Equilibrium points and their linear stability in the planar equilateral restricted four-body problem: a review and new results

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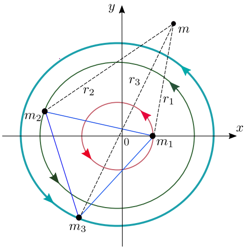
Casa abierta al tiempo

The planar equilateral restricted four-body problem

The planar equilateral restricted four-body problem (ERFBP) describes the motion of an infinitesimal mass m, moving under the Newtonian gravitational attraction of three bodies (primaries) with positive masses m_1, m_2 and m_3 lying at every moment at the vertices of an equilateral triangle, while each one describes a circular orbit around their common center of mass.



The problem has a rich literature that goes back at least to the work of Pedersen (1944, 1952), and has been treated from different points of view.

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In this talk we focus on summarizing some results obtained previously, as well as others obtained by us, on the existence and linear stability of equilibrium points, either as relative equilibria or central configurations of the $(3+1)$-body problem, where the primaries are forming an equilateral triangle configuration.

Special cases

- $m_i = m_j = 0$, we obtain the rotating Kepler's problem with $m_k = 1$, $k \neq i, j$, at the origin of coordinates.
- If $m_i = 0$, we obtain the circular R3BP, with the other two masses m_j, $m_k \neq 0$.
- $m_1 = m_2 = m_3$, we obtain the symmetric case with three masses equal.

Synodic frame (also referred as co-rotating frame)

We choose a synodic coordinate frame which places the triangle in the xy plane and fixes the center of mass at the origin. We orient the triangle so that the primary m_1 is on the positive x-axis.

The masses of the primaries are normalized so that $m_1 + m_2 + m_3 = 1$. Define $K = m_2(m_3 - m_2) + m_1(m_2 + 2m_3)$. The locations of the three primaries are given as

$$x_1 = \frac{|K|\sqrt{m_2^2 + m_2 m_3 + m_3^2}}{K}, \qquad y_1 = 0,$$

$$x_2 = -\frac{|K|\left[(m_2 - m_3)m_3 + m_1(2m_2 + m_3)\right]}{2K\sqrt{m_2^2 + m_2 m_3 + m_3^2}}, \qquad y_2 = \frac{\sqrt{3}}{2}\frac{m_3}{\sqrt{m_2^2 + m_2 m_3 + m_3^2}},$$

$$x_3 = -\frac{|K|}{2\sqrt{m_2^2 + m_2 m_3 + m_3^2}}, \qquad y_3 = -\frac{\sqrt{3}}{2}\frac{m_2}{\sqrt{m_2^2 + m_2 m_3 + m_3^2}}.$$

The equations of motion of m

$$\ddot{x} - 2\dot{y} = \Omega_x,$$

$$\ddot{y} + 2\dot{x} = \Omega_y,$$

where

$$\Omega = \Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3}$$

and $r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$, for $i = 1, 2, 3$.

The motion of the infinitesimal mass m is then studies in a similar manner, as in the restricted three-body problem.

Computation of the equilibrium points

To find the equilibrium points, as usual, we have to put the right motion equation to zero and solving for the variable value x, y:

$$\frac{\partial\Omega}{\partial x} = x - \frac{m_1(x-x_1)}{[(x-x_1)^2+y^2]^{3/2}} - \frac{m_2(x-x_2)}{[(x-x_2)^2+(y-y_2)^2]^{3/2}} - \frac{m_3(x-x_3)}{[(x-x_3)^2+(y-y_3)^2]^{3/2}} = 0,$$

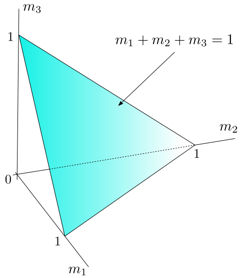
$$\frac{\partial\Omega}{\partial y} = y - \frac{m_1 y}{[(x-x_1)^2+y^2]^{3/2}} - \frac{m_2(y-y_2)}{[(x-x_2)^2+(y-y_2)^2]^{3/2}} - \frac{m_3(y-y_3)}{[(x-x_3)^2+(y-y_3)^2]^{3/2}} = 0.$$

The search for equilibrium points and the examination of their linear stability have been carried out by means of analytical and numerical methods.

