On local Gevrey integrability of differential systems

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Outline of the talk

- Background on local integrability
- New results on local Gevrey integrability
- Sketch proofs to the new results

Background on local integrability

Local integrability for analytic vector fields

- is on existence, number and regularity of functionally independent local first integrals.

As we know: at a regular point

- an autonomous C^r vector field is C^r completely integrable.

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- an autonomous C^r vector field is C^r completely integrable.

For a singularity,

- the situation is completely different.
- the problem becomes much difficulty

The study on this problem has a long history, which

- can be traced back to Poincaré in 1891.

In this direction,

- there have appeared lots of published papers and books.

Here lists some books related to our next study, see e.g.

- Bibikov [Lecture Notes Math. 702, 1979]

  Local theory of nonlinear analytic ODE

- Weigu LI [Science Press (in Chinese), 2000]

  Normal form theory and its applications

- Romanovski and Shafer [Birkhäuser 2009]

  The center and cyclicity problems: a CAA

- **Z.** [Springer, 2017]

  Integrability of Dynamical Systems: Algebra and Analysis

In this direction, the most classical one:

- Center–focus problem of planar analytic vector fields.

This problem is still open even for cubic systems.

Equivalent characterization for planar analytic VF,

- Existence of linear center

  $\Updownarrow$

  Existence of local analytic first integral.

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  Analytically orbital linearization at the singularity

- Degenerate center could have no analytic first integrals.

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For the local analytic differential system

$$\dot{x} = Ax + f(x), \qquad x \in (\mathbb{R}^n, 0) \tag{1}$$

with

- A an $n \times n$ real matrix,
- $f(x) = O(\|x\|^2) \in C^\omega(\mathbb{R}^n, 0)$ an analytic function

Denote by $\mathscr{X}$

- the vector field associated to system (1)

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Concrete descriptions on the progress.

Set

$$\mathscr{M}_\lambda := \left\{ m \in \mathbb{Z}_+^n \,|\, \langle m, \lambda \rangle = 0,\ |m| \geq 2 \right\},$$

where

- $\mathbb{Z}_+$ is the set of nonnegative integers,
- $|m| = m_1 + \ldots + m_n$ for $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$.

Definition:

- If $\mathscr{M}_\lambda = \emptyset$, we call λ $\mathbb{Q}_+$*-non-resonant*.
- If $\mathscr{M}_\lambda \neq \emptyset$, we call λ $\mathbb{Q}_+$*-resonant*.

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Poincaré in 1891 proved the next result in nonresoant case.

Theorem (Poincaré Theorem)

If system (1) is analytic, and

- the eigenvalues λ of A are non–resonant,

then

- the system has neither analytic nor formal first integrals.

In non–resonant case, there are some related results, see e.g.

- Furta [ZAMP, 1996]
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Remark:

- Non-resonance cannot prohibit existence of C^∞ first integrals

Proposition 1 [Wu, Xu, Z. preprint, 2023]

The following statements hold.

(i) If H is a C^∞ local first integral of a C^∞ vector field F, then it generates a C^∞ ∞-flat local first integral $\hat{H}$ for F.

(ii) There exists $F \in C^\omega(U)$, which has no a formal first integral but a C^∞ ∞-flat one.

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In order for existence of local analytic or formal first integral,

- the eigenvalues λ must be resonant

In two dimension, the nondegenerate case is

$$\lambda = (\sqrt{-1}, -\sqrt{-1}), \text{ or } \lambda = (q, -p), \ q, p \in \mathbb{N}.$$

- The analytic integrability was completely characterized only for quadratic differential systems in the cases of center and weak saddle.

The degenerate case

- One eigenvalue is equal to zero
- Two eigenvalues both vanish: nilpotent case, $A = 0$ case

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For higher dimensional system (1) with λ resonant, there appeared some necessary conditions:

- Chen, Yi and **Z.** [JDE 2008] provided

  ♠ an optimal upper bound on the numbers of functionally independent analytic first integrals.

- Shi [JMAA 2007] proved nonexistence of

  ♠ meromorphic first integrals in $\mathbb{Q}$–nonresoant.

- Cong, Llibre and **Z.** [JDE 2011] provided

  ♠ an optimal upper bound on the numbers of functionally independent meromorphic first integrals in $\mathbb{Q}$-resonant.

On equivalent characterization of integrability via normalization:

- Zung [Math. Res. Lett. 2002] provided a relation between analytic integrability and convergence of normalization to Poincaré-Dulac normal form.
- Z. [JDE 2013] established necessary and sufficient conditions on existence of analytic normalization and local analytic integrability
- Du, Romanovski and Z. [JDE 2016] proved the existence of analytic normalization of partly analytic integrable systems at a singularity with some additional conditions.
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Necessary and sufficient conditions:

Li, Llibre and Z. [ZAMP 2003] under the condition:

$$\lambda_1 = 0 \text{ and } \sum_{j=2}^{n} m_j\lambda_j \neq 0 \text{ for } m_j \in \mathbb{Z}_+ \text{ and } \sum_{j=2}^{n} m_j \geq 1. \tag{2}$$

obtained the next result.

Theorem A (Li, Llibre and Z. ZAMP 2003)

Assume that system (1) is analytic and the conditions (2) hold.

(a) For $n = 2$, system (1) has an analytic first integral in $(\mathbb{R}^n, 0)$ $\Longleftrightarrow$ the singular point $x = 0$ is not isolated.

