Ejection/Collision Orbits in the RTBP: why, what and how

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Outline

- Binary collisions.
  - Celestial mechanics: the RTBP (planar and 3d cases). O. Rodríguez's PhD Thesis.
  - Key ingredients: regularization, collision manifold, ejection-collision orbits (n-EC orbits)
  - How do ejection orbits interact with "other"? Equilibrium points, periodic orbits, stable/unstable manifolds, invariant tori
  - Atomic chemistry: the hydrogen atom in a circularly polarized microwave field. The CP problem. Ionization. Joint work with E. Barrabés and O. Rodríguez.
- Multiple collisions.
  - Celestial mechanics: some N-body problems.
  - Joint work with M. Álvarez-Ramírez, E. Barrabés and M. Medina.

Why do we study ejection/collision orbits?

- They are a kind of solutions of a system of ODE
- ODE with singularities, they require regularization
- The ejection orbits may be regarded as some kind of skeleton of close by orbits
- The EC orbits are actually the termination of continuation of families of periodic orbits
- Ballistic transport (zero energy transfer) using the natural dynamics of the system.

Part I

The planar RTBP

The PRTBP

RTBP

The restricted three-body problem (RTBP) consists in the description of the motion of an infinitesimal body P under the attraction of two bodies (P_1 and P_2) called primaries.

Circular RTBP. Synodic coordinate system

The primaries P_1 and P_2 have masses $1-\mu$ and μ, $\mu \in (0,1)$, their positions are fixed at $(\mu, 0)$ and $(\mu-1, 0)$, respectively, and the period of their motion is 2π. .

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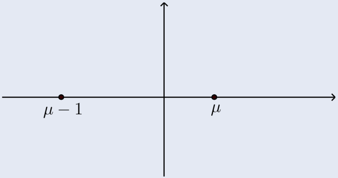

The RTBP

Equation of motion

The motion of the third body is given by

$$\begin{aligned} \ddot{x} - 2\dot{y} &= D_x\Omega(x,y) \\ \ddot{y} + 2\dot{x} &= D_y\Omega(x,y), \end{aligned} \tag{1}$$

where

$$\Omega(x,y) = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{\sqrt{(x-\mu)^2+y^2}} + \frac{\mu}{\sqrt{(x-\mu+1)^2+y^2}} + \frac{1}{2}\mu(1-\mu).$$

Position Primaries

⚠ The equation of motion is not well defined in the position of the primaries.

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Jacobi Integral & symmetry

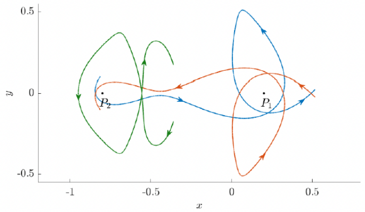
Jacobi Integral

$$C = 2\Omega(x, y) - \dot{x}^2 - \dot{y}^2. \quad (2)$$

Lemma

The equation satisfy the symmetry

$$(t, x, y, x', y') \longrightarrow (-t, x, -y, -x', y'). \quad (3)$$



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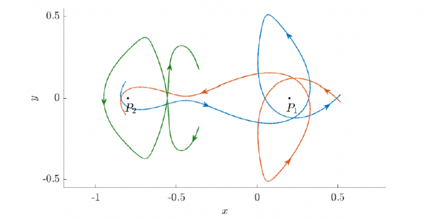
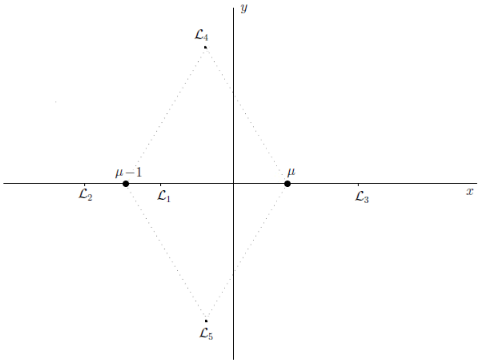


Equilibrium points

Five eq. points

- Triangular points: $L_{4,5} = (\mu - \frac{1}{2}, \pm\frac{\sqrt{3}}{2})$
- Collinear points: L_1, L_2 i L_3 s.t. $x_{L_2} \leq \mu - 1 \leq x_{L_1} \leq \mu \leq x_{L_3}$

