
4. Fuchs indices or Kovalevskaya exponents: $V(j) = 0$

$$\frac{\delta U}{\delta y}[b_0 x^{r_0}, x] = \lim_{s \to 0} \frac{U[b_0 x^{r_0} + s x^{r_0 - j}, x] - U[b_0 x^{r_0}, x]}{s} = V(j) x^{\tilde{r}_0}$$

Finding algebraic invariants

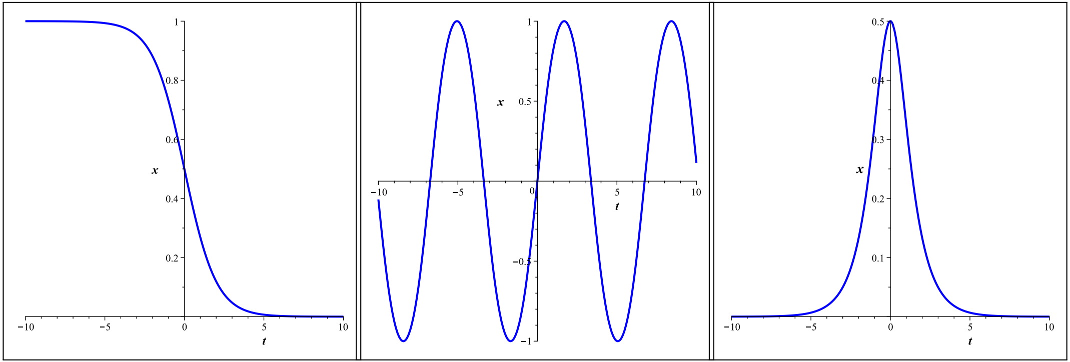
Computational aspects

- finite number of admissible Puiseux series:
$$\{y_{j,\infty}(x) \in \mathbb{C}_\infty\{x\}, j = 1, \ldots, N\} \Rightarrow \deg_y F \leq N$$
- infinite number of admissible Puiseux series:
$$\sum_{m=1}^{M} (\beta_m)^k = M\varrho_k, \quad k \in \mathbb{N}$$

Lemma (M.V. Demina, 2021). If for some $M_0 \in \mathbb{N}$ this system has a solution $(\beta_1, \ldots, \beta_{M_0})$ with $\beta_{m_1} \neq \beta_{m_2}$ whenever $m_1 \neq m_2$, then all other solutions of this system exist only when $M = lM_0$, where $l \in \mathbb{N} \setminus \{1\}$, and in such case involve l multiple roots for each element of the tuple $(\beta_1, \ldots, \beta_{M_0})$.

Exact solutions

$$P(u, u_\tau, u_s, u_{\tau\tau}, u_{s\tau}, u_{ss}, \ldots) = 0, \qquad u(s,\tau) = x(t), \qquad t = s + v_0\tau$$



(a) Kink (b) Periodic wave (c) Solitary wave

Figure: Examples of traveling waves

Meromorphic solutions

$\mathbb{W}$-meromorphic functions

- Elliptic functions
- Meromorphic simply-periodic functions of the form
  $x(t) = R(\exp\{\alpha t\}), \qquad R(s) \in \mathbb{C}(s), \qquad \alpha \in \mathbb{C} \setminus \{0\}$

Theorem 6 (C. Briot, T. Bouquet)

Any $\mathbb{W}$-meromorphic function $x(t)$ satisfies an algebraic first order ordinary differential equation $F(x, x_t) = 0$, $F(x, y) \in \mathbb{C}[x, y]$.

Conclusion:

$\mathbb{W}$-meromorphic solutions are algebraically invariant solutions

Meromorphic solutions

$$(E):\quad \sum_j E_j[x(t)] = 0, \qquad E_j[x(t)] = \alpha_j x^{j_0} \left\{\frac{dx}{dt}\right\}^{j_1} \dots \left\{\frac{d^M x}{dt^M}\right\}^{j_M}$$

- Degree of the differential monomial $E_j[x(t)]$: $\quad \deg E_j = \sum_{m=0}^{M} j_m$

The finiteness property

There exists only a finite number of formal Laurent series of the form $x(t) = \sum_{k=0}^{+\infty} a_k t^{k-p},\ p \in \mathbb{N}$ that satisfy equation (E).

Meromorphic solutions

Theorem 7 (A. Eremenko, 2007)

All transcendental meromorphic solutions of equation (E) are $\mathbb{W}$-meromorphic functions whenever (E) has the finiteness property and only one dominant differential monomial.

