On the study of limit cycles in piecewise smooth generalized Abel equations via a new Chebyshev criterion

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This is a work derived from

- A. Gasull, A. Geyer, F. Mañosas, *On the number of limit cycles for perturbed pendulum equations*, **Journal of Differential Equations**, 261(2016), 2141-2167.
- J. Huang, J. Torregrosa, J. Villadelprat, *On the Number of Limit Cycles in Generalized Abel Equations*, **SIAM Journal of Applied Dynamical systems**, 19(2020), 2343-2370.
- M. Grau, F. Mañosas, J. Villadelprat, *A Chebyshev criterion for Abelian integrals*, **Transactions of the American Mathematical Society**, 363(2011), 109-129.
- A. Gasull, A. Geyer, F. Mañosas, *A Chebyshev criterion with applications*, **Journal of Differential Equations**, 269(2020), 6641-6655.

1. Introduction and background

- Consider a generalized Abel equation:

$$\dot{x} = \sum_{i=l_1}^{l_2} a_i(\theta)x^i, \quad l_1 \leq l_2 \in \mathbb{Z}, \quad a_{l_1}, \cdots, a_{l_2} \in \mathrm{C}^\infty(\mathbb{S}^1). \quad (1)$$

  - $(l_1, l_2) = (0, 2)$: Riccati equation;
  - $(l_1, l_2) = (0, 3)$: Abel equation.

  - Periodic solution $x(\theta)$: The solution with $x(0) = x(2\pi)$.
  - Periodic orbit (resp. limit cycle) $x = x(\theta)$: The orbit on the cylinder $\mathbb{S}^1 \times I$ where $x(\theta)$ is a periodic solution (resp. isolated periodic solution).

- Background 1: Hilbert's 16th problem.
  - Some planar differential systems, e.g. Rigid systems, systems defined by the sum of two homogeneous vector fields, can be changed into equation (1).

1. Introduction and background

- An important problem for (1):
  - Estimate its number of limit cycles (posed by Smale and Pugh).
- Denote by $\mathcal{N}$ the number of limit cycles of (1).
- Some classical works (since 1980s):
  - $(l_1, l_2) = (0, 1), (0, 2)$: $\mathcal{N} \leq 1$ (resp. $N \leq 2$) (Lins-Neto, *Invent. Math.*; Lloyd, *J. London Math. Soc.*).
  - $(l_1, l_2) = (0, 3)$ (i.e. Abel equation):
    - $\mathcal{N} \leq 3$ if $a_3(t) \neq 0$ or $a_2(t) \neq 0$ and $a_0(t) = 0$ (Lins-Neto, Lloyd, Gasull and Llibre).
    - $\mathcal{N}$ is not bounded for general case (Lins-Neto).
  - $l_2 > 3$: Similar to $(l_1, l_2) = (0, 3)$.

- For this reason, a more specific problem arises:
  - Given fixed l_1 and l_2, whether the maximum number of $\mathcal{N}$ for trigonometrical equation (1), denoted by $\mathcal{H}(m)$, is bounded in terms of the degree m of the coefficients?

- Some works on the lower bound of $\mathcal{H}(m)$ in this line:
  - Neto, A. L., *On the number of solutions of the equation* $\frac{dx}{dt} = \sum_{j=0}^{n} a_j(t)x^j, 0 \leqslant t \leqslant 1$, *for which* $x(0) = x(1)$, **Invent. Math.**, 1(1980), 67-76.
  - Álvarez, M. J., Gasull, A. & Yu, J., *Lower bounds for the number of limit cycles of trigonometric Abel equations*, **JMAA**, 342(2008), 682-693.
  - A. Gasull, C. Li, J. Torregrosa, *A new Chebyshev family with applications to Abel equations*, **JDE**, 252(2012), 1635-1641.
  - J. Huang, J. Torregrosa, J. Villadelprat, *On the Number of Limit Cycles in Generalized Abel Equations*, **SIADS**, 19(2020), 2343-2370.