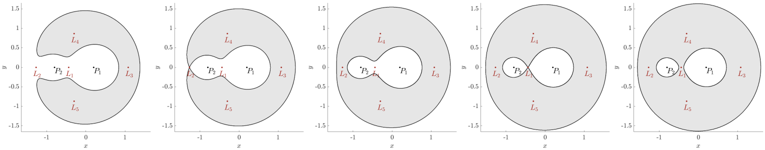


Hill's region

Hill's region:

From the Jacobi Integral C we can define the associated Hill's region as

$$\mathcal{R}(C) = \{(x,y) \in \mathbb{R}^2 \mid 2\Omega(x,y) \geq C\}.$$



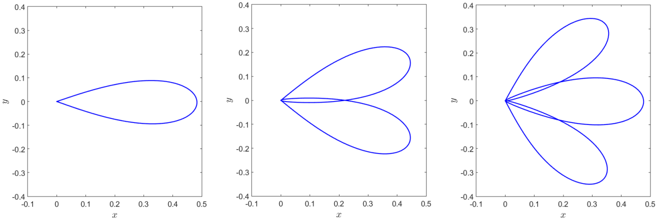
(a) $C < C_{L_2}$ (b) $C = C_{L_2}$ (c) $C_{L_2} < C < C_{L_1}$ (d) $C = C_{L_1}$ (e) $C > C_{L_1}$

n-Ejection collision orbits (ECO)

Definition

EC orbit such that ejects from the primary P_1, reaches n times a relative maximum in the distance with respect to P_1 and finally collides with it.

Examples: 1, 2 and 3 ECO



1st necessary step: regularization

McGehee's regularization

Idea:

- Polar change of coordinates with origin at the position of the primary that we want to regularize. ($P_1 = (0,0)$ and $P_2 = (1,0)$)
- New variables: $v = \dot{r} r^{1/2} \qquad u = r^{3/2}\dot{\theta}$
- Change of time: $dt/d\tau = r^{3/2}$

Equation of motion:

$$
\begin{aligned}
r' &= vr \\
\theta' &= u \\
v' &= \frac{1}{2}v^2 + u^2 + 2ur^{3/2} + r^3 - (1-\mu) - \mu r^2 \left(\cos\theta + \frac{r - \cos\theta}{(1 + r^2 - 2r\cos\theta)^{3/2}} \right) \\
u' &= -\frac{1}{2}uv - 2vr^{3/2} + \mu r^2 \sin\theta \left(1 - \frac{1}{(1 + r^2 - 2r\cos\theta)^{3/2}} \right),
\end{aligned}
$$

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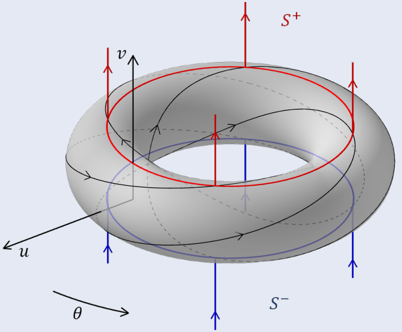
McGehee's regularization

Collision Manifold



$\Lambda = \{r = 0, v^2 + u^2 = 2(1-\mu)\}$

- $S^+ = \{(r = 0, \theta, v = v_0, u = 0)\}$
- $S^- = \{(r = 0, \theta, v = -v_0, u = 0)\}$

with $v_0 = \sqrt{2(1-\mu)}$

- $\forall\, p^+ \in S^+$:
  - $\dim(W^u(p^+)) = 2$
  - $\dim(W^s(p^+)) = 1$
- $\forall\, p^- \in S^-$:
  - $\dim(W^u(p^-)) = 1$
  - $\dim(W^s(p^-)) = 2$

An ejection orbit: an angle θ_0

Levi-Civita's regularization

Idea:

- New variables u and v s.t.:

$$\begin{cases} x - x_0 = u^2 - v^2 \\ y = 2uv \end{cases}$$

  where x_0 is the x-coordinate of the position of the primary that we want to regularize.