(b) For $n > 2$, system (1) has a formal first integral in $(\mathbb{R}^n, 0)$ $\Longleftrightarrow$ the singular point $x = 0$ is not isolated.

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Remark:

- For $n \geq 3$, the next problem remains open since 2003:

Is it true that

- *the analytic differential system* (1) *under Theorem A(b) has an analytic first integral in* $(\mathbb{R}^n, 0)$*?*

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Theorem B [Z. JDE 2017]

For the analytic system (1), satisfying the condition (2).

(a) If the real parts of $\lambda_2, \ldots, \lambda_n$ all have the same sign, then system (1) has an analytic first integral in $(\mathbb{R}^n, 0)$

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the singular point $x = 0$ is not isolated.

(b) If $\lambda_2, \ldots, \lambda_n$ have both positive and negative real parts, there exist analytic differential systems of form (1) which have no analytic first integrals in $(\mathbb{R}^n, 0)$.

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According to Theorems A and B, under

Conditions (2) + singularity nonisolated

Problems to be solved:

1. Does there always exist a C^∞ first integral?
2. Provide a measure on the set of analytic systems which have an analytic first integral.
3. Characterize the class of analytic differential systems which have an analytic first integral.

Answer to Problem 1:

Theorem C (Z. JDE 2021)

Under the conditions (2),

- the analytic system (1) has a C^∞ **first integral** in $(\mathbb{R}^n, 0)$

$$\Updownarrow$$

the singularity at the origin is non-isolated, and

the formal first integral is nontrivial.

Answer related to Problem 2.

Let $\mathfrak{K}$ be the set of analytic differential systems of type (1)

- with the same linear part
- satisfying the conditions (2).
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Theorem D [Z. JDE 2021]

Let: $\mathscr{K}$ be any finite dimensional subspace of $\mathfrak{K}$.

The following statements hold.

(a) If $\mathscr{K}$ contains an element, which has only formal but not analytic first integral near the origin, then all elements in $\mathscr{K}$ except a **pluripolar subset** also have this property.

(b) If $\mathscr{K}$ has a **nonpluripolar subset** whose any element has an analytic first integral near the origin, then all systems in $\mathscr{K}$ have this property.

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(b) If $\mathscr{K}$ has a **nonpluripolar subset** whose any element has an analytic first integral near the origin, then all systems in $\mathscr{K}$ have this property.

Recall that

- If systems (1) are polynomials of a bounded degree, then $\mathfrak{K}$ is finite dimensional.

Remark:

- A pluripolar set is a subset of $\mathbb{C}^m$ for some $m \in \mathbb{N}$
- A pluripolar set is of Lebesgure measure zero
- Countable union of pluripolar sets is also a pluripolar set

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Local Gevrey first integrals

As we know, between analytic C^ω and formal $\mathbb{F}^n[[x]]$ classes

- there exist Gevrey class $\mathscr{G}_s$ $(s \geq 1)$ and C^∞
- $\mathscr{G}_1 \subseteq \mathscr{G}_s$ $(s \geq 1) \subseteq C^\infty$, with $\mathscr{G}_1$ analytic class

  and $C^\infty = \mathbb{F}^n[[x]]/\sim$ with $\sim$ the set of C^∞ ∞-flat ones

Question: In the previous setting on the eigenvalues

- what about Gevrey first integrals?

Recall that a Gevrey first integral

- is a first integral, which is a Gevrey function

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Definition: For $s \geq 1$,

a Gevrey-s function defined on an open set $\Omega \subset \mathbb{R}^n$

- is a smooth complex-valued function, satisfying that for any compact set $K \subset \Omega$, $\exists\, M, C > 0$ such that for all $k \in \mathbb{Z}_+^n$

$$\sup_{x \in K} \left| D^k f(x) \right| \leq M C^{|k|} (|k|!)^s$$

Denoted by $\mathscr{G}_s(\Omega)$

- the class of Gevrey-s functions defined on Ω.

According to the conditions on the eigenvalues

one is zero and others are nonresonant

for simplicity, we write the system in

$$\frac{dx}{dt} = Ax + f_1(x,y), \quad \frac{dy}{dt} = f_2(x,y) \tag{3}$$

with

- $x \in \mathbb{R}^d$, $y \in \mathbb{R}$, and
- $f = (f_1, f_2) = O(|x|^2 + |y|^2)$ analytic as $(x,y) \to 0$.
- A has eigenvalues λ, which are nonresonant

$$k \cdot \lambda \neq 0, \quad k \in \mathbb{Z}_+^d \tag{4}$$

By non-isolate of the singularity,

♠ system (3) can be turned to

$$\frac{dx}{dt} = Ax + \hat{f}_1(x,y), \quad \frac{dy}{dt} = \hat{f}_2(x,y), \tag{5}$$

with

$$\hat{f}_1(0,y) \equiv 0 \quad \text{and} \quad \hat{f}_2(0,y) \equiv 0.$$

The corresponding formal normal form is

$$\frac{dx}{dt} = Ax + g(x,y), \quad \frac{dy}{dt} = 0, \tag{6}$$

where $g(x,y) = \sum_{k,j,l\in\Lambda_r} g_{(k,j),l} x^k y^j e_l$ with e_l the l-th unit vector.