- Background 2: Reduction of differential equations from real-world models:
  - Van der Pol equation, Liénard equation;
  - Some single-input, second-order bilinear control system;
  - Einstein-Friedmann equation;
  - Reaction-diffusion equation (from a brain tumors model);
  - Pendulum-like systems (Gasull et.al., 2016, 2020; Yang, 2021):

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\sin x + \varepsilon Q(x) y^p, \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} = y, \\ \dot{y} = \sin x\,(\cos x - \gamma) + \varepsilon Q(x) y^p, \end{cases}$$

which can be changed into

$$\frac{dy}{dx} = A(x) y^{-1} + \varepsilon Q(x) y^{p-1}, \quad A(x) = -\sin x \text{ or } \sin x\,(\cos x - \gamma)\,.$$

- The consideration of the real-world factors is necessary.
  - For instance, sudden behaviors and discontinuous phenomena.

- Background 3: The problem of planar piecewise smooth differential systems.
  - The study goes back to several works of the authors A. Andronov, A. Vitt and S. Khaikin (1930s).
  - More multifarious behaviors of the orbits.
  - For instance, systems with a separation straight line:
    - Continuous piecewise linear systems: at most 1 limit cycle (reachable) (Fieire et.al., 1998; Llibre et.al., 2013).
    - Discontinuous piecewise linear systems: examples of 3 limit cycles (Huan et.al., 2013; Llibre et.al., 2012), the maximum number is at most 8 (Novaes et. al., 2022).
  - Note: The separation boundary plays an important role in determining the number of limit cycles (Braga et.al., 2014; Novaes et. al., 2015; Cardin et. al., 2016).

- Stimulated by backgrounds 1-3, we focus on the following type:

$$\frac{dx}{d\theta} = a_p(\theta)x^p + a_q(\theta)x^q, \quad p, q \in \mathbb{Z}\backslash\{1\}, \ \frac{q-1}{p-1} \notin \mathbb{Z}_{\leq 1}, \quad (2)$$

  where a_p, a_q are piecewise 2π-trigonometrical polynomials of degree m with two zones $0 \leq \theta < \theta_1$ and $\theta_1 \leq \theta \leq 2\pi$.

- Denote by $\mathcal{H}_{\theta_1}(m)$ the maximum number of positive and negative limit cycles of equation (2).
- Our goal:
  - Provide a lower bound for $\mathcal{H}_{\theta_1}(m)$;
  - Study how the number of limit cycles is affected by θ_1, i.e., the location of the separation line $\theta = \theta_1$ (closely related to the work of Cardin et. al., 2016).

1. Introduction and background

- A perturbed equation to be observed:

$$\frac{dx}{d\theta} = \sin\theta x^p + \sum_{i=1}^{N} Q_i(\theta)\varepsilon^i x^q, \qquad (3)$$

where $p, q \in \mathbb{Z}\backslash\{1\}$, $\frac{q-1}{p-1} \notin \mathbb{Z}_{\leq 1}$ and

$$Q_i(\theta) := \begin{cases} Q_i^+(\theta) = \sum_{k=0}^{m} \left(c_{ik}^+ \sin k\theta + d_{ik}^+ \cos k\theta\right), & 0 \leq \theta < \theta_1, \\ Q_i^-(\theta) = \sum_{k=0}^{m} \left(c_{ik}^- \sin k\theta + d_{ik}^- \cos k\theta\right), & \theta_1 \leq \theta \leq 2\pi, \end{cases}$$

$i = 1, \cdots, n$.

- The smooth case with $p = -1$ and $N = 1$: perturbed pendulum (Gasull et. al., JDE, 2016, 2020).