- Change of time: $dt/ds = 4(u^2 + v^2)$

Levi-Civita's regularization

Equation of motion:

$$\begin{cases} u'' - 8(u^2+v^2)v' = \left(4U(u^2+v^2)\right)_u \\ v'' + 8(u^2+v^2)u' = \left(4U(u^2+v^2)\right)_v \end{cases}$$

where $' = d/ds$ and

$$U = \frac{1}{2}\left[(1-\mu)\left(u^2+v^2\right)^2 + \mu r_1^2\right] + \frac{1-\mu}{u^2+v^2} + \frac{\mu}{r_2} - \frac{C}{2},$$

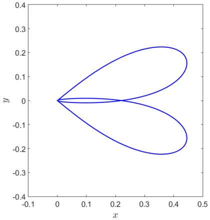
with $r_2 = \sqrt{(1+u^2-v^2)^2 + 4u^2v^2}$.

McGehee's regularization vs Levi-Civita's regularization

McGehee's regularization	Levi-Civita's regularization
"easy" equations	"difficult" equations
Collision Manifold	-
Collision/ejection in infinite time	Collision/ejection in finite time

From now on: Levi-Civita regularization

Fix μ and C
Initial condition of an ejection orbit given by θ_0 ($\theta_0 \in [0, 2\pi)$ in original coord. or $\theta_0 \in [0, \pi)$ in Levi Civita coord.)



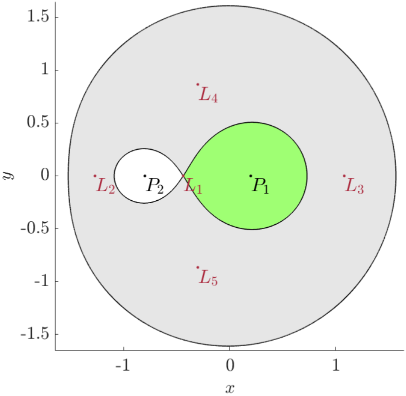
Analytical results for n-EC orbits

Some considerations:

- Hill region, Value of C such that

$$\{C \,|\, C \geq C_{L_1}(\mu)\}.$$

- We just need to regularize the position of the first primary.



n-EC orbits

Previous results: $n = 1$ [Llibre 1982, Lacomba and Llibre 1988, Chenciner and Llibre 1988]

Given a value of the mass parameter $\mu \in (0, 1)$ there exists a $\hat{C}(\mu)$ big enough such that for all values of the Jacobi constant $C \geq \hat{C}(\mu)$ the are exactly four 1-EC orbits.

$\Downarrow$

Goal

Extend this result for any $n > 1$

Theorem 1 (2021)

For all $n \in \mathbb{N}$, there exists a $\hat{K}(n)$ such that for $K \geq \hat{K}(n)$ and for any value of $\mu \in (0,1)$ and $C = 3\mu + K(1-\mu)^{2/3}$, there exist exactly four n-EC orbits, which can be characterized by:

- Two n-EC orbits both symmetric with respect to the x axis.
- Two n-EC orbits symmetric to each other with respect to the x axis.

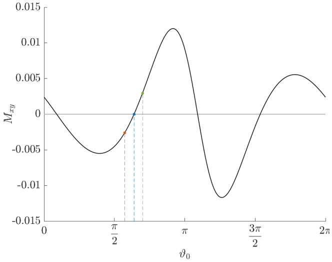
Theorem 2 (2021)

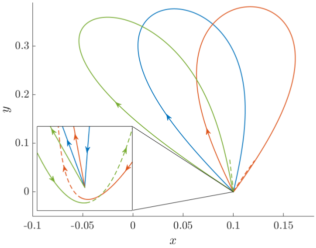
There exists an $\hat{L}$ such that for $L \geq \hat{L}$ and for any value of $\mu \in (0,1)$, $n \in \mathbb{N}$ and $C = 3\mu + Ln^{2/3}(1-\mu)^{2/3}$, there exist exactly four n-EC orbits, with the same characterization.

Idea of the proof:
characterization of an n-ECO: at the n-th minimum distance (with the primary) the angular momentum is equal to zero:
$M = uv' - vu' = 0$. So

$M(\theta_0) = 0$
$+$ a perturbative approach
$+$ the implicit function theorem

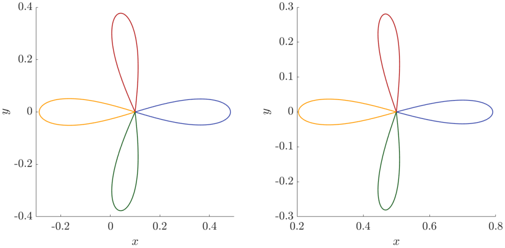
In order to compute the number of n-EC orbits we need to obtain the 0's of the angular momentum at the n-th minimum distance (with respect to the primary) of the ejection orbits.