Denote the resonant set by

$$\Lambda_r = \left\{(k,j,l) \;\middle|\; k\cdot\lambda = \lambda_l,\ \ |k|+j \geq 2,\quad k \in \mathbb{Z}_+^d,\ \ j \in \mathbb{Z}_+,\ \ l \in \{1,\ldots,d\}\right\}$$

Define the numbers

$$q = \min\{|k| \;\big|\; (k,j,l) \in \Lambda_r,\ \ g_{(k,j),l} \neq 0,\ \ \exists j,l\}, \tag{7}$$

and

$$q^* = \min\{|k|+j \;\big|\; (k,j,l) \in \Lambda_r,\ \ g_{(k,j),l} \neq 0,\ \ \exists l\}. \tag{8}$$

Remark:

- These two quantities are invariant by near-identity local coordinate substitutions

Formulating the function

$$c^{-1}\Phi(t) = \max\{|k\cdot\lambda|^{-1} \mid |k| \le t,\ k \in \mathbb{Z}_+^d\} \tag{9}$$

with

- Φ an increasing positive function
- c normalizes Φ such that $\Phi(1) = 1$.

Remark:

- When $\Phi(t) = t^\mu$, it is of the diophantine type.
- $\Phi(t)$ satisfying

$$\int_1^\infty \frac{\ln\Phi(t)}{t^2}dt < \infty,$$

the small divisor condition, is of the Bruno-Rüssmann type.

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Theorem 1 [Wu, Xu, Z. preprint, 2023]

Assume that

- system (3) is Gevrey-s smooth, with $s \geq 1$
- λ is non-resonant, i.e. the condition (4)
- the singularity at the origin is non-isolated

The following statements hold.

(a) If the real parts of λ have the same sign, then system (3) has local Gevrey-s smooth first integrals with non-zero formal parts.

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Theorem 1 (Continued)

(b) Assume that

$\diamondsuit$ A is in the diagonal form, and

$\diamondsuit$ the divisor $\Phi(t) = t^\mu$ for some constant $\mu > 0$.

One has the next results.

(b_1) If $\partial_x \hat{f}_1(0, y) \equiv 0$ in (5) and $q < \infty$ given by (7), there exist

$\spadesuit$ local Gevrey-s^* smooth first integrals with non-zero formal parts, where $s^* = \max\left\{s, \dfrac{\mu + q}{q - 1}\right\}$.

(b_2) If $q^* < \infty$ given by (8), there exist

$\spadesuit$ formal Gevrey-s^* first integrals with non-zero formal parts, where $s^* = \max\left\{s - 1, \dfrac{\mu + 1}{q^* - 1}\right\}$.

Remark:

- Theorem 1(a) is inherited from the analytic integrability property, which admits no loss of regularity.
- Theorem 1(b) shows that
  ♠ difference of linear parts affect loss of Gevrey regularity.
- (b_1) implies that the divisor condition leads to no shrinking of the region for the variable x.
- (b_2) indicates that for the higher-order perturbation, we have to shrink the whole region.
- At this moment, we cannot explain what exactly happens between (b_1) and (b_2).

Preparation to prove Theorem 1

For the Taylor expansion of f at $P = (0, a)$

$$f(X) = \sum f_{k,l}(X-P)^k e_l,$$

- the weighted majorant operator is defined as

$$\mathscr{M}_P f(X) = \sum |f_{k,l}| E(|k|)(X-P)^k e_l$$

with the weight function $E(t) = e^{\omega(t)}$, where $\omega(t) = -\tau t \ln t$ satisfying

♠ $\omega(0) = \omega(1) = 0$, $\omega'(t) \le 0$, and $\omega''(t) \le 0$ for $t \ge 1$

♠ $\tau \ge 0$ describes the indices of Gevrey smoothness.

Notice that

$$\mathscr{M}_P f = (\mathscr{M}_P f_1, \cdots, \mathscr{M}_P f_d)$$

Following Pöschel [JDDE 2021],

♠ introduce the norm for f

$$|f|_{U,\tau,\rho} = \sup_{P\in U} \sum_{l} \mathscr{M}_P f_l(\rho,\cdots,\rho) < \infty, \tag{10}$$

with U the domain, $\rho > 0$ a number, and .

$$\mathscr{M}_P f(\rho,\ldots,\rho) = \sum |f_{k,l}| E(|k|)\rho^{|k|} e_l,$$

Remark:

- For the case $\tau = 0$, ρ is related to analytical radius
- For the case $\tau > 0$, it makes no real geometry meaning.

  ♠ So U and ρ can be independent.
- $|\cdot|_{P,\tau,\rho}$ is just the formal Gevrey-τ norm, provided that

  ♠ U degenerates to a point P.

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  ♠ U degenerates to a point P.

Here, the partial formal Gevrey norms is needed. Set

- $X = (x, y)$ with $x \in \mathbb{C}^d$ and $y \in \mathbb{C}$
- $U_r = \{z \mid |z| \leq r\} \subset \mathbb{C}$ for $r > 0$
- $\hat{U}_\rho = \{0\} \times U_\rho \subseteq \mathbb{C}^d \times \mathbb{C}$ for $\rho > 0$.