Theorem 8 (M. V. Demina, 2019)

All transcendental meromorphic solutions of equation (E) are $\mathbb{W}$-meromorphic functions whenever (E) has the finiteness property and only two dominant differential monomials of the form $x^l(x_t - \beta x)$, $l \in \mathbb{N}$, $\beta \in \mathbb{C}$.

Meromorphic solutions

Theorem 9 (M. V. Demina, 2022)

Let $x(t)$ be a $\mathbb{W}$-meromorphic solution of equation (E). Then there exist an irreducible in $\mathbb{C}[x,y]\setminus\mathbb{C}[x]$ polynomial $F(x,y)$ and a number $N\in\mathbb{N}$ such that $x(t)$ satisfies the algebraic first-order ordinary differential equation $F(x,x_t)=0$ and the polynomial $F(x,y)$ takes the form

$$F(x,y)=\left\{\prod_{j=1}^{N}\{y-y_{j,\infty}(x)\}\right\}_{+}.$$

In this expression $y_{1,\infty}(x)$, ..., $y_{N,\infty}(x)$ are pairwise distinct Puiseux series centered at the point $x=\infty$ that

Meromorphic solutions

(A): solve equation (H);

(B): possess the leading-order terms given either by $b_0^{(j)}x$ or by $b_0^{(j)}x^{(p_j+1)/p_j}$, where $b_0^{(j)} \neq 0$ and $p_j \in \mathbb{N}$ is an order of a pole of $x(t)$;

(C): satisfy the conditions

$$\left\{\sum_{j=1}^{N} y_{j,\infty}^{k}(x)\right\}_{-} = 0, \quad 1 \leq k \leq N.$$

Meromorphic solutions

Explicit expressions of $\mathbb{W}$-meromorphic functions

1. genus 0

$$w(z) = \sum_{k=K_1}^{K_2} h_k \exp\left(2\omega k z\right) - \omega \sum_{m=1}^{M} \left\{ \sum_{k=1}^{p_m} \frac{(-1)^k a_{p_m-k}^{(m)}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \right\} \coth\left(\omega\{z - z_m\}\right)$$

2. genus 1

$$w(z) = \sum_{m=1}^{M} \left\{ \sum_{k=2}^{p_m} \frac{(-1)^k a_{p_m-k}^{(m)}}{(k-1)!} \frac{d^{k-2}}{dz^{k-2}} \right\} \wp(z - z_m) + \sum_{m=1}^{M} a_{p_m-1}^{(m)} \zeta(z - z_m) + h_0,$$

$$\sum_{m=1}^{M} a_{p_m-1}^{(m)} = 0.$$

The integrability problem $(2\mathcal{D})$

- Polynomial vector fields $V \subset \mathbb{C}^{(m+2)(m+1)-l} \times (\mathbb{C} \setminus \{0\})^l$

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}, \quad P(x,y), Q(x,y) \in \mathbb{C}[x,y]$$

- Polynomial systems of ordinary differential equations

$$x_t = P(x,y), \quad y_t = Q(x,y)$$

Problems

1. Find the functional classes of first integrals that vector fields from V can have.
2. Find all the vector fields from V having a first integral from some functional class.

The integrability problem $(2\mathcal{D})$

Functional classes of first integrals

- rational;
- meromorphic;
- Darboux;
- Liouvillian

Darboux functions

$$G(x,y) = \prod_{j=1}^{K} F_j^{d_j}(x,y)\exp\{R(x,y)\}, \quad R(x,y)\in\mathbb{C}(x,y),$$

$$F_1(x,y),\ldots,F_K(x,y)\in\mathbb{C}[x,y], \quad d_1,\ldots,d_K\in\mathbb{C}$$

The integrability problem $(2\mathcal{D})$

Liouvillian functions

belong to the following differential field extension of the field of rational functions $\mathbb{C}(x,y)$:

$$\mathbb{C}(x,y) = K_0 \subset K_1 \subset \ldots \subset K_M = L, \quad K_{j+1} = K_j(s), \quad \Delta = \{\partial_x, \partial_y\}$$

- s is an algebraic element over K_j;
- s is a transcendental element over K_j such that $\forall \delta \in \Delta \Rightarrow \delta s \in K_j$;
- s is a transcendental element over K_j such that $\forall \delta \in \Delta \Rightarrow \dfrac{\delta s}{s} \in K_j$.