Numerical results

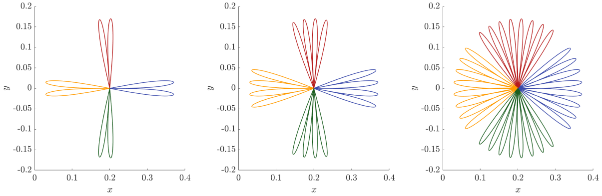
1-EC orbits



Trajectories of the four 1-EC orbits for $\mu = 0.1$, 0.5 and $C = 5$.

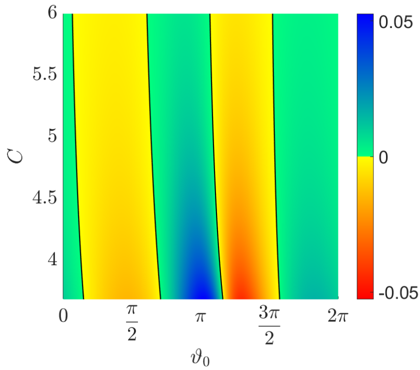
n-EC orbits

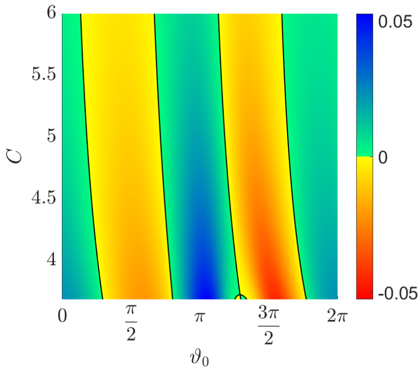
Numerically we have computed: For all $\mu \in (0,1)$ and n from 1 to 100, $C \geq \hat{C}(\mu, n)$ there exist exactly four n-EC orbits,



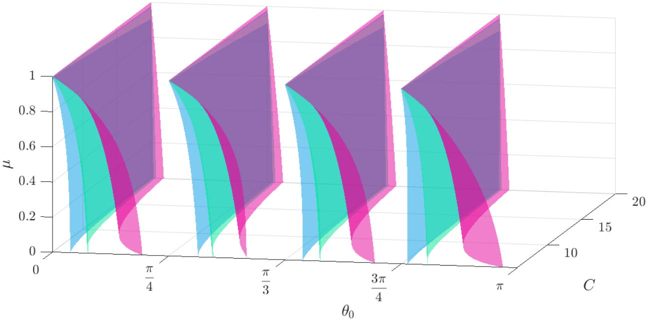
Trajectories of the four n-EC orbits for $\mu = 0.2$ and $C = 10$ for $n = 2,\ 4,\ 8$.

Computation 1-EC orbits, 2-EC orbits, varying C ($\mu = 0.1$)





n-EC orbits

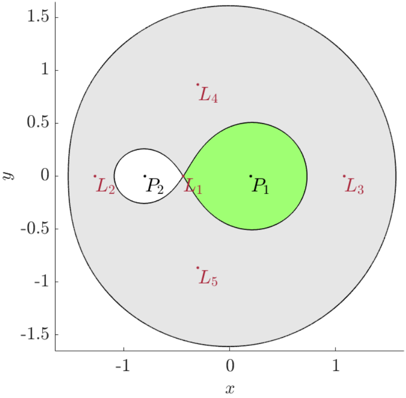


Continuation of families γ_n, δ_n, α_n and β_n of n-EC orbits for $n = 1,\ 2,\ 5$ (blue, green and purple colors respectively) in θ_0 and $C \in [5.5, 20]$ when varying $\mu \in (0, 1)$.

QUESTION: what happens if we decrease the Jacobi constant value?