- Remark for equation (3).
    - The solution $x = x_\varepsilon(\theta, \rho)$, is smooth with respect to ε and ρ, and piecewise smooth with respect to θ, respectively;
    - Displacement function:

$$x_\varepsilon(2\pi, \rho) - \rho = \sum_{i=1}^{\infty} \varepsilon^i M_i(\rho) \quad \text{with} \quad M_i(\rho) = \frac{1}{i!} \partial_\varepsilon^i \left(x_\varepsilon(2\pi, \rho) \right) |_{\varepsilon=0}.$$

Theorem 1 (Huang & Li, NA-RWA, 2022)

Assume that M_n is the first non-vanishing Melnikov function of equation (3) with $n \leq N$. Let $Z_n(m)$ be the maximum number of isolated zeros of M_n on I, counted with multiplicity. Then, the value of $Z_n(m)$ with respect to the value of θ_1 and the parity of p, are given in Table 30.

	$\theta_1 \in (0,\pi) \cup (\pi, 2\pi)$	$\theta_1 = \pi$	$\theta_1 = 2\pi$
p is even	$3m+1$	$2m$	m
p is odd	$2(3m+1)$	$4m$	$2m$

Table: The values of $Z_n(m)$.

Theorem 2 (Huang & Li, NA-RWA, 2022)

The maximum number of positive and negative limit cycles $\mathcal{H}_{\theta_1}(m)$ for equation (2), with respect to the value of θ_1 and the parities of p and q, verifies the estimates in Table 2.

	$\theta_1 \in (0,\pi)\cup(\pi,2\pi)$	$\theta_1=\pi$	$\theta_1=2\pi$
Both p and q are even	$\geq 3m+1$	$\geq 2m$	$\geq m$
Either p or q is odd	$\geq 2(3m+1)$	$\geq 4m$	$\geq 2m$

Table: Estimates of $\mathcal{H}_{\theta_1}(m)$.

- To illustrate the expression of the Melnikov function M_n, we introduce two families of integrals

$$\begin{aligned} \mathcal{C}_k^E(z) &= \int_E \cos k\theta \left(\frac{1}{1 + z(1-p)\,(1-\cos\theta)} \right)^{\frac{q-p}{p-1}} d\theta, \\ \mathcal{S}_k^E(z) &= \int_E \sin k\theta \left(\frac{1}{1 + z(1-p)\,(1-\cos\theta)} \right)^{\frac{q-p}{p-1}} d\theta. \end{aligned} \tag{4}$$

Lemma 3

If $n \leq N$ and $M_1 = M_2 = \cdots = M_{n-1} \equiv 0$, then the n-th order Melnikov function of equation (3), is given by $M_n(\rho) = \rho^p \int_0^{2\pi} Q_n(\theta) x_0(\theta, \rho)^{q-p} \, d\theta$. Furthermore, $M_n(\rho)$ can be written as

$$M_n(\rho) = \rho^q \left(\sum_{k=0}^{m} \left(d_{nk}^{+} + d_{nk}^{-}\right) \mathcal{C}_k^{E_1}(\rho^{p-1}) + \sum_{k=1}^{m} \left(c_{nk}^{+} - c_{nk}^{-}\right) \mathcal{S}_k^{E_1}(\rho^{p-1}) + 2 \sum_{k=0}^{m} d_{nk} \mathcal{C}_k^{E_2}(\rho^{p-1}) \right),$$

where $E_1 = (0, \theta_1)$, $E_2 = (\theta_1, \pi)$ and $d_{nk} = d_{nk}^{-}$ (resp. $E_1 = (0, 2\pi - \theta_1)$, $E_2 = (2\pi - \theta_1, \pi)$ and $d_{nk} = d_{nk}^{+}$) when $\theta_1 \in (0, \pi]$ (resp. $\theta_1 \in (\pi, 2\pi]$).

- Difficult and key point: Analyze the structures of the families of functions

$$\mathcal{C}_0^{E_1}, \ldots, \mathcal{C}_m^{E_1};\ \mathcal{S}_0^{E_1}, \ldots, \mathcal{S}_m^{E_1};\ \mathcal{C}_0^{E_2}, \ldots, \mathcal{C}_m^{E_2}.$$

- There are some known results:
  - $(\mathcal{C}_0^{E_i}, \ldots, \mathcal{C}_m^{E_i})$ is ECT-system (Gasull et. al., JDE 2012).
  - $(\mathcal{S}_0^{E_1}, \ldots, \mathcal{S}_m^{E_1})$ is ECT-system (Gasull et. al., JDE 2020).