The integrability problem $(2\mathcal{D})$

Differential form: $\omega = Q(x,y)dx - P(x,y)dy$

Integrating factor: $M(x,y) : D \subset \mathbb{C}^2 \to \mathbb{C}$

- $M(x,y)\{Q(x,y)dx - P(x,y)dy\} = dI(x,y)$;
- $M(x,y) \in \mathbb{C}^1(D) \;\Rightarrow\; \mathcal{X}M = -\mathrm{div}(\mathcal{X})\mathrm{M}, \quad \mathrm{div}(\mathcal{X}) = P_x + Q_y$;
- symplectic form: $\Omega = M(x,y)dx \wedge dy, \quad (x,y) \in D$.

The Darboux theory of integrability ($2D$)

Theorem 10 (J. Chavarriga et al., 2003; C. Christopher et al., 2019)

A polynomial vector field $\mathcal{X}$ is Darboux integrable if and only if it has a rational integrating factor.

Theorem 11 (M. F. Singer, 1992)

A polynomial vector field $\mathcal{X}$ is Liouvillian integrable if and only if it has a Darboux integrating factor.

The Darboux theory of integrability ($2\mathcal{D}$)

Darboux functions

$$M(x,y) = \prod_{j=1}^{K} F_j^{d_j}(x,y)\exp\{R(x,y)\}, \quad R(x,y) \in \mathbb{C}(x,y),$$

$$F_1(x,y),\dots,F_K(x,y) \in \mathbb{C}[x,y], \quad d_1,\dots,d_K \in \mathbb{C}$$

Theorem 12 (C. Christopher, 1999)

If a Darboux function $M(x,y)$ is an integrating factor of a polynomial vector field $\mathcal{X}$, then $F_1(x,y)$,..., $F_K(x,y)$, $\exp\{R(x,y)\}$ are invariants of the vector field $\mathcal{X}$.

Invariants

Invariants of a polynomial vector field $\mathcal{X}$

- Algebraic invariants (Darboux polynomials)

$$F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C} : \mathcal{X}F = \lambda(x,y)F, \quad \lambda \in \mathbb{C}[x,y]$$

$\lambda(x,y)$ is called the cofactor of $F(x,y)$

- Exponential invariants (multiple algebraic invariants)

$$E(x,y) = \exp\left\{\frac{S(x,y)}{T(x,y)}\right\} : \mathcal{X}E = \varrho(x,y)E, \ S, T, \varrho \in \mathbb{C}[x,y]$$

$\varrho(x,y)$ is called the cofactor of $E(x,y)$

The integrability problem $(2\mathcal{D})$

Integrability conditions

- Darboux first integrals: $$I = \prod_{j=1}^{K} F_j^{d_j}(x,y) \exp\left\{\frac{S(x,y)}{T(x,y)}\right\}$$

$$\sum_{j=1}^{K} d_j \lambda_j(x,y) + \varrho(x,y) = 0;$$

- Darboux integrating factors: $$M = \prod_{j=1}^{K} F_j^{d_j}(x,y) \exp\left\{\frac{S(x,y)}{T(x,y)}\right\}$$

$$\sum_{j=1}^{K} d_j \lambda_j(x,y) + \varrho(x,y) = -\mathrm{div}\mathcal{X}$$

Finding the cofactor of an algebraic invariant

$$(H): \quad P(x,y)y_x - Q(x,y) = 0$$

Theorem 13 (M. V. Demina, 2021)

The cofactor $\lambda(x,y)$ *of an algebraic invariant* $F(x,y)$ *reads as*

$$\lambda(x,y) = \left\{ \sum_{m=0}^{+\infty} \sum_{j=1}^{N} \frac{\{Q(x,y) - P(x,y)(y_{j,\infty})_x\}(y_{j,\infty})^m}{y^{m+1}} + P(x,y) \sum_{m=0}^{+\infty} \sum_{l=1}^{L} \frac{\nu_l x_l^m}{x^{m+1}} \right\}_+ ,$$

where $y_{1,\infty}, \ldots, y_{N,\infty} \in \mathbb{C}_\infty\{x\}$ *and satisfy equation* (H), $x_1, \ldots, x_L$ *are pairwise distinct zeros of the polynomial* $\mu(x) \in \mathbb{C}[x]$ *with multiplicities* $\nu_1, \ldots, \nu_L \in \mathbb{N}$ *and* $L \in \mathbb{N} \cup \{0\}$.