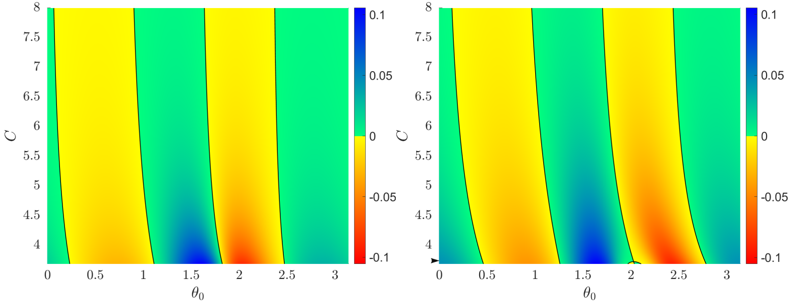
- $C_{L_1} \leq C$, bifurcations
- $C_{L_2} \leq C \leq C_{L_1}$

Still $C_{L_1} \leq C$



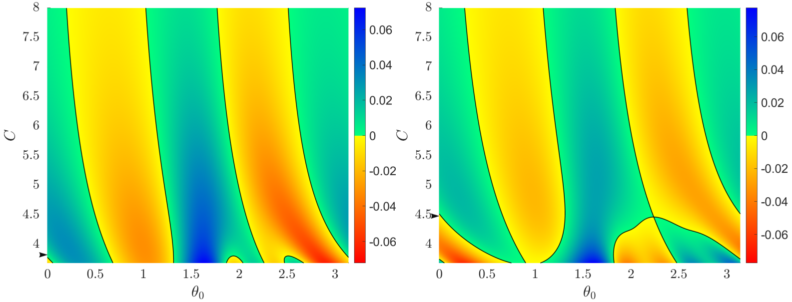
n-EC orbits ($\mu = 0.1$), still $C_{L_1} \le C$

$$n = 1, 2$$



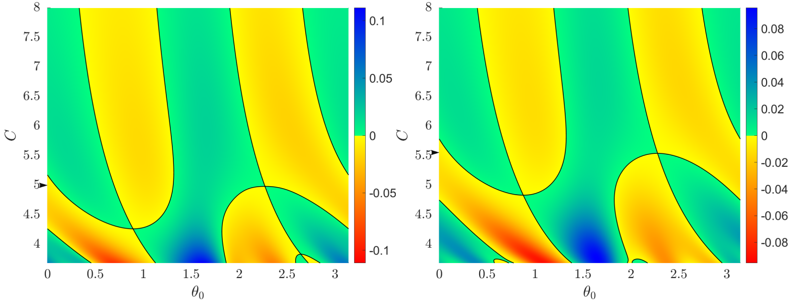
n-EC orbits ($\mu = 0.1$), still $C_{L_1} \leq C$. Bifurcations

$$n = 3, 4$$



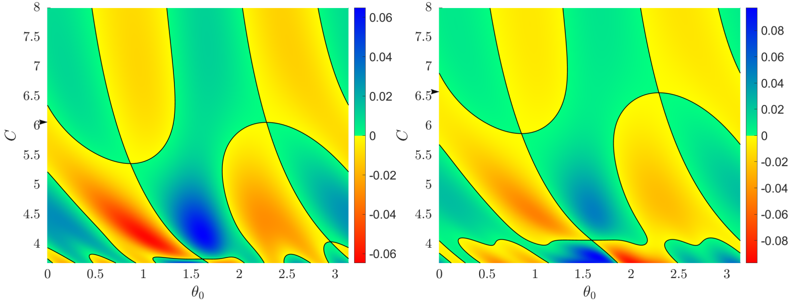
n-EC orbits ($\mu = 0.1$), still $C_{L_1} \leq C$. Bifurcations