The condition $m_1 + m_2 + m_3 = 1$ implies that ERFBP depends only on two mass parameters. In particular, we take $m_3 = 1 - m_1 - m_2$, so the two free parameters will be m_1 and m_2. The parameter space of the ERFBP is then reduced to the 2-simplex, called the triangle of masses,

$$\Sigma = \{(m_1, m_2, m_3) \in \mathbb{R}^3_+ \mid m_1 + m_2 + m_3 = 1,\ 0 \leq m_k \leq 1,\ k = 1, 2, 3\}$$

The sides correspond to mass values of the $(2+2)$-body problem (two large and two massless), while the vertices are the masses of the $(1+3)$-body problem (rotating Kepler's problem)



Pedersen (1944) made a combination of numerical and analytical methods to compute the number of central configurations for the infinitesimal mass of the $(1+3)$-body problem, when the three large masses form a Lagrangian equilateral triangle.

He found that, there can be 8, 9, or 10 equilibrium positions, depending on the values of the primary masses, and also proved that the set of degenerate equilibrium points is a simple closed curve contained in the interior of the simplex Σ. We denote this curve by $\mathfrak{B}$. He proved that, on the bifurcation curve $\mathfrak{B}$, there are 9 equilibrium points.

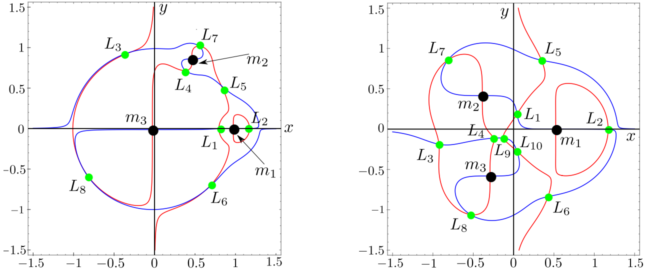
Pedersen's numerical calculations were later confirmed in a paper due to Simó (1978), where a numerical study was done for the number of relative equilibrium solutions in the four-body problem for arbitrary masses.

Arenstorf (1982) outlines some analytical proofs of the main results obtained by Pedersen. He stressed that a careful mathematical analysis and rigorous calculations required to prove these results are contained in the Ph.D. thesis of his student Gannaway (1981). As it turns out, in Gannaway's dissertation, there are only a few analytical evidences for particular assertions, namely, bifurcations and counting are verified once again only by a thorough numerical analysis.

Later on, Barros and Leandro (2011, 2014) were able to give a mathematically rigorous computer-assisted proof proving that, $\mathfrak{B}$ is a simple, closed, continuous curve, which lies inside the triangle Σ. They also confirmed that there are either 8, 9, or 10 equilibrium solutions (depending on the primary masses), and proved that 6 of them are outside of the Lagrange equilateral triangle formed by the primary bodies.

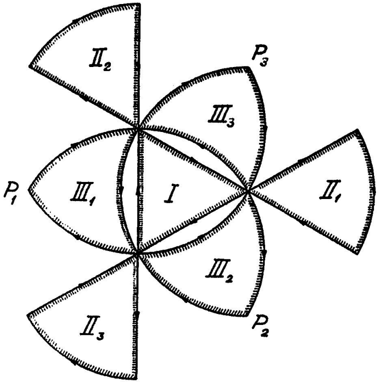
In 2022, Figueras et al. gave a new proof to the one performed by Barros and Leandro, about number of relative equilibria. Their proof is also based in computer-assisted methods.

Baltagiannis and Papadakis (2011) study numerically the ERFBP. They provided an extended list of possible combinations of primary bodies masses and their respective number of points of equilibrium. In the same vein, it is a recent work by Zotos (2020) where previously known results are retrieved.



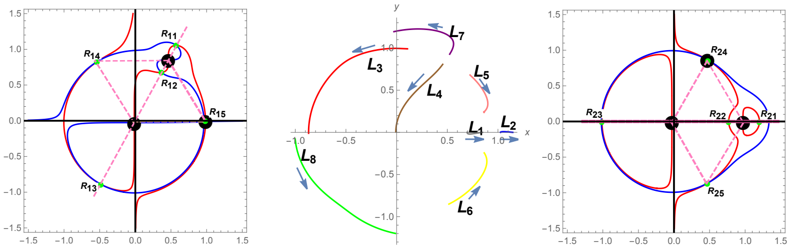
(a) 8 points (b) 10 points

Figure: Equilibrium points in the x-y plane located at the intersection of the curves $\Omega_x = 0$ (red) and $\Omega_y = 0$ (blue) for: (a) $m_1 = 0.02$ and $m_2 = 0.015$ with 8 equilibria and (b) $m_1 = 0.4$ and $m_2 = 0.35$ with 10 equilibria. The green dots denote the position of the equilibrium points and the positions of the primaries are marked by black dots.