Finding exponential invariants

Theorem 14 (M. V. Demina, 2018)

Suppose that a polynomial vector field $\mathcal{X}$ *admits an exponential invariant* $E = \exp(g/f)$ *related to the algebraic invariant* $f(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$ *with the cofactor* $\lambda(x,y) \in \mathbb{C}[x,y]$, *then for each non-zero Puiseux series* $y_{j,\infty}(x)$ *centered at the point* $x = \infty$ *that satisfies the equation* $f(x,y) = 0$ *there exists a number* $q \in \mathbb{Q}$ *such that the Puiseux series for the function* $\lambda(x, y_{j,\infty}(x))/P(x, y_{j,\infty}(x))$ *centered at the point* $x = \infty$ *is*

$$\frac{\lambda(x, y_{j,\infty}(x))}{P(x, y_{j,\infty}(x))} = \sum_{k=n}^{+\infty} b_k x^{-\frac{k}{n}}, \quad b_n = q.$$

The Puiseux integrability

Local invariants of a polynomial vector field $\mathcal{X}$

- Elementary algebraic invariants

$$F(x,y) = y - y_{j,x_0}(x) \in \mathbb{C}_{x_0}\{x\}[y],\ F(x,y) = y_{j,x_0}(x) \in \mathbb{C}_{x_0}\{x\},$$

$$\mathcal{X}F = \lambda(x,y)F, \quad \lambda(x,y) \in \mathbb{C}_{x_0}\{x\}[y]$$

- Elementary exponential invariants

$$E(x,y) = \exp\left[g_l(x)y^l\right], \quad g_l(x) \in \mathbb{C}_{x_0}\{x\}, \quad l \in \mathbb{N} \cup \{0\};$$

$$E(x,y) = \exp\left[\frac{u(x,y)}{\{y - y_{j,x_0}(x)\}^n}\right],\ y_{j,x_0}(x) \in \mathbb{C}_{x_0}\{x\},$$

$$u(x,y) \in \mathbb{C}_{x_0}\{x\}[y],\ n \in \mathbb{N};\ \mathcal{X}E = \varrho(x,y)E,\ \varrho(x,y) \in \mathbb{C}_{x_0}\{x\}[y]$$

The Puiseux integrability

Definition (M. V. Demina, J. Giné, C. Valls, 2022)

A polynomial vector field $\mathcal{X}$ is called Puiseux integrable near a line $\{x = x_0, y \in \overline{\mathbb{C}}\}$, $x_0 \in \overline{\mathbb{C}}$ if it has a formal integrating factor

$$M(x, y) = \exp\left\{\frac{g(x,y)}{f(x,y)}\right\} \prod_{j=1}^{K} F_j^{d_j}(x,y), \quad K \in \mathbb{N} \cup \{0\},$$

where $F_1(x,y)$, …, $F_K(x,y)$, $g(x,y)$, and $f(x,y)$ are Puiseux polynomials from the ring $\mathbb{C}_{x_0}\{x\}[y]$ and d_1, …, $d_K \in \mathbb{C}$.

Polynomial Liénard equations

$$x_{tt} + f(x)x_t + g(x) = 0, \quad f(x), g(x) \in \mathbb{C}[x], \quad f(x)g(x) \not\equiv 0;$$

$$x_t = y, \qquad y_t = -f(x)y - g(x).$$

- Polynomial vector fields

$$\mathcal{X} = y\frac{\partial}{\partial x} - (f(x)y + g(x))\frac{\partial}{\partial y}$$

- Abel differential equations

the second kind : $\quad yy_x + f(x)y + g(x) = 0,$

the first kind : $\; w_x - g(x)w^3 - f(x)w^2 = 0, \; w(x) = \dfrac{1}{y(x)}$

Polynomial Liénard equations

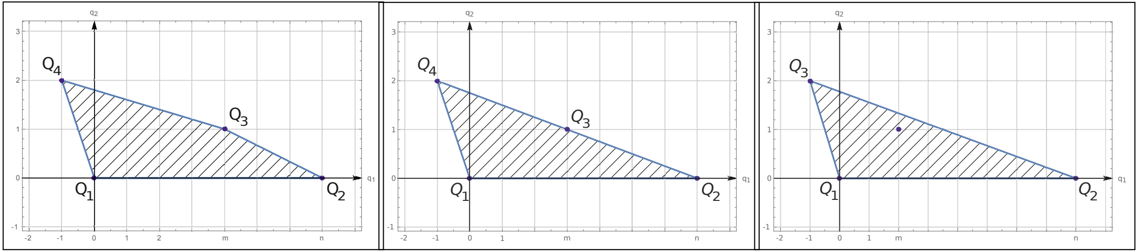
$$L_{n,m} = \left\{ y\frac{\partial}{\partial x} - (f(x)y + g(x))\frac{\partial}{\partial y} : \ \deg f = m, \ \deg g = n \right\}$$

$$m \geq n, \quad (m,n) \neq (0,0)$$

1. Vector fields from $L_{n,m}$ do not have algebraic invariants provided that $g(x) \neq Cf(x)$, $C \in \mathbb{C}$; [K. Odani, 1995].
2. Vector fields from $L_{n,m}$ are not Liouvillian integrable provided that $g(x) \neq Cf(x)$, $C \in \mathbb{C}$; [J. Llibre, C. Valls, 2013].