The norm utilized here is of the mixing type

$$\|f\|_{\tau,\rho} = \sup_{(x,y)\in\hat{U}_\rho} \sum_l \mathscr{M}_{(x,y)} f_l(\rho, \cdots, \rho) < \infty. \tag{11}$$

Denoted by

$$\mathscr{X}_\rho = \left\{ f(x,y) = \sum_{|j|\geq 1, l} f_{j,l}(y) x^j e_l \,\middle|\, f_{j,l}(y) \in \mathscr{G}_{\tau+1}(U_\rho),\ \|f\|_{\tau,\rho} < \infty \right\},$$

which is the set of functions admitting

formal Gevrey-τ in $x \in \mathbb{C}^d$ and Gevrey-$(\tau + 1)$ in $y \in \mathbb{C}$

Note that this definition is equivalent to the classical one

$$[]f[]_{\tau,\rho} := \sum_{|j|\geq 1,l} |f_{j,l}|_{U_\rho,\tau,\rho} \frac{\rho^{|j|}}{(|j|!)^\tau} = \sum_{i,|j|\geq 1,l} \sup_{y\in U_\rho} |\partial_y^i f_{j,l}(y)| \frac{\rho^{i+|j|}}{(i!)^{\tau+1}(|j|!)^\tau}, \tag{12}$$

with

$f(x,y) = \sum_{j,l} f_{j,l}(y) x^j e_l$, and $j! = j_1!\cdots j_d!$ for $j = (j_1,\ldots,j_d)$,

where

$$f_{j,l}(y) = \frac{1}{j!}\partial_x^j f(x,y) e_l|_{x=0} \in \mathscr{G}_{\tau+1}(U_\rho).$$

Here

$$|f_{j,l}|_{U_\rho,\tau,\rho} = \sum_i \sup_{y\in U_\rho} |\partial_y^i f_{j,l}(y)| \frac{\rho^i}{(i!)^{\tau+1}},$$

is the classical Gevrey-$(\tau+1)$ norm.

$\mathscr{X}_\rho$ has the next property.

Lemma 1

The space $\{\mathscr{X}_\rho, \|\cdot\|_{\tau,\rho}\}$ is complete.

Proof: For any $f \in \mathscr{X}_\rho$, we build

$$\hat{f}(x,y) = \sum_{|j|\geq 1,l} \hat{f}_{j,l}(y)x^j e_l, \quad \hat{f}_{j,l} = |f_{j,l}|_{U_\rho,\tau,\rho} \frac{\rho^{|j|}}{(|j|!)^\tau},$$

which yields a complete Banach space l^1, with the norm

$$\|\hat{f}\| = \sum_{j,l} |\hat{f}_{j,l}|.$$

So, the space $\{\mathscr{X}_\rho, \|\cdot\|_{\tau,\rho}\}$ is a weighted l^1. □

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The next is a key point for general ultra-differential norms.

Lemma 2

Assume that $\omega(u)$ is a C^2 function satisfying

$$\omega(1) = 0 \text{ and } \omega''(u) \leq 0 \text{ for } u \geq 1.$$

Let $E(u) = e^{\omega(u)}$ and $v_i \geq 1$ for all i. Then we have

$$E(v_1 + v_2) \leq E(v_1)E(v_2)$$

$$E\left(\sum_{i=1}^{t} v_i\right) \leq E(t) \prod_{i=1}^{t} E(v_i).$$

When ω' is non-positive decreasing and $|\omega''| \leq M$

$$E(u + v - \gamma) \leq c\kappa(u + v - \gamma)E(u)E(v),$$

for $u \geq \beta \geq 1$, $v \geq \beta \geq 1$, and $0 \leq \gamma < \beta$, where

$$\kappa(u) = e^{\omega'(u-\gamma)(\beta-\gamma)},\ c = c(\beta,\gamma) = \exp(\omega(\beta) + M(\beta - \gamma)^2/2).$$

By this lemma, one can prove the next properties on norms.

Lemma 3

For f, $g \in \mathscr{X}_\rho$, the following statements hold.

(i) $\|f \cdot g\|_{\tau,\rho} \leq \|f\|_{\tau,\rho}\|g\|_{\tau,\rho}$, where $\cdot$ denotes the inner product.

(ii) $\|f \circ (\mathrm{id} + g)\|_{\tau,\rho} \leq \|f\|_{\tau,\kappa}$, provided $(d+2)\rho + \|g\|_{\tau,\rho} \leq \kappa < \infty$, where $\circ$ represents composition.

Finally, handling the Cauchy type estimate.

For any $f \in \mathscr{X}_\rho$, we define the power shift operator $\mathscr{P}_\mu$:

$$\mathscr{P}_\mu f = \sum_{j,l} |j|^\mu |f_{j,l}(y)| x^j e_l. \tag{13}$$

for the expansion $f(x,y) = \sum f_{j,l}(y) x^j e_l$.

Lemma 4

Assume that

- $f, g \in \mathscr{X}_\rho$ are scale functions,
- $0 < \delta < 1$, and c is that in Lemma 2.

The following statements hold.

(i) If $\|f\|_{\tau,\rho}, \|g\|_{\tau,\rho e^{-\delta}} < \infty$, then

$$\|\partial_y f \cdot g\|_{\tau,\rho e^{-\delta}} \le c\delta^{-(\tau+1)}\rho^{-1}\|f\|_{\tau,\rho}\|g\|_{\tau,\rho e^{-\delta}}.$$

(ii) If $f(0,y) = g(0,y) = 0$, $\partial_x f(0,y) = \partial_x g(0,y) = 0$, and $\|f\|_{\tau,\rho}, \|g\|_{\tau,\rho e^{-\delta}} < \infty$, then

$$\|\partial_{x_i} f \cdot g\|_{\tau,\rho e^{-\delta}} \le c\delta^{-1}\rho^{-1}\|f\|_{\tau,\rho}\|g\|_{\tau,\rho e^{-\delta}}.$$

Lemma 4 (Continued)