$\Longrightarrow$ An interesting problem: Whether the union of some Chebyshev families also possesses the Chebyshev property?

Definitions of Chebyshev systems.

- Let
  - $f_0, f_1, \cdots, f_m$: smooth functions an interval E.
  - Z_k: Maximum number of isolated zeros of the linear combination of $f_0, f_1, \cdots, f_k$ on E, $0 \leq k \leq m$.
- Then $\{f_0, f_1, \cdots, f_m\}$ is called
  - Chebyshev system (T-system) on E, if $Z_m = m$.
  - Extended Chebyshev system (ET-system) on E, if $Z_m = m$ counted with multiplicity.
  - Complete Chebyshev system (CT-system) on E, if $Z_k = k$ for each $k = 0, 1, \cdots, m$.
  - Extended complete Chebyshev system (ECT-system) on E, if $Z_k = k$ counted with multiplicity for each $k = 0, 1, \cdots, m$.

Equivalent definitions of CT/ECT-system.

- Notations: $f_0, \cdots, f_k := \mathbf{f}_k$, $t_0, \cdots, t_k := \mathbf{t}_k$ and

  $D[\mathbf{f}_k; \mathbf{t}_k] = \det(f_j(t_i);\ 0 \le i, j \le k)$ (discrete Wronskian),
  $W[\mathbf{f}_k](t) = \det(f_j^{(i)}(t);\ 0 \le i, j \le k)$ (continuous Wronskian).

- Then $\{f_0, f_1, \cdots, f_m\}$ is
  - CT-system on E $\Leftrightarrow$ For each $k = 0, 1, \cdots, m$,

    $$D[\mathbf{f}_k; \mathbf{t}_k] \neq 0 \text{ for all } \mathbf{t}_k \in E^{k+1} \text{ s.t. } t_i \neq t_j \text{ for } i \neq j;$$

  - ECT-system on E $\Leftrightarrow$ For each $k = 0, 1, \cdots, m$,

    $$W[\mathbf{f}_k](t) \neq 0 \text{ for all } t \in E.$$

4. Chebyshev systems and a new family

- Some related works on efficiently determining the Chebyshev property of the families:
  - [Gasull, Li, Llibre & Zhang, *PJM*, 2002] studies the elliptic integrals by the argument principle.
  - [Grau, Mañosas & Villadelprat, *TAMS*, 2011] and [Gasull, Geyer & Mañosas, *JDE*, 2020] studies some Abelian integrals via Chebyshev properties of the integrands.
  - [Gasull, Li & Torregrosa, *JDE*, 2012] studies based on Gram determinant.
  - [Gasull, Lázaro & Torregrosa, *JMAA*, 2012] studies via Derivation-Division algorithm.
  - [Cen, Liu & Zhao, *JDE*, 2020] studies some Abelian integrals according to asymptotic expansions of the Wronskians.
  - [Liu & Xiao, *JDE*, 2020] gives a new methods motivated by the idea of criterion functions.

Theorem 4 (Huang & Li, NA-RWA, 2022)

Let $\mathcal{C}_k^E$ *and* $\mathcal{S}_k^E$ *be defined in* (4). *For any fixed* $m \in \mathbb{Z}^+$, $J_1 = [0, \vartheta]$ *and* $J_2 = [\vartheta, \pi]$ *with* $\vartheta \in (0, \pi]$, *the ordered set of functions*

$$\left(\mathcal{C}_0^{J_1}, \cdots, \mathcal{C}_m^{J_1}, \mathcal{S}_m^{J_1}, \cdots, \mathcal{S}_1^{J_1}, \mathcal{C}_0^{J_2}, \cdots, \mathcal{C}_m^{J_2}, \mathcal{S}_m^{J_2}, \cdots, \mathcal{S}_1^{J_2}\right) \tag{5}$$

is an ECT-system on $(-\infty, \frac{1}{2p-2})$ *(resp.* $(\frac{1}{2p-2}, +\infty)$*) when* $p > 1$ *(resp.* $p < 1$*). Here we have used the convention: when* $\vartheta = \pi$ *the set is*

$$\left(\mathcal{C}_0^{J_1}, \cdots, \mathcal{C}_m^{J_1}, \mathcal{S}_m^{J_1}, \cdots, \mathcal{S}_1^{J_1}\right).$$

Key points of the proof:

- "Commutativity" of integration and determinant (inspired by the spirit of [Grau et al, *TAMS*, 2011]).
- Decomposition of the "huge" determinant.