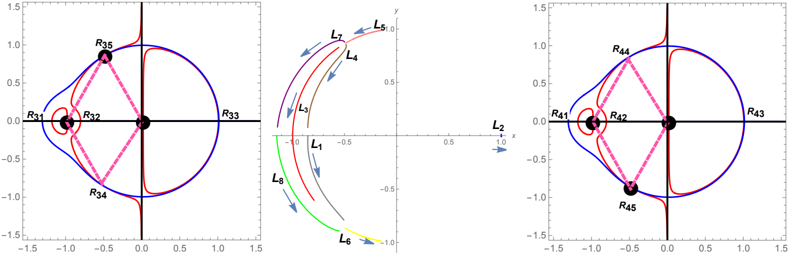
Pedersen (1944), Simó (1978), Leandro (2006)

$m_1 = 0.01, m_2, m_3$



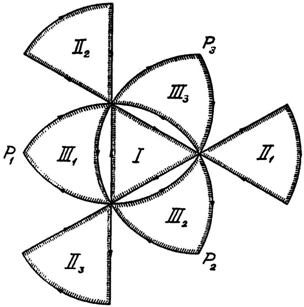
$m_1 = 0$	$m_1, m_2 \neq 0$	$m_2 = 0$
R_{13}	L_8	R_{23}
R_{11}	L_7	R_{24}
R_{12}	L_4	R_{24}
R_{14}	L_3	R_{24}
R_{15}	L_1	R_{21}
R_{15}	L_2	R_{22}
R_{15}	L_5	R_{24}
R_{15}	L_6	R_{25}

$m_1 = 0.99, m_2, m_3$

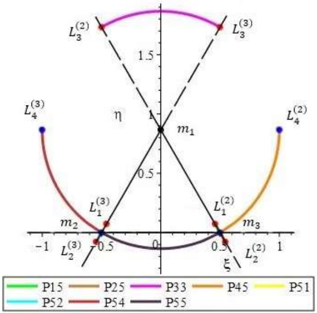


$m_1 = 0$	$m_1, m_2 \neq 0$	$m_1 + m_2 = 1$
R_{35}	L_5	R_{44}
R_{35}	L_4	R_{42}
R_{35}	L_3	R_{45}
R_{35}	L_7	R_{41}
R_{31}	L_8	R_{45}
R_{32}	L_1	R_{45}
R_{33}	L_2	R_{43}
R_{34}	L_6	R_{45}

Pedersen (1944)



Bardin and Volkov (2021)



Once the coordinates of the equilibrium conditions (x_0, y_0) have been determined, its linear stability can also be studied. We start by moving the equilibria to the origin of a coordinate system. The characteristic equation can be written as:

$$\lambda^4 + (4 - A_{11} - A_{22})\lambda^2 + A_{11}A_{22} - A_{12}^2 = 0, \tag{1}$$

$$A_{11} = 1 + \sum_{i=1}^{3} \frac{m_i[2(x_0 - x_i)^2 - (y_0 - y_i)^2]}{[(x_0 - x_i)^2 - (y_0 - y_i)^2]^{5/2}},$$

$$A_{12} = 3\sum_{i=1}^{3} \frac{m_i[(x_0 - x_i)(y_0 - y_i)]}{[(x_0 - x_i)^2 - (y_0 - y_i)^2]^{5/2}}, \tag{2}$$

$$A_{22} = 1 - \sum_{i=1}^{3} \frac{m_i[(x_0 - x_i)^2 - 2(y_0 - y_i)^2]}{[(x_0 - x_i)^2 - (y_0 - y_i)^2]^{5/2}}.$$

By virtue of Lyapunov's theorem on stability of equilibria for autonomous Hamiltonian systems with two degrees of freedom, we have that the equilibria are linearly stable if (1) has four pure imaginary roots.