Polynomial Liénard equations

$$yy_x + f(x)y + g(x) = 0, \qquad \deg f = m, \deg g = n$$



(a) : $m < n < 2m+1$ (b) : $n = 2m+1$ (c) : $2m+1 < n$

Figure: Newton polygons

Polynomial Liénard equations

$$L_{n,m} = \left\{ y\frac{\partial}{\partial x} - (f(x)y + g(x))\frac{\partial}{\partial y} : \ \deg f = m, \deg g = n \right\}$$

$$m < n, \quad (m,n) \neq (0,1)$$

1. A generic vector field from $L_{n,m}$ is not Liouvillian integrable.
2. Vector fields from $L_{n,m}$ are not Darboux integrable provided that $n \neq 2m+1$.
3. For any n and m there exist vector fields from $L_{n,m}$ that are Liouvillian integrable.
4. The problem of Liouvillian integrability is solved completely provided that $n \neq 2m+1$. In the case $n = 2m+1$ our results are complete in the non-resonant case.

Polynomial Liénard equations

Example: a family of Liouvillian integrable vector fields from $L_{n,m}$

$$f(x) = \frac{(k+2l)}{4} w^{l-1} w_x,\ g(x) = \frac{k}{8}\left(w^{2l-1} + 4\beta w^{k-1}\right) w_x,\ w(x) \in \mathbb{C}[x]$$

$$\beta \in \mathbb{C},\ \deg w = \frac{m+1}{l},\ \frac{n+1}{m+1} = \frac{k}{l},\ (l,k) = 1$$

- Liouvillian first integral:

$$I(x,y) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2} + \frac{l}{k}; \frac{3}{2}; -\frac{(2y+w^l)^2}{4\beta w^k}\right) \frac{(2l-k)(2y+w^l)}{4kw^{\frac{k}{2}}\beta^{\frac{1}{2}+\frac{l}{k}}} + z^{\frac{1}{2}-\frac{l}{k}}$$

- Darboux integrating factor: $M(x,y) = z^{-\left(\frac{1}{2}+\frac{l}{k}\right)}$, $z = \left(y + \frac{w^l}{2}\right)^2 + \beta w^k$

Invariant algebraic manifolds of codimension $n-2$

- Reduction of order

$$E: \quad E\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^n x}{\mathrm{d}t^n}\right) = 0$$

$$\Downarrow \; \tfrac{\mathrm{d}^2 x}{\mathrm{d}t^2} = y(x, x_t)$$

$$H: \quad H\left(x, s, y_x, y_s, \ldots\right) = 0, \; s = \frac{\mathrm{d}x}{\mathrm{d}t}$$

- Compatible equations: $F\left(x, \dfrac{\mathrm{d}x}{\mathrm{d}t}, \dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right) = 0 \;\Rightarrow\; F(x, s, y) = 0$

- $F(x, s, y) \in \mathbb{C}[x, s, y]$ is called an algebraic invariant

  $x(t)$ such that $F\left(x, \dfrac{\mathrm{d}x}{\mathrm{d}t}, \dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right) = 0$ is called an algebraically invariant solution of equation (E)

Invariant algebraic manifolds of codimension $n-2$

- Functional Puiseux series

$$\mathbb{C}_{\infty}^{x}\{s\} = \left\{ y(x,s) = \sum_{k=0}^{+\infty} b_k(x) s^{\frac{l_0}{n}-\frac{k}{n}}, \quad x_0 = \infty \right\};$$

$$\mathbb{C}_{s_0(x)}^{x}\{s\} = \left\{ y(x,s) = \sum_{k=0}^{+\infty} c_k(x) (s - s_0(x))^{\frac{l_0}{n}+\frac{k}{n}}, \quad x_0 \in \mathbb{C} \right\}$$

- Factorization

$$F(x,s,y) = \mu(x,s) \prod_{j=1}^{N} \left(y - y_{j,\infty}(x,s) \right), \quad y_{j,\infty}(x,s) \in \mathbb{C}_{\infty}^{x}\{s\}$$

Summary

1. The method of Puiseux series is a power and visual method of finding algebraic invariants and solving the Poincaré problem.
2. The Darboux theory of integrability combined with the method of Puiseux series provides the necessary and sufficient conditions of Liouvillian integrability for polynomial systems in the plane.
3. The method of Puiseux series admits a generalization to higher dimensions.