$$n = 5, 6$$

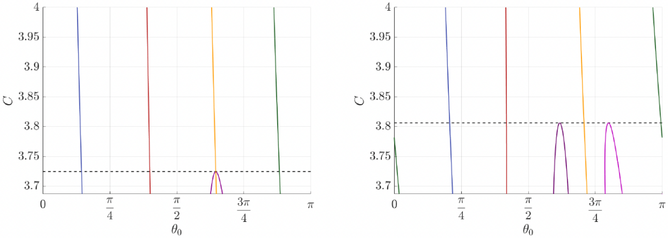


n-EC orbits ($\mu = 0.1$), still $C_{L_1} \leq C$. Bifurcations

$$n = 7, 8$$

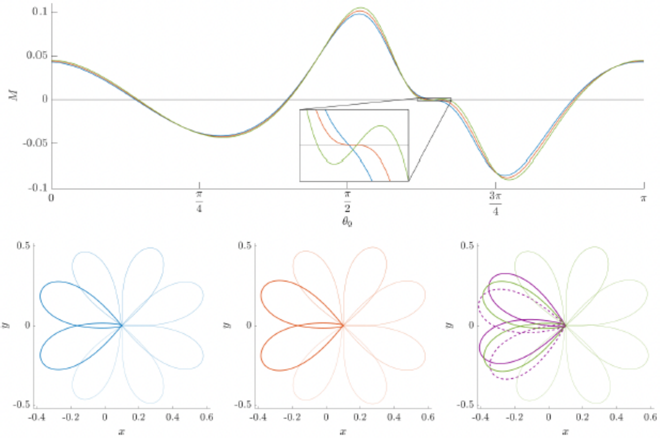


Bifurcation of n-EC orbits



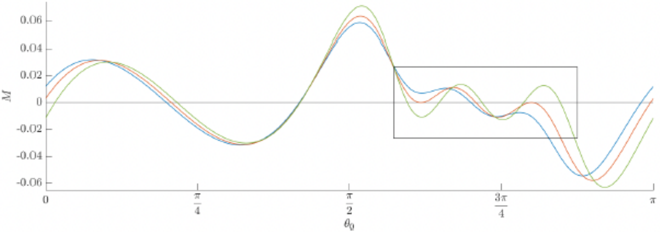
Bifurcation of n-EC orbits ($\mu = 0.1$, $n = 2$)

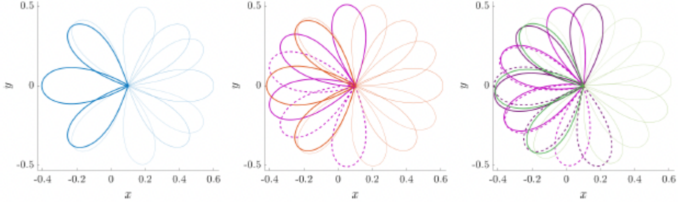
Zeros of the angular momentum



Bifurcation of n-EC orbits ($\mu = 0.1$, $n = 3$)

Zeros of the angular momentum



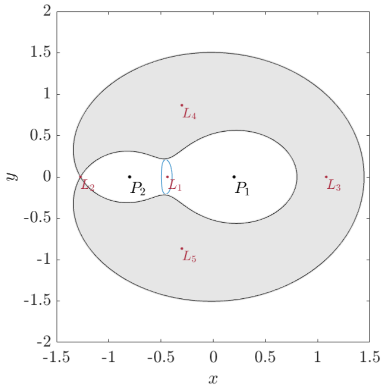


-So far $C_{L_1} \leq C$

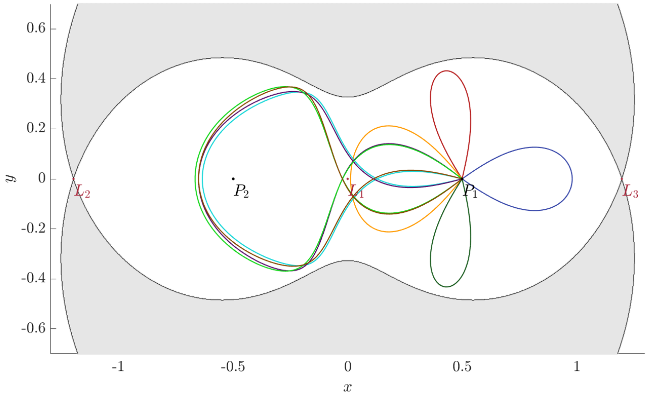
-Now $C_{L_2} \leq C \leq C_{L_1}$

Now $C_{L_2} \leq C \leq C_{L_1}$

- Interaction between ejection orbits and other?
- Other (not so simple) EC orbits?