(iii) If $f(0,y) = g(0,y) = 0$, $\partial_x^s f(0,y) = \partial_x^s g(0,y) = 0$ for

$$s = 1, \ldots, q-1 \text{ and } 2 \leq q \in \mathbb{Z}_+,$$

$\|f\|_{\tau,\rho}$, $\|g\|_{\tau,\rho} < \infty$, and $\tau \geq \frac{\mu+1}{q-1}$, then

$$\|\mathscr{P}_\mu(\partial_{x_i} f \cdot g)\|_{\tau,\rho} \leq c\rho^{-1}\|f\|_{\tau,\rho}\|g\|_{\tau,\rho}.$$

The proof follows by using Lemmas 2 and 3, together with some technique estimate

To apply the homological equation, we need the next property

Proposition 2

Under the condition (4), the resonant set

$$\Lambda_r = \{(j,l) \mid j\cdot\lambda = \lambda_l,\ j \in \mathbb{Z}_+^d,\ |j| \geq 2,\ l = 1,\cdots,d\},$$

has finitely many elements, i.e. $\sharp\Lambda_r < \infty$.

The proof follows by contrary and the drawer principle

Recall that

- system (3) can be written in

$$\frac{dx}{dt} = Ax + \hat{f}_1(x,y), \quad \frac{dy}{dt} = \hat{f}_2(x,y), \tag{14}$$

where $\hat{f}_1(0,y) \equiv 0$ and $\hat{f}_2(0,y) \equiv 0$.

- an admissible transformation is of the form

$$(x,y) \mapsto (x + h_1(x,y), y + h_2(x,y))$$

with

$$\mathscr{A}: \quad h_1(0,y) \equiv 0, \quad h_2(0,y) \equiv 0, \tag{15}$$

which persists $x = 0$ as the center manifold.

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which persists $x = 0$ as the center manifold.

For $F = Ax + f(x,y)$, and f and $g \in \mathscr{X}_\rho$,

♠ consider the homological equation in h

$$Ad_F(h) = g, \tag{16}$$

where

$$Ad_F(h) := \partial_x h F$$

Specially, when

- $Ad_A h := \partial_x h Ax$
- $A = \mathrm{diag}(\lambda)$ is in the diagonal form

then

$$h = Ad_A^{-1} g = \sum_{|j| \geq 1, l} \frac{g_{j,l}(y)}{j \cdot \lambda} x^j e_l. \tag{17}$$

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Lemma 5

Assume that Reλ_j's of λ have the same sign, and

$$\hat{c}_2 \|f\|_{\tau,\rho} \rho^{-1} \leq 1, \tag{18}$$

then the equation

$$Ad_F(h) = \mathscr{P}_\mu g \tag{19}$$

has the unique solution $h = Ad_F^{-1} \circ \mathscr{P}_\mu g$ for any $g \in \mathscr{X}_\rho$, satisfying

$$\|h\|_{\tau,\rho} \leq \hat{c}_1 \|g\|_{\tau,\rho} \text{ uniformly for } 0 \leq \mu \leq 1,$$

where $\hat{c}_1 = \hat{c}_2 = 4\kappa^{-1}$, with $\kappa = \min_i\{|\mathrm{Re}\lambda_i|\} > 0$, and $\mathscr{P}_\mu$ is the power shifted operator, given in (13)

Idea of the proof: Set

- $A = D + \varepsilon N$ with D diagonal, N nilpotent, $\varepsilon > 0$ small
- $f = B(y)x + \hat{f}$, with $B(y) = \partial_x f(0,y) \in \mathscr{G}_{\tau+1}(U_\rho)$
- $\hat{B} = \varepsilon N + B(y)$

Then

$$Ad_F = Ad_D + Ad_{\hat{B}} + Ad_{\hat{f}}$$

So equation (19) can be written in

$$(Ad_D + Ad_{\hat{B}})(h) = Ad_D(I + Ad_D^{-1} \circ Ad_{\hat{B}})h = \mathscr{P}_\mu g - Ad_{\hat{f}}(h).$$

In case of invertibility of $I + Ad_D^{-1} \circ Ad_{\hat{B}}$, one further has

$$h = (I + Ad_D^{-1} \circ Ad_{\hat{B}})^{-1}(Ad_D^{-1} \circ \mathscr{P}_\mu g - Ad_D^{-1} \circ Ad_{\hat{f}}(h)).$$

Next is to prove

- the operator $I + Ad_D^{-1} \circ Ad_{\hat{B}}$ is invertible
- estimate the norm of the inverse operator.

Taking the classical operator norm $|\cdot|_o$ on $\mathscr{X}_\rho$, i.e.

$$|f|_o = \sup_{\|g\|_{\tau,\rho}=1} \|f\cdot g\|_{\tau,\rho}.$$

Then

$$|Ad_D^{-1}|_o \le \kappa^{-1},$$

and for properly small $\varepsilon > 0$ and $\rho > 0$

$$|Ad_D^{-1}\circ Ad_{\hat{B}}|_o \le 1/2$$

Hence $I + Ad_D^{-1}\circ Ad_{\hat{B}}$ is invertible and

$$|(I + Ad_D^{-1}\circ Ad_{\hat{B}})^{-1}|_o \le 2$$

$$\|Ad_D^{-1}\circ Ad_{\hat{f}}(h)\|_{\tau,\rho} \le \|h\|_{\tau,\rho}/4$$

for $4\|f\|_{\tau,\rho}\rho^{-1} \le \kappa$. Hence

$$\|h\|_{\tau,\rho} \le 2\kappa^{-1}\|g\|_{\tau,\rho} + \frac{1}{2}\|h\|_{\tau,\rho} \implies \|h\|_{\tau,\rho} \le 4\kappa^{-1}\|g\|_{\tau,\rho}.$$

This completes the proof by setting $\hat{c}_1 = \hat{c}_2 = 4\kappa^{-1}$.