Remark for the Theorem 4:

- Applicable to simultaneously analyzing the first non-vanishing Melnikov function in both the smooth and the piecewise smooth cases.

Actually, Theorem 4 is a corollary of a result of our recent work (Huang, Liang & Zhang, JDE, 2023).

- Suppose that $E_0, \ldots, E_n$ are non-intersecting intervals, and E is an open interval that contains $\bigcup_{i=0}^{n} E_i$. Consider a family

$$\mathcal{F} = \bigcup_{i=0}^{n} \{I_{i,0}, I_{i,1}, \ldots, I_{i,m_i}\}, \tag{6}$$

where

$$I_{i,j}(y) := \int_{E_i} \frac{f_{i,j}(t)}{(1 - yg(t))^{\alpha}} dt,$$

with g being monotonic on E and $\alpha \in \mathbb{R} \backslash \mathbb{Z}_0^-$.

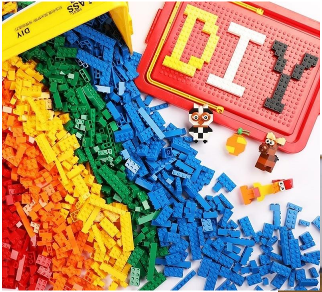
Theorem 5 (Huang, Liang & Zhang, JDE, 2023)

The set of ordered functions, $\mathcal{F}$ in (6), forms an ECT-system on an open interval $U \subseteq \{y \in \mathbb{R} : (1 - yg(t))|_{t\in E} > 0\}$, if the following hypothesis holds:
(H) *For each $i \in \{0, \ldots, n\}$ the ordered functions $\{f_{i,0}, \ldots, f_{i,m_i}\}$ form a CT-system on E_i.*

Highlight of Theorem 5:

- A criterion on how the Chebyshev property of subfamilies of functions can be continued to their union.

5. Further discussion on the Chebyshev criterion



- We introduce two more families of integrals

$$
\begin{aligned}
C_k^E(y) &= \int_E \frac{\xi_E(\theta)\cos k\theta}{(1 - y\cos\nu\theta)^\alpha} d\theta, \\
S_k^E(y) &= \int_E \frac{\xi_E(\theta)\sin k\theta}{(1 - y\cos\nu\theta)^\alpha} d\theta,
\end{aligned}
\tag{7}
$$

where $\nu \in \mathbb{Z}^+$, $\alpha \in \mathbb{R}$ and ξ_E is an analytic non-vanishing function defined on E.

Some new ECT-systems from Theorem 5:

- $\bigcup_{i=0}^{n} \mathcal{F}_i$ on $(-1,1)$, where $\mathcal{F}_i$ is one of the ordered sets

$$\left\{S_1^E, S_2^E, \ldots, S_{m+1}^E\right\}, \quad \left\{C_0^E, C_1^E, \ldots, C_m^E\right\},$$
$$\left\{C_0^E, C_1^E \ldots, C_m^E, S_m^E, S_{m-1}^E, \ldots, S_1^E\right\},$$

  with $E = E_i$ and $m = m_i \in \mathbb{Z}_0$, and $E_0, E_1, \ldots, E_n$ being non-intersecting open intervals contained in $(0,\pi)$.