Motivation: other 1-EC orbits for $\mu = 0.5$ and $C = C_{L_2}(\mu)$

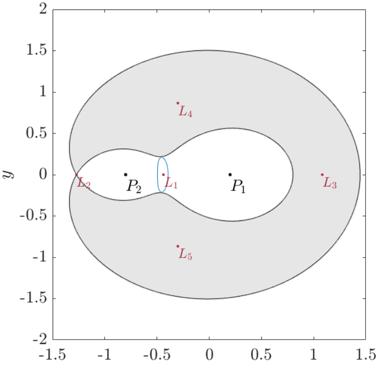


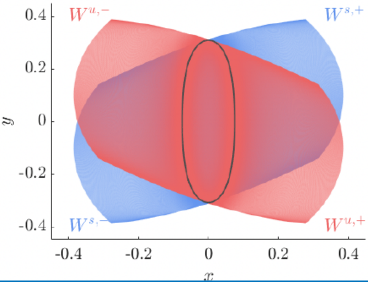
Transit regions and ejection/collision orbits

Transit regions and ejection/collision orbits

Goals:

- Evolution of ejection orbits along time.
- Role of other invariant objects, in particular: LPO_1.
- Study the EC orbits for $C \in [C_{L_{2,3}}, C_{L_1})$.

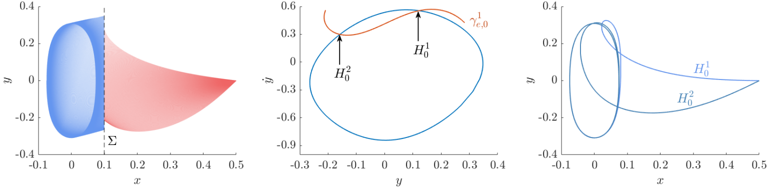




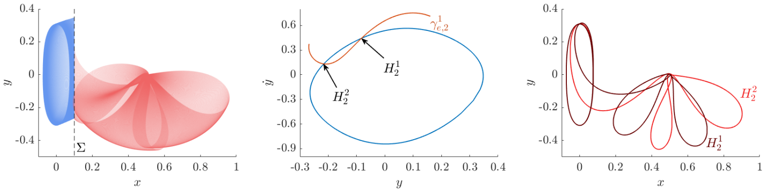
Transit regions

Question: How do we find "big" transit regions?

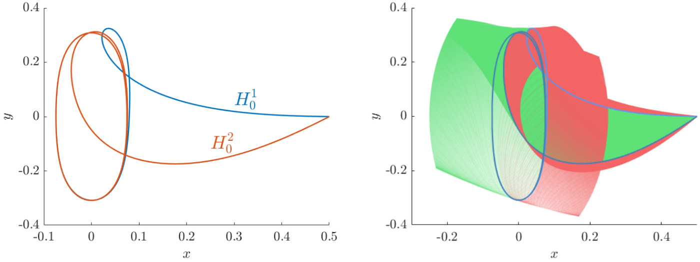
- Through the heteroclinic connections between a primary and the LPO_1, $E - O_1$.



Transit regions



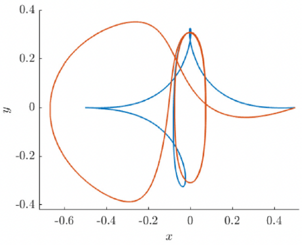
Transit regions

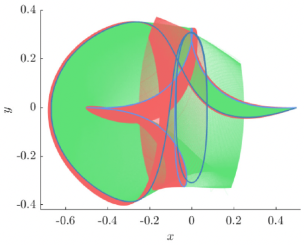


Transit regions - transition intervals

Is the determination of such transition intervals this simple? No!!!

Transit regions





Transit regions - transition intervals

How many heteroclinic E-O_1 do exist?
Infinitely many!!!

Transit regions - transition intervals

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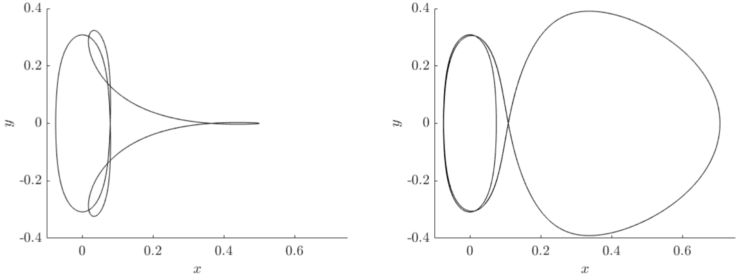
Infinitely many!!!