Preparation to proof of Theorem 1(b)

Lemma 6

Assume that $A = D$ is diagonal, $0 < \delta < 1$,

- the divisor $\Phi(t) = t^{\mu}$, $\mu > 0$,
- q is in (7) and q^* is in (8).

If the norm $\|\cdot\|_{\tau,\rho}$ is associated with

(b_1) $(x,y) \in \{0\} \times U_\rho$, $2 \le q < \infty$, and $\tau \ge (\mu+q)/(q-1)$, or

(b_2) $(x,y) \in \{0\} \times \{0\}$, $q^* < \infty$, and $\tau \ge (\mu+1)/(q^*-1)$,

Eq (16) has the unique solution h satisfying

$$\|h\|_{\tau,\rho e^{-\delta}} \le \hat{c}_1 \delta^{-\mu} \|g\|_{\tau,\rho} \quad \text{for} \ \ \hat{c}_2 \|f\|_{\tau,\rho} \rho^{-1} \le 1,$$

where $\hat{c}_1 > 0$, $\hat{c}_2 = 2ec^{-1}c_2$, c is from the small divisor condition (9), and c_2 is the c in Lemma 4.

Idea of the proof: Eq (16) is turned into

$$h = Ad_D^{-1} g - Ad_D^{-1} \circ Ad_f h.$$

For (b_1), by Lemma 4(iii), we get that

$$\|Ad_D^{-1} \circ Ad_f(h)\|_{\tau,\rho} \le c_1 \rho^{-1} \|f\|_{\tau,\rho} \|h\|_{\tau,\rho}, \tag{20}$$

for $\tau \ge (\mu + q)/(q-1)$, and

$$\|Ad_D^{-1} g\|_{\tau,\rho e^{-\delta}} \le c_3 \delta^{-\mu} \|g\|_{r,\rho}.$$

For (b_2), similar estimates hold. So in both of the cases,

$$\|h\|_{\tau,\rho e^{-\delta}} \le c_3 \delta^{-\mu} \|g\|_{\tau,\rho} + \frac{1}{2} \|h\|_{\tau,\rho e^{-\delta}},$$

for $c_1 e \rho^{-1} \|f\|_{\tau,\rho} < 1/2$, which implies that

$$\|h\|_{\tau,\rho e^{-\delta}} \le \hat{c}_1 \delta^{-\mu} \|g\|_{\tau,\rho}, \quad \text{with } \hat{c}_1 = 2c^{-1} \mu^\mu e^{-\mu}.$$

Proof of the main theorem

Main tool is the KAM methods to do cancellations

The admissible coordinates substitution

$$x \mapsto x, \quad y \mapsto y + h(x,y), \tag{21}$$

sends system (3) to

$$\frac{dx}{dt} = Ax + f_1(x, y+h), \quad \frac{dy}{dt} = g(x,y), \tag{22}$$

where

$$g = -\partial_x h(Ax + f_1(x,y)) + f_2(x,y) + \mathscr{R},$$

and $\mathscr{R} = \mathscr{S}_1 + \mathscr{S}_2 + \mathscr{S}_3$ with

$$\mathscr{S}_1 = f_2(x, y+h) - f_2(x,y),$$

$$\mathscr{S}_2 = -\partial_x h(f_1(x,y+h) - f_1(x,y)),$$

$$\mathscr{S}_3 = ((1+\partial_y h)^{-1} - 1)(\mathscr{S}_1 + \mathscr{S}_2)$$

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$$\mathscr{S}_3 = ((1 + \partial_y h)^{-1} - 1)(\mathscr{S}_1 + \mathscr{S}_2)$$

By Lemmas 5 and 6, the equation

$$Ad_F(h) := \partial_x h(Ax + f_1(x,y)) = f_2(x,y) \tag{23}$$

has a solution h satisfying the desired norm estimate.

Taking h as the solution of (23), and writing system (22) in

$$\frac{dx}{dt} = Ax + f_1(x,y) + f_1^+(x,y), \quad \frac{dy}{dt} = f_2^+(x,y), \tag{24}$$

where $f_1^+(x,y) = f_1(x,y+h) - f_1(x,y)$ and $f_2^+(x,y) = \mathscr{R}$.

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Set $f = B(y)x + \hat{f}$, with

$$B(y) = \partial_x f(0,y) \in \mathcal{G}_{\tau+1}(U_\rho),$$

$\hat{f}$ the higher order terms in x.

Lemma 7

Assume that

- there exists $\rho_0 > 0$ such that $\|f\|_{\tau,\rho_0} < \infty$.

Then

$$\|B(y)x\|_{\tau,\rho} \le \widetilde{c}_1\rho^2 \quad \text{and} \quad \|\hat{f}\|_{\tau,\rho} \le \widetilde{c}_1\rho^2 \text{ for } \rho \le \rho_0/2$$

with $\widetilde{c}_1 > 0$ to be determined.

Now comes the iterative lemma.