- The set of ordered functions

$$\{1, y, \ldots, y^{m_0}\} \bigcup \left(\bigcup_{i=1}^{n} \{(y+a_i)^\beta, y(y+a_i)^\beta, \ldots, y^{m_i}(y+a_i)^\beta\} \right)$$

  on $(-a_n, +\infty)$, where $a_1 > a_2 > \ldots > a_n \in \mathbb{R}$, $m_1 \geq m_2 \geq \ldots \geq m_n \in \mathbb{Z}_0^+$, $m_0 \in \mathbb{Z}_0^+$ and $\beta \in (\mathbb{R} \backslash \mathbb{Z}_0^+) \cap (-\infty, m_0 - m_1 + 1)$.

Example 1: Piecewise smooth planar differential systems separated by rays

$$\Sigma := \bigcup_{i=0}^{n} \left\{ (x, y) = (r\cos\vartheta_i, r\sin\vartheta_i) : r \in \mathbb{R}^+ \right\},$$
$$n \geq 1,\ \vartheta_0 < \vartheta_1 < \cdots < \vartheta_n \in [-\pi, \pi).$$

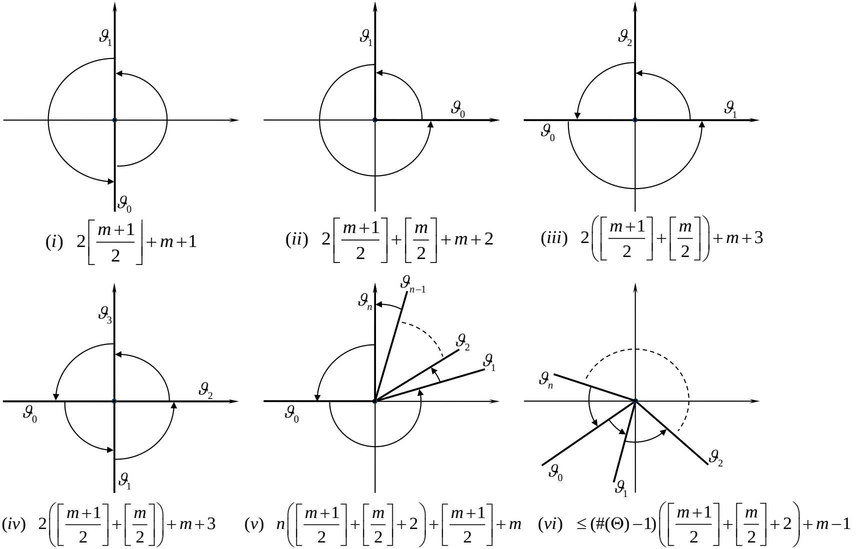
- A series of previous works (Buzzi, Cardin, Li, Liu, Llibre, Novaes, Torregrosa, Zhang, et al.).

Consider

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y(1-ax) + \varepsilon P_m(x,y) \\ x(1-ax) + \varepsilon Q_m(x,y) \end{pmatrix}, \tag{8}$$

where $a \neq 0$ and P_m, Q_m are piecewise polynomials of degree m separated by Σ.

5. Further discussion on the Chebyshev criterion



- Derivative of the modified first order Melnikov function:

$$y^{m+1}M_1^{(m)}(y)=\sum_{s=0}^{n}\left(\sum_{p=0}^{\left[\frac{m+1}{2}\right]}\eta_{s,p}C_{m+1-2p}^{E_s}(y^{-1})+\sum_{p=0}^{\left[\frac{m}{2}\right]}\lambda_{s,p}S_{m+1-2p}^{E_s}(y^{-1})\right),$$

where $E_s=(\vartheta_s,\vartheta_{s+1})$, $\xi_{E_s}=1$ and $\vartheta_{n+1}=2\pi+\vartheta_0$.

- Set $Z(M_1)$: Maximum number of isolated positive zeros of M_1.

Then,

- An efficient and unified way to estimate $Z(M_1)$ for arbitrary separation rays Σ is realized.
- Some mechanism that how $Z(M_1)$ is affected by the symmetry (distribution) of $\vartheta_0,\cdots,\vartheta_n$ is exhibited. In fact,

$$Z(M_1)\le(\#(\Theta)-1)\left(\left[\tfrac{m+1}{2}\right]+\left[\tfrac{m}{2}\right]+2\right)+m-1,$$
$$\Theta=\{0,\tfrac{\pi}{2},\pi,|\vartheta_0|,\cdots,|\vartheta_n|\}.$$

Example 2: Smooth planar differential systems with homogeneous nonlinearities of degree m.