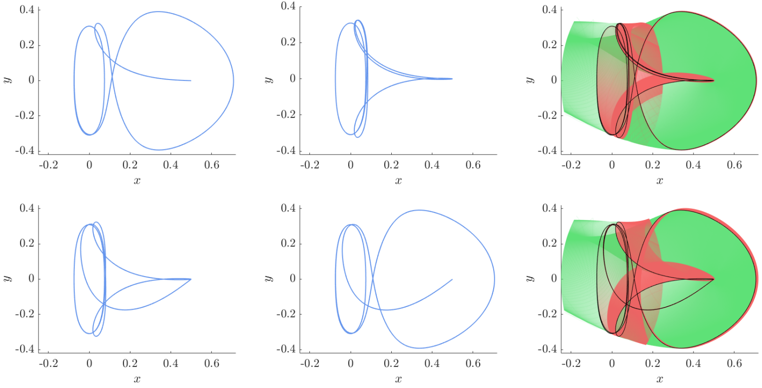
Transit regions - transition intervals

How many heteroclinic E-O_1 do exist?
Infinitely many!!!
Why?
Crucial: existence of homoclinic orbits to the LPO_1.

Let us play with two homoclinic Connections of the LPO_1...



...infinitely many Heteroclinic Connections $E - O_1$

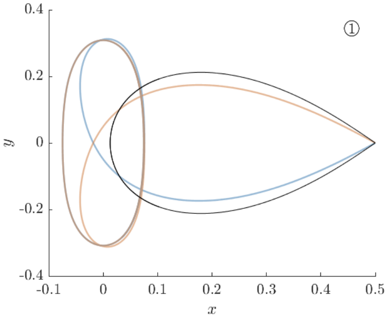


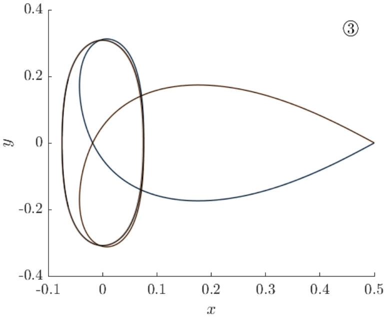
So, chaotic classification!!!

Transit regions - transition intervals

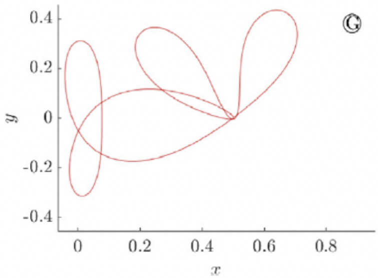
There are infinitely many homoclinic orbits to LPO_1!!!
So, crazy chaotic classification!!!

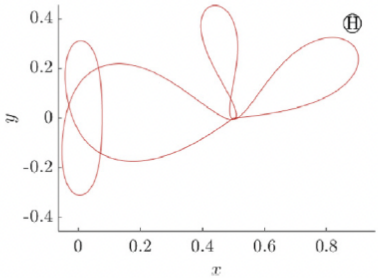
...infinitely many EC orbits to P_1



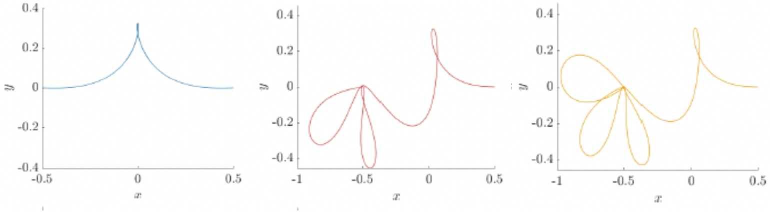


...infinitely many EC orbits to P_1





Orbits from P_1 to P_2



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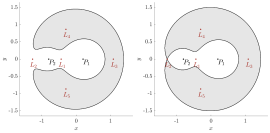
Future work

- Smaller values of C. Connections ejection-infinity?



(f) $C < C_{L_2}$ (g) $C = C_{L_2}$

- The spatial 3D case.
  - J. Llibre, J. Martinez Alfaro, Cel. Mech, Dyn. Astron., 35, 113–128, 1985
  - M. O., O. Rodriguez, J. Soler, CNSNS, 106410, 1–21, 2022. 3D case and $n = 1$
  - n-EC orbits, $n \geq 1$. Analytically and numerically.
- The elliptic RTBP.

Thank you very much for your attention!!!!