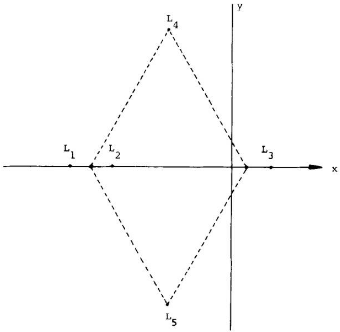
The stability is secured by the three following conditions:

$$\begin{aligned}
&(4 - A_{11} - A_{22})^2 - 4(A_{11}A_{22} - A_{12}^2) \geq 0,\\
&4 - A_{11} - A_{22} > 0,\\
&A_{11}A_{22} - A_{12}^2 > 0
\end{aligned}$$

which must be fulfilled simultaneously, whose frequencies ω_1 and ω_2 are given by

$$\omega_{1,2} = \frac{1}{\sqrt{2}}\sqrt{-4 + A_{11} + A_{22} \pm \sqrt{(4 - A_{11} - A_{22})^2 - 4(A_{11}A_{22} - A_{12}^2)}}.$$

In the circular restricted three-body problem (in short, CR3BP), there are two solutions in which the bodies move along the equilateral triangular solutions L_4 and L_5 (now known as the Lagrange solutions), and there are also three collinear solutions, attributed to Euler, denoted by L_1, L_2 and L_3.



The Routh's criterion for linear stability states that

$$\frac{m_1m_2 + m_2m_3 + m_3m_1}{m_1 + m_2 + m_3} < \frac{1}{27}$$

In the CR3BP with $m_1 = 1 - \mu, \quad m_2 = \mu, \quad 0 < \mu \leq \frac{1}{2}$.
At the Lagrange equilateral triangular equilibrium points L_4 and L_5, the Routh's critical mass ratio is

- $\mu_1 = \frac{1}{2}(1 - \sqrt{69}/9) \approx 0.038520896504551$

L_4 and L_5 points are stable for $\mu < \mu_1$.

This critical value μ_1 lies on the boundary of stability of L_4 and corresponds to 1:1 resonance between frequencies of the system linearized in a neighborhood of the point.

Considering that $m_3 = 1 - m_1 - m_2$

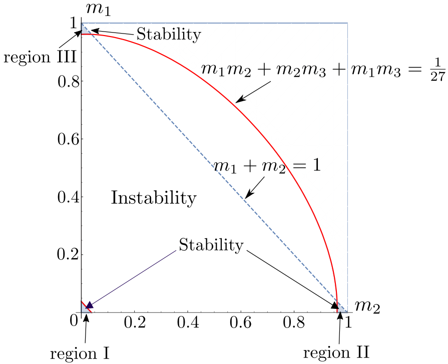


Figure: The three small "triangular" shaded regions I, II and III are stability domains of the Lagrange triangle configuration, and white region below the line $m_1 + m_2 = 1$ is instability domain. The mass parameter of the third primary is $m_3 = 1 - m_1 - m_2$. The red lines correspond to the Routh's critical curve.

It is known from numerical studies by, among others, Pedersen, Arenstorf, Simó and Baltagiannis-Papadakis, that the region on the plane (m_1, m_2) where the triangular configuration of the three primaries is stable, there exist eight equilibrium points.

It is noteworthy that Barros and Leandro used analytical and computational techniques to prove that, for all triples $(m_1, m_2, m_3) \in \Sigma$ which are close enough to $\partial\Sigma$, the number of central configurations is eight.

This means that we will have eight equilibrium points on regions I, II and III.

Zotos (2020)

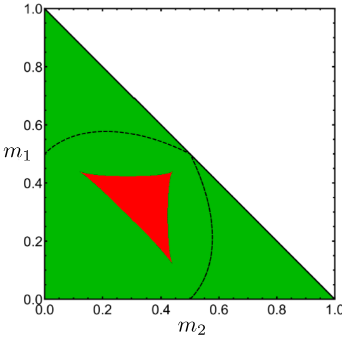
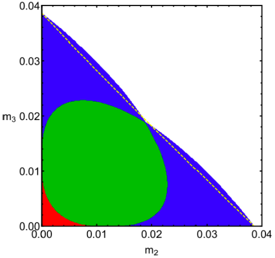
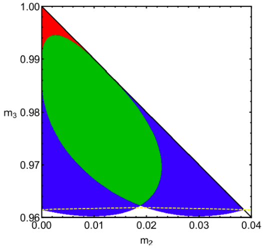
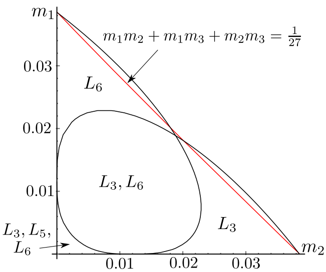