- Set $H(m)$: Maximum number of limit cycles surrounding the origin that such systems can have.
- A problem posed in [Gasull, Yu & Zhang, *JDE*, 2015]:

	Node	Saddle	Strong focus	Weak focus	Nilpotent singularity
m is odd	$\geq \left[\frac{m}{2}\right]+1$	$\geq \left[\frac{m}{2}\right]+1$	$\geq \left[\frac{m}{2}\right]+1$	$\geq \left[\frac{m}{2}\right]$	$\geq \left[\frac{m}{2}\right]$
m is even	$=0$	$=0$	?	?	$=0$

Table: Estimates of $H(m)$.

Problem: "*Whether the result for the remaining cases is similar to the ones when n is odd.*"

A positive answer to the problem:

- Consider the system (with a weak focus when $\varepsilon \neq 0$):

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y + \dfrac{1}{2m-1}x^2(x^2+y^2)^{m-1} + \varepsilon P_{2m}^H(x,y) \\ x + \dfrac{1}{2m-1}xy(x^2+y^2)^{m-1} + \varepsilon Q_{2m}^H(x,y) \end{pmatrix}. \quad (9)$$

- First order Melnikov function of (9):

$$M_1(y) = 2y^{2m-1}\sum_{k=0}^{m}\lambda_k C_{2k}^E(y^{2m-1}), \quad \text{with } E=(0,\pi) \text{ and } \xi_E = \sin^2\theta.$$

Then,

  - Weak focus: $H(2m) \geq m$ (i.e. $H(m) \geq [\frac{m}{2}]$ when m is even).
  - Strong focus: $H(2m) \geq m+1$ (i.e. $H(m) \geq [\frac{m}{2}]+1$ when m is even) taking Hopf bifurcation into account.

Example 3: Smooth perturbation for a system having a center and a family of vertical and horizontal lines of singularities [Gasull, Lázaro & Torregrosa, *NA*, 2012]:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y\prod_{i=1}^{n_1}(x-a_i)\prod_{j=1}^{n_2}(y-b_j)+\varepsilon P_m(x,y) \\ x\prod_{i=1}^{n_1}(x-a_i)\prod_{j=1}^{n_2}(y-b_j)+\varepsilon Q_m(x,y) \end{pmatrix}, \qquad (10)$$

- Modified first order Melnikov function:

$$M_1(y) = \sum_{a\in A}\left(P_{a,[\frac{m}{2}]+\#(D)}(y)(y+a)^{-\frac{1}{2}} + R_{[\frac{m-1}{2}]+\#(D)}(y)\right),$$

where $P_{a,k}$, R_k are polynomials of degree k, and D, A are number sets related to a_i, b_i.

- M_1 is analyzed in by Derivation-Division algorithm [Gasull et. al., *NA*, 2012].
- Instead, note that

$$M_1 \in \left\{1, y, \cdots, y^{\left[\frac{m-1}{2}\right]+\#(D)}\right\} \bigcup \left(\bigcup_{a \in A} \left\{(y+a)^{-\frac{1}{2}}, y(y+a)^{-\frac{1}{2}}, \cdots, y^{\left[\frac{m}{2}\right]+\#(D)}(y+a)^{-\frac{1}{2}}\right\}\right)$$

which can be analyzed using the last new family.

Our published papers

- J. Huang, j. Li, *On the number of limit cycles in piecewise smooth generalized Abel equations with two asymmetric zones*, **Nonlinear Analysis: Real World Applications**, 66(2022), 103551.
- J. Huang, H. Liang, X. Zhang, *A new Chebyshev criterion and its application to planar differential systems*, **Journal of Differential Equations**, 344(2023), 658-695.

Thank you!