Lemma 8

By the conditions of Theorem 1, if $0 < \delta < 1$, $0 < \rho < 1$,

$$\widetilde{c}_2 \|f_1\|_{\tau,\rho}\rho^{-1} \leq 1, \qquad \|f_2\|_{\tau,\rho} \leq \widetilde{c}_3 \rho \delta^{\mu+\tau+1},$$

with $\widetilde{c}_2 = \hat{c}_2$, $\widetilde{c}_3 = 1/((1+2c)\hat{c}_1 e^2)$, then in system (24),

$$\|f_2^+\|_{r,\rho e^{-\delta}} \leq K\rho^{-1}\delta^{-(\tau+2\mu+2)}\|f_2\|^2_{\tau,\rho}, \tag{25}$$

$$\|f_1^+\|_{r,\rho e^{-\delta}} \leq K\delta^{-(\tau+\mu+1)}\|f_2\|_{\tau,\rho}, \tag{26}$$

Here

- $\mu = 0$, $\tau \geq 0$ is in Theorem 1(a);
- $\mu \geq 0$, $\tau \geq (\mu+q)/(q-1)$ is in Theorem 1(b_1);
- $\mu \geq 0$, $\tau \geq (\mu+1)/(q^*-1)$ is in Theorem 1(b_2).

Summarizing arguments above, we first prove the next one.

Theorem 2

By the conditions of Theorem 1, if

$$\|f\|_{\tau,\rho_0} < \infty \text{ in system (3) of form (14),}$$

then there exits $\hat{\rho} > 0$ such that

$$\text{the change of (23) satisfying } \|h\|_{\tau,\hat{\rho}} < \infty$$

turns system (3) into (22) satisfying $\|g\|_{\tau,\hat{\rho}} = 0$

Here the norm $\|\cdot\|_{\tau,\rho}$ is for

- $\mu = 0$, $\tau \geq 0$, and $(x,y) \in \{0\} \times U_\rho$ in Theorem 1(a);
- $\mu \geq 0$, $\tau \geq \dfrac{\mu+q}{q-1}$ and $(x,y) \in \{0\} \times U_\rho$ in Theorem 1(b_1);
- $\mu \geq 0$, $\tau \geq \dfrac{\mu+1}{q^*-1}$ and $(x,y) \in \{0\} \times \{0\}$ in Theorem 1(b_2).

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- $\mu \geq 0$, $\tau \geq \dfrac{\mu+1}{q^*-1}$ and $(x,y) \in \{0\} \times \{0\}$ in Theorem 1(b_2).

Proof of Theorem 2

Here using the KAM methods, assume:

$$\|f\|_{\tau,\rho_0} = \varepsilon_0 \rho \text{ with } \varepsilon_0 > 0 \text{ sufficiently small.}$$

Set

$$\delta_0 < \tfrac{1}{2}, \quad \rho_0 = \rho, \quad \delta_n = \delta_0 2^{-n}, \quad \text{and} \quad \rho_n = \rho_{n-1} e^{-\delta_{n-1}}.$$

By induction on the iteration, let

$$f^{(0)} = f = (f_1^{(0)}, f_2^{(0)}).$$

In the nth step, it begins at system (14) with

$$f^{(n-1)} \quad \text{in the norm } \|\cdot\|_{\tau,\rho_{n-1}},$$

Solving the homological equation (23) gives

$$h = \hat{h}_n \quad \text{in the norm } \|\cdot\|_{\tau,\rho_n}$$

which brings system (14) to system (24) with

$$f^{+} = f^{(n)} \quad \text{in the norm } \|\cdot\|_{\tau,\rho_n}.$$

Precisely, for the homological equation (23) in the different cases,

- its solution $\hat{h}_n$ exists by

  ♢ Lemma 5 as $\rho = \rho_{n-1}$ and

  ♢ Lemma 6 as $\rho = \rho_{n-1} e^{-\delta_{n-1}/2}$ and $\delta = \delta_{n-1}/2$

  with the common norm estimate

$$\|\hat{h}_n\|_{\tau,\rho_n} \le \hat{c}_1 \left(\frac{\delta_{n-1}}{2}\right)^{-\mu} \|f_2^{(n-1)}\|_{\tau,\rho_{n-1}}.$$

By the control (25) and (26) of Lemma 8, it follows

$$\|f_2^{(n)}\|_{\tau,\rho_n} \leq K\rho_{n-1}^{-1}\delta_{n-1}^{-\mu'}\|f_2^{(n-1)}\|_{\tau,\rho_{n-1}}^2$$

and

$$\|f_1^{(n)}\|_{\tau,\rho_n} \leq K\delta_{n-1}^{-\mu'}\|f_2^{(n-1)}\|_{\tau,\rho_{n-1}}$$

as $\rho = \rho_{n-1}$ and $\delta = \delta_{n-1}$, where $\mu' = \tau + 2\mu + 2$.

By induction gives

$$\begin{aligned} \|f_2^{(n)}\|_{\tau,\rho_n} &\leq (K{\delta_0}^{-\mu'}2^{\mu'}\varepsilon_0)^{2^n}\rho, \\ \|f_1^{(n)}\|_{\tau,\rho_n} &\leq (K{\delta_0}^{-\mu'}2^{\mu'}\varepsilon_0)^{2^{n-1}}\rho. \end{aligned} \tag{27}$$

♣ By (27), one can get the norm estimate on g

By the control (25) and (26) of Lemma 8, it follows

$$\|f_2^{(n)}\|_{\tau,\rho_n} \leq K\rho_{n-1}^{-1}\delta_{n-1}^{-\mu'}\|f_2^{(n-1)}\|_{\tau,\rho_{n-1}}^2$$

and

$$\|f_1^{(n)}\|_{\tau,\rho_n} \leq K\delta_{n-1}^{-\mu'}\|f_2^{(n-1)}\|_{\tau,\rho_{n-1}}$$

as $\rho = \rho_{n-1}$ and $\delta = \delta_{n-1}$, where $\mu' = \tau + 2\mu + 2$.