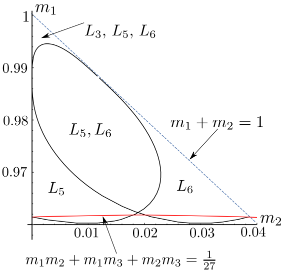
Figure: Regions on the (m_1, m_2) plane, corresponding to eight (green) and ten (red) points of equilibrium. The black dashed lines are indicating the set of mass values on which the sign of the relative positions coordinates of the primaries m_1 and m_2 is changing. The border of the red region corresponds to the bifurcation curve $\mathfrak{B}$.

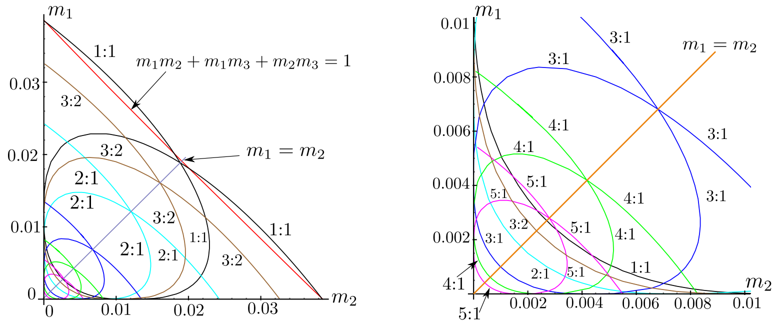
L_3, L_5, L_6 THE ONLY STABLE EQUILIBRIUM POINTS







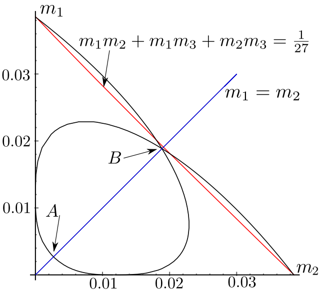




(a) Resonance curves in region I.

(b) The enlarged graph.

Figure: Plots of all resonance curves in region I for equilibrium points L_3, L_5 and L_6.



$A(0.002716, 0.002716)$ is precisely the point with resonance 1:1 for L_5 where Burgos and Delgado (2013) (L_2 in their notation) established the existence of a "blue sky catastrophe", and the other is $B(0.01883, 0.01883)$ resonance values for L_3 and L_6, whose stability has not hitherto been studied.

Curves and points obtained by Simó (1978)

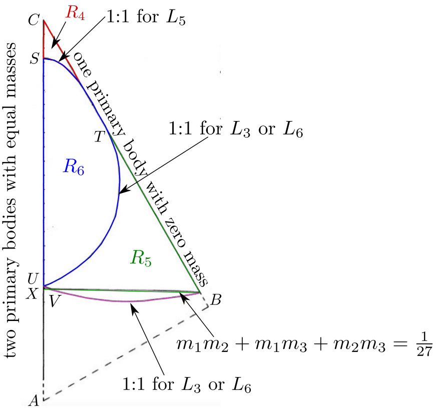
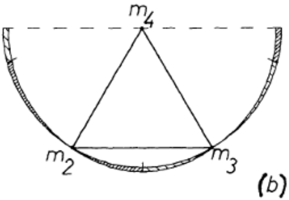


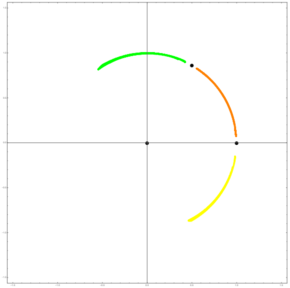
Figure: The region $CYVX$ is the linear stability area of the primaries in the mass space, where $R_4 = CST$, $R_6 = CUT$ and $R_5 = CUVY$ are stability regions of equilibrium points L_5, L_3 and L_6.

Regions of stability

In 1978, Simó claimed:

"The stable relative equilibria of m are confined to a small belt around the greater mass at approximately the same distance from the two smaller finite bodies m_2, m_3 and in the same semiplane".





Belt not to scale

$L_6 \rightarrow$ amarillo $\quad L_3 \rightarrow$ verde $\quad L_5 \rightarrow$ naranja

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