By induction gives

$$\begin{aligned} \|f_2^{(n)}\|_{\tau,\rho_n} &\leq (K\delta_0{}^{-\mu'}2^{\mu'}\varepsilon_0)^{2^n}\rho, \\ \|f_1^{(n)}\|_{\tau,\rho_n} &\leq (K\delta_0{}^{-\mu'}2^{\mu'}\varepsilon_0)^{2^{n-1}}\rho. \end{aligned} \tag{27}$$

♣ By (27), one can get the norm estimate on g

At last, set

$$h_n = \mathrm{Id} + \hat{h}_n, \qquad h^{(n)} = h_n \circ h_{n-1} \circ \cdots \circ h_1.$$

One has

$$h^{(n)} - h^{(n-1)} = \hat{h}_n \circ h^{(n-1)}.$$

And for all $n \in \mathbb{N}$, $h^{(n)}$'s are well defined, and

- have a uniform bound norm

$$\|h^{(t)}\|_{\tau,\hat{\rho}} \leq \frac{(t+1)\hat{\rho}_2}{N+1} < \gamma\rho \quad \text{for all} \quad t \leq N$$
$$\|h^{(n)}\|_{\tau,\hat{\rho}} \leq \frac{2\gamma\rho}{3} \quad \text{for all} \quad n > N$$

with $\gamma = e^{-2\delta_0}$, $N \in \mathbb{N}$ such that $\sum_{n>N} 2^{-n} \leq \gamma/3$

and $\hat{\rho} = \min\left\{\frac{\gamma\rho}{3(d+2)}, \frac{r}{(N+1)(d+2)}\right\}$, $r \in (0, \gamma\rho/3]$

Furthermore, since

- the sequence $\{h^{(n)}\}$ is fundamental, following

$$\|h^{(n)} - h^{(n-1)}\|_{\tau,\hat{\rho}} = \|\hat{h}_n \circ h^{(n-1)}\|_{\tau,\hat{\rho}} \le \frac{\rho}{2^n}, \quad n > N,$$

- the space $(\mathscr{X}, \|\cdot\|_{\tau,\hat{\rho}})$ is complete,

it follows that

- $\{h^{(n)}\}$ is convergent in $(\mathscr{X}, \|\cdot\|_{\tau,\hat{\rho}})$
- Its limit h satisfies the requirement of the theorem.

Theorem 2 is proved □

Proof of Theorem 1

Since $f = (f_1, f_2)$ is of Gevrey-s, it follows

- $\|f\|_{\tau,\rho_0} < \infty$ with $\tau = s - 1$ for the $\rho_0 > 0$.

♠ Theorem 1(b_2): By Theorem 2 yields that

- the formal norm of g in system (22) vanishes.

So it admits a formal Gevrey-τ first integral, with

$$\tau = s^* \geq (\mu + 1)/(q^* - 1).$$

This confirms Theorem 1(b_2).

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This confirms Theorem $1(b_2)$.

Now, take

- $\mu = 0$ and $\tau \geq 0$ in Theorem 1(a),
- $\mu \geq 0$ and $\tau \geq (\mu + q)/(q-1)$ in Theorem 1(b_1).

with the norm $\|\cdot\|_{\tau,\rho}$ about $(x,y) \in \{0\} \times U_\rho$.

By Theorem 2, one can find the change h in (21)

- turning the original system into (22) with $\|g\|_{\tau,\hat{\rho}} = 0$.

Using the Borel type lemma for the Gevrey functions,

- $\exists\ \widetilde{h}(x,y)$ of Gevrey-$(\tau+1)$ satisfying $\text{Jet}^{\infty}_{x=0}(\widetilde{h}-h)=0$

which replaces h in (21), and sends system (3) to

$$\frac{dx}{dt}=Ax+f_1(x,y+\widetilde{h}),\quad \frac{dy}{dt}=\widetilde{g}(x,y), \tag{28}$$

where $\text{Jet}^{\infty}_{x=0}\widetilde{g}=0$.

Finally, we prove that system (28) is

- Gevrey-$(\tau+1)$ conjugated to the one with $\widetilde{g}(x,y)=0$

via Theorem K below.

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Theorem K

For the system

$$\frac{dx}{dt} = Ax + f_1(x,y) + r_1(x,y), \quad \frac{dy}{dt} = By + f_2(x,y) + r_2(x,y), \qquad (29)$$

with A hyperbolic and B center, assume that

- f, $r = O(\|x\|^2 + \|y\|^2)$ as $(x,y) \to (0,0)$,
- $f_1(0,y) \equiv 0$ for all y (local center manifold is strengthened)
- $\mathrm{Jet}^{\infty}_{(0,y)} r = 0$ for all y.

If f and r are both of Gevrey-α, then

- a Gevrey-α coordinates substitution annihilates r.

Its proof can be done by Belitskii and Kopanskii [JDDE, 2002], Stolovitch [Ann. Inst. Fourier 2013], Z. [JDE 2021]

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By Theorem K

- the original system is Gevrey-$(\tau+1)$ conjugated to

$$\frac{dx}{dt} = Ax + \hat{f}_1(x,y), \quad \frac{dy}{dt} = 0.$$

So

- the original system has a Gevrey-s^* first integral

Recall that $\tau = s^*$. Theorem 1 is proved. □

谢　　谢!

Thanks for